Geodesic flows on quotients of Bruhat-Tits buildings

and Number Theory

Anish Ghosh

University of East Anglia

joint with J. Athreya (Urbana) and A. Prasad (Chennai)

- Diagonal Actions

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- Recurrence

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- Buildings

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- Recurrence
- Buildings
- Shrinking Targets

Shrinking

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Target

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Problems

Shrinking

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- $0 - 1$ laws

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- Return time asymptotics

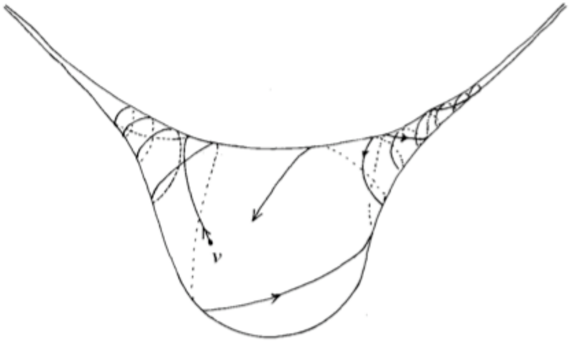
Shrinking

Target

Problems

- $0 - 1$ laws
- Return time asymptotics
- Heirarchy

Excursions in hyperbolic 3 manifolds



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- if and only if

$$\sum_{t=1}^{\infty} e^{-kr_t} = \infty$$

Logarithm Laws

Corollary

For almost every $y \in Y$ and almost all $\xi \in T_y(Y)$

$$\limsup_{t \to \infty} \frac{\operatorname{dist}(y, \gamma_t(y, \xi))}{\log t} = \frac{1}{k}$$

Some Number theory

- For almost every z, there exist infinitely many pairs $(p, q) \in O(\sqrt{-d})$ such that

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- And then apply the Borel-Cantelli Lemma

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Khintchine-Groshev Theorem

- For almost all $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$, there exist infinitely many $\mathbf{q} \in \mathbb{Z}^n, \mathbf{p} \in \mathbb{Z}^m$ such that

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If time permits

More general *multiplicative* results

- $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ can be identified with the space X_n of unimodular lattices in $\mathbb{R}^n$

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- The systole of a lattice

$$s(\Lambda) := \min_{v \in \Lambda} \|v\|$$

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- Read Diophantine properties of x from the g_t (semi) orbits of $u_x\mathbb{Z}^2$
- For example, x is badly approximable if and only if the orbit of $u_x\mathbb{Z}^2$ is bounded

More generally

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- if and only if there are infinitely many $t > 0$ such that

$$s(g_t u_x \mathbb{Z}^2) \leq r(t)$$

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- The speed is governed by convergence of

$$\sum_{t=0}^{\infty} \mathrm{vol}(X_n(t))$$

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- To force quasi-independence on average

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- Which is crucial in G-Gorodnik-Nevo

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- For $g, h \in S$

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- In fact, any lattice in a p-adic Lie group is necessarily cocompact (Tamagawa)

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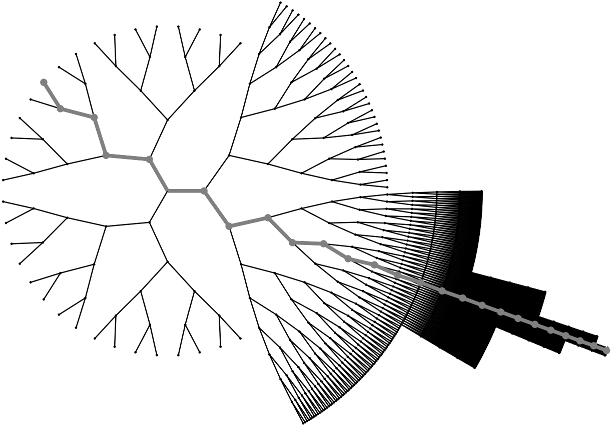
- And $\Gamma = \mathrm{PGL}_2(\mathbb{F}_p[t, t^{-1}])$
- Which turns out to be a non-uniform lattice in G

The Tree of $\mathsf{SL}_2(\mathbb{F}_2((X^{-1})))$



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- The graph $K\backslash G/\Gamma$ where G is an algebraic group
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- K is a compact open subgroup and Γ is a lattice in G

Local fields of positive characteristic

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- $\mathbb{F}_p((X^{-1})) \leftrightsquigarrow \mathbb{R}$

Euclidean buildings

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- And G acts by isometries

Borel-Harish-Chandra, Behr-Harder

$G(\mathbb{F}_p[X])$ is a lattice in $G(\mathbb{F}_p((X^{-1})))$

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Lubotzky

G (rank 1) has non-uniform and uniform, arithmetic and non-arithmetic lattices.

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- Namely lattices in rank 1 algebraic groups

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- And reminiscent of Sullivan

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- This includes any irreducible lattice in higher rank

And their quotients

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- And several results in Diophantine Approximation

Main Result

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- $\mathcal{B}$ is *Borel-Cantelli for $\mathcal{F}$* if for every sequence $\{A_n \ : \ n \in \mathbb{N}\}$ of sets from $\mathcal{B}$

$$\mu(\{x \in G/\Gamma \mid f_n(x) \in A_n \text{ for infinitely many } n \in \mathbb{N}\}) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \mu(A_n) < \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} \mu(A_n) = \infty \end{cases}$$

- Δ a function on G/Γ define the tail distribution function

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- Call Δ "distance-like" if

$$\Phi_\Delta(n) \asymp s^{-\kappa n} \quad \forall\, n \in \mathbb{Z}$$

- Let $\mathcal{F} = \{f_n \mid n \in \mathbb{N}\}$ be a sequence of elements of G satisfying

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- is Borel-Cantelli for $\mathcal{F}$

Ergodic ideas

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Effective mixing

Let G be a connected, semisimple, linear algebraic group, defined and quasi-split over k and Γ be an non-uniform irreducible lattice in G. For any $g \in G$ and any smooth functions $\phi, \psi \in L_0^2(G/\Gamma)$,

$$|\langle \rho_0(g)\phi, \psi \rangle| \ll \|\phi\|_2 \|\psi\|_2 H(g)^{-1/b}.$$

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- We use the Burger-Sarnak method as extended to p-adic and positive characteristic fields by
- Clozel and Ullmo
- Along with the solution of the Ramanujan conjecture for GL_2 due to Drinfield

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- Building on work of Harder, Soulé and others

Picture credits

- D. Sullivan, Acta Math 1982

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- Ulrich Görtz, Modular forms and special cycles on Shimura curves, S. Kudla, M. Rapoport and T. Yang