Symbolic coverings for general Pisot β-transformations

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Introduction

Let $\beta > 1$ and $A = \{a_1, \dots, a_m\}$ a set of real numbers with $a_1 < a_2 < \cdots < a_m$. Expressions of the form

$$x = \sum_{n=1}^{\infty} \frac{b_n}{\beta^n},$$

with $b_n \in A$ for all $n \geq 1$, are called β-expansions with arbitrary digits.

This gives numbers in the interval $[\frac{a_1}{\beta - 1}, \frac{a_m}{\beta - 1}]$.

β is called the base, A is the digit set and elements of A are called digits. The sequence $b_1 b_2 \cdots$ is a digit sequence for x.

Allowable digit sets

If, for a given $\beta > 1$, a set of real numbers $A = \{a_1, \ldots, a_m\}$ satisfies

(i) $a_1 < \cdots < a_m$,

(ii) $\max_{2 \leq j \leq m} (a_j - a_{j-1}) \leq \frac{a_m - a_1}{\beta - 1}$,

it is called an allowable digit set.

Theorem (Pedicini, 2005)

If A is an allowable digit set for β, then every $x \in \left[\frac{a_1}{\beta - 1}, \frac{a_m}{\beta - 1}\right]$ has a β-expansion with digits in A.

Outline

- Introduce a class of transformations that generate β-expansions.
- Characterize the set of digit sequences given by such a transformation.
- For specific β's (Pisot units) give a construction of a natural extension for the transformation.
- From the natural extension, get an absolutely continuous invariant measure.
- Under a further assumption, construct a symbolic covering of the torus that is almost everywhere finite-to-one.
- Give an example in which this map is not one-to-one.

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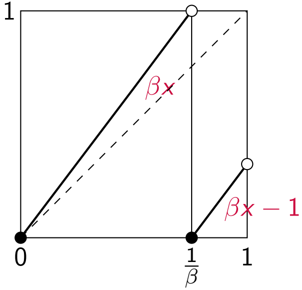
Transformations

For each $\beta > 1$ and allowable digit set $A = \{a_1, \ldots, a_m\}$ there exist transformations that generate β-expansions with digits in A by iteration.

Example: $x \mapsto \beta x \pmod 1$

Consider a non-integer $1 < \beta < 2$ and digit set $A = \{0, 1\}$. One transformation that generates β-expansions with digits in this set is the map $Tx = \beta x \pmod 1$.

The classical β-expansions



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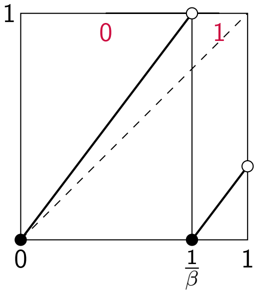

Assign a digit to each interval.

Make a digit sequence by setting

$$b_1(x) = \begin{cases} 0, & \text{if } x < \frac{1}{\beta}, \\ 1, & \text{otherwise.} \end{cases}$$

and $b_n(x) = b_1(T^{n-1}x)$ for $n \geq 1$. Then we have $Tx = \beta x - b_1$ and $T^2x = \beta Tx - b_2$, etc.

$$x = \frac{b_1}{\beta} + \frac{Tx}{\beta} = \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \frac{T^2x}{\beta^2} = \cdots = \sum_{k=1}^{n} \frac{b_k}{\beta^k} + \frac{T^n x}{\beta^n}.$$

In the limit $x = \sum_{k=1}^{\infty} \frac{b_k}{\beta^k}$.

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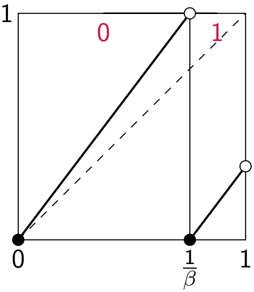

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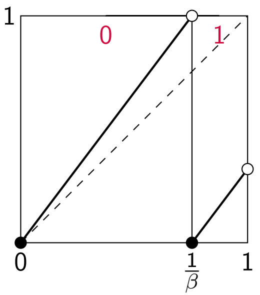

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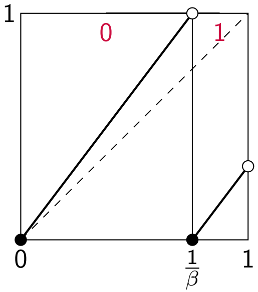

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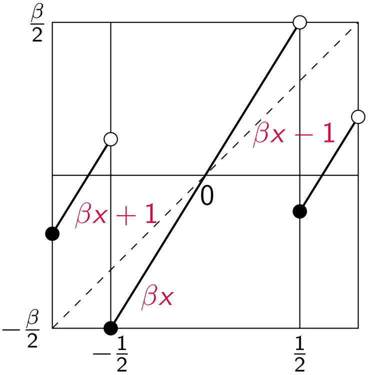
In the limit $x = \sum_{k=1}^{\infty} \frac{b_k}{\beta^k}$.

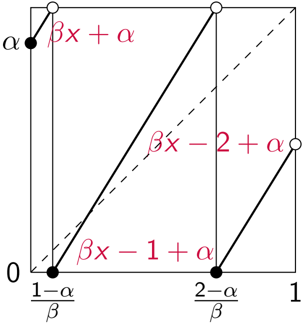
Other transformations: the minimal weight transformation

Take β to be the golden mean and $A = \{-1, 0, 1\}$. This is a minimal weight transformation, i.e., if an x has a finite β-expansion, then the expansion generated by this transformation has the highest number of 0's. [Frougny & Steiner, 2009]



Other transformations: the linear mod 1 transformation

Take $\beta > 1$ and $0 \leq \alpha < 1$. Suppose $n < \beta + \alpha \leq n + 1$. The linear mod 1 transformation below ($Tx = \beta x + \alpha \pmod 1$) gives β-expansions with digits in $\{j - \alpha : 0 \leq j \leq n\}$.



The class of transformations

Given a real number $\beta > 1$ and a digit set $A = \{a_1, \ldots, a_m\}$, we consider the class of transformations that have the following properties.

- For each digit in the digit set a_i, there is a bounded interval Z_i and if $i \neq j$, then $Z_i \cap Z_j = \emptyset$. We assume $Z_i = [b_i, c_i)$ for $b_i, c_i \in \mathbb{R}$.
- On the interval Z_i the transformation is given by $Tx = \beta x - a_i$.
- If $X = \bigcup_{i=1}^{m} Z_i$, then $TX = X$.

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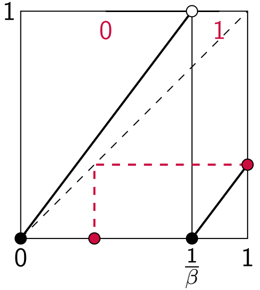
Admissible sequences



Expansions $\sum_{k=1}^{\infty} \frac{b_k}{\beta^k}$ are uniquely determined by the digit sequences $(b_k)_{k\geq 1}$.

A transformation T with digit set A does not produce all sequences in $A^{\mathbb{N}}$.

Here, for example, the block 11 never occurs.

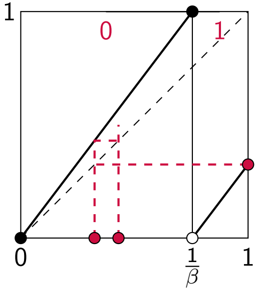
The set of admissible sequences

Given a transformation T for a $\beta > 1$ and digit set A, we call a sequence $u_1 u_2 \cdots \in A^{\mathbb{N}}$ admissible for T if there is an $x \in X$ such that $u_1 u_2 \cdots = b_1(x) b_2(x) \cdots$.

A two-sided sequence $\cdots u_{-1} u_0 u_1 \cdots$ is called admissible if for each $n \in \mathbb{Z}$ there is an $x \in X$, such that $u_n u_{n+1} \cdots = b_1(x) b_2(x) \cdots$.

Notation: $\mathcal{S}^+$ is the set of one-sided admissible sequences and $\mathcal{S}$ is the set of two-sided ones.

Admissible sequences for $x \mapsto \beta x \pmod 1$



For the map $Tx = \beta x \pmod 1$ there is a characterisation of all the generated sequences.

Consider the map $\tilde{T}$, given by

$$\tilde{T}x = \begin{cases} \beta x, & \text{if } x \leq \frac{1}{\beta}, \\ \beta x - 1, & \text{if } \frac{1}{\beta} < x \leq 1. \end{cases}$$

Theorem (Parry, 1960)

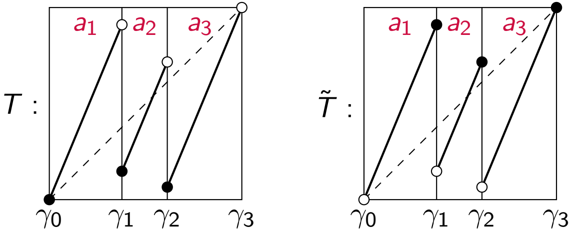
Let $\tilde{b}(1)$ be the expansion of 1 generated by $\tilde{T}$. Then a sequence $u_1 u_2 \cdots \in \{0,1\}^{\mathbb{N}}$ is generated by T iff for each $n \geq 1$,

$$u_n u_{n+1} \cdots \prec \tilde{b}(1),$$

where $\prec$ is the lexicographical ordering.

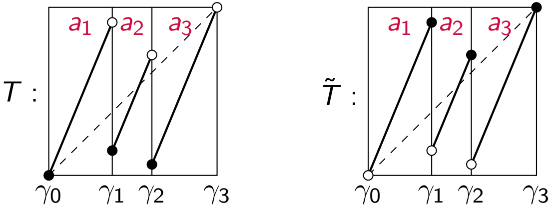
Admissible sequences

We can characterise the digit sequences generated by any transformation similarly.



Let $b(x)$ be a digit sequence given by T and $\tilde{b}(x)$ the one given by $\tilde{T}$. Then we have the following characterization in terms of the sequences $b(\gamma_j)$ and $\tilde{b}(\gamma_j)$.

Admissible sequences



Admissible sequences

A sequence $u_1 u_2 \cdots \in \{a_1, \ldots, a_m\}^{\mathbb{N}}$ is generated by T iff for each $n \geq 1$, if $u_n = a_j$, then

$$b(\gamma_j) \preceq u_n u_{n+1} \cdots \prec \tilde{b}(\gamma_{j+1}),$$

where $\preceq$ denotes the lexicographical ordering.

Shift space

Let $A = \{a_1, \ldots, a_m\}$. Define the map $\xi\big((b_k)_{k\geq 1}\big) = \sum_{k=1}^{\infty} \frac{b_k}{\beta^k}$.

Let σ denote the left shift on $\mathcal{S}^+$. Then ξ gives the commuting diagram:

$$\begin{array}{ccc} \mathcal{S}^+ & \xrightarrow{\ \sigma\ } & \mathcal{S}^+ \\ \xi \big\downarrow & & \big\downarrow \xi \\ X & \xrightarrow{\ T\ } & X \end{array}$$

Using the symbolic space $(\mathcal{S}, \sigma)$, we find a 'nice' natural extension of the dynamical system (X, T).

Natural extensions

Consider the non-invertible system $(X, \mathcal{B}, \mu, T)$, where $\mathcal{B}$ is the Lebesgue σ-algebra on X and μ an invariant measure for T. Then a version of the natural extension of $(X, \mathcal{B}, \mu, T)$ is an invertible system $(\hat{X}, \hat{\mathcal{B}}, \nu, \hat{T})$, such that

- There is a map $\pi : \hat{X} \to X$ that is surjective, measurable and such that $\pi \circ \hat{T} = T \circ \pi$.
- For all measurable sets $E \in \mathcal{B}$, $\mu(E) = (\nu \circ \pi^{-1})(E)$. We can define the measure μ in this way.
- This system is the smallest in the sense of σ-algebras: $\bigvee_{n \geq 0} \hat{T}^n(\pi^{-1}(\mathcal{B})) = \hat{\mathcal{B}}$.

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Pisot β's

$X = \bigcup_{i=1}^{m} Z_i$ where $Z_i = [b_i, c_i)$ are disjoint intervals and $Tx = \beta x - a_i$ on Z_i.

From now on we assume that the real number $\beta > 1$ is a Pisot unit:

- β is an algebraic unit: it is a root of a minimal polynomial of the form $x^d - c_1 x^{d-1} - \cdots - c_d$, with $c_i \in \mathbb{Z}$ for all i and $c_d \in \{-1, 1\}$.
- Denote all the other roots of the polynomial $x^d - c_1 x^{d-1} - \cdots - c_d$ by β_j, then $|\beta_j| < 1$ for all j.

We also assume that $a_i \subset \mathbb{Q}(\beta)$ for all $1 \leq i \leq m$. For convenience, we take $a_i \subset \mathbb{Z}$.

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Hyperbolic toral automorphism

Let $\beta > 1$ be a Pisot unit with minimal polynomial $x^d - c_1 x^{d-1} - \cdots - c_d$, $c_i \in \mathbb{Z}$ and $c_d \in \{-1, 1\}$. Consider the companion matrix M:

$$M = \begin{pmatrix} c_1 & c_2 & \cdots & c_{d-1} & c_d \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

The eigenvalues are β and $\beta_2, \ldots, \beta_d$, the Galois conjugates of β. Also, $|\det M| = 1$, so M is invertible.

Hyperbolic toral automorphism

β is a Pisot unit with minimal polynomial $x^d - c_1 x^{d-1} - \cdots - c_d$ and Galois conjugates $\beta_2, \ldots, \beta_d$.

$$\begin{pmatrix} c_1 & c_2 & \cdots & c_{d-1} & c_d \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} \beta_j^{d-1} \\ \beta_j^{d-2} \\ \vdots \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 \beta_j^{d-1} + \cdots + c_d \\ \beta_j^{d-1} \\ \vdots \\ \beta_j \end{pmatrix} = \begin{pmatrix} \beta_j^d \\ \beta_j^{d-1} \\ \vdots \\ \beta_j \end{pmatrix} = \beta_j \mathbf{v}_j.$$

The natural extension space

We use the eigenvectors of M to define the natural extension space by mapping the admissible sequences into $\mathbb{R}^d$.

Let $w \cdot u = \cdots w_{-1} w_0 u_1 u_2 \cdots \in A^{\mathbb{Z}}$. Define the map $\psi : A^{\mathbb{Z}} \to \mathbb{R}^d$ by:

$$\psi(w \cdot u) = \sum_{n=1}^{\infty} \frac{u_n}{\beta^n} \mathbf{v}_1 - \sum_{j=2}^{d} \sum_{n=0}^{\infty} w_{-n} \beta_j^n \mathbf{v}_j.$$

$$(\beta > 1) \qquad (|\beta_j| < 1)$$

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The natural extension transformation

For the natural extension transformation $\hat{T} : \hat{X} \to \hat{X}$ we want:

- $\hat{T}$ is a.e. invertible.
- $\hat{T}$ preserves the dynamics of T.
- $\hat{T}$ is invariant wrt the Lebesgue measure.

Partition $\hat{X} = \bigcup_{i=1}^{m} \hat{Z}_i$ with $\hat{Z}_i = \{\psi(w \cdot u) \mid u_1 = a_i\}$. For $\mathbf{x} \in \hat{X}$, write $\mathbf{x} = x\mathbf{v}_1 - \sum_{j=2}^{d} y_j \mathbf{v}_j$. If $\mathbf{x} \in \hat{Z}_i$, take

$$\begin{aligned} \hat{T}\mathbf{x} &= \overbrace{(\beta x - a_i)}^{Tx}\mathbf{v}_1 - \sum_{j=2}^{d} (\beta_j y_j + a_i)\mathbf{v}_j \\ &= M\mathbf{x} - \sum_{j=1}^{d} a_i \mathbf{v}_j. \end{aligned}$$

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An invariant measure for T

The Lebesgue measure λ^d on $\mathbb{R}^d$ is invariant for $\hat{T}$ (recall $|\det M| = 1$).

Let $\pi : \hat{X} \to X$ be given by $\pi\left(x\mathbf{v}_1 - \sum_{j=2}^{d} y_j \mathbf{v}_j\right) = x$.

Define the measure μ on X by $\mu(E) = (\lambda^d \circ \pi^{-1})(E)$ for each Borel measurable set E.

Then μ is invariant for T.

Purely periodic points

Denote by H the subspace of $\mathbb{R}^d$ spanned by the real and imaginary parts of $\mathbf{v}_2, \ldots, \mathbf{v}_d$.

Let $\Gamma_j : \mathbb{Q}(\beta) \to \mathbb{Q}(\beta_j) : \beta \mapsto \beta_j$.

Define the function $\Phi : \mathbb{Q}(\beta) \to H$ by $\Phi(x) = \sum_{j=2}^{d} \Gamma_j(x)\mathbf{v}_j$.

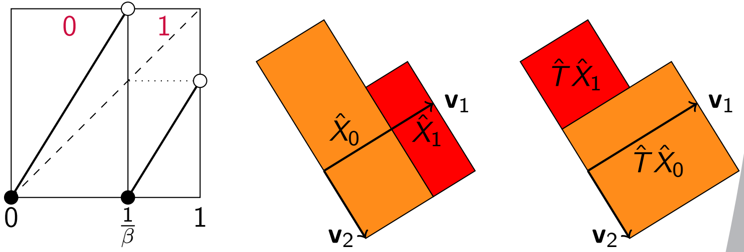
Theorem

The expansion of x generated by T is purely periodic iff $x \in \mathbb{Q}(\beta)$ and $x\mathbf{v}_1 + \Phi(x) \in \hat{X}$.
(For $x \mapsto \beta x \pmod 1$, Ito and Rao(2005))

An example: the golden mean

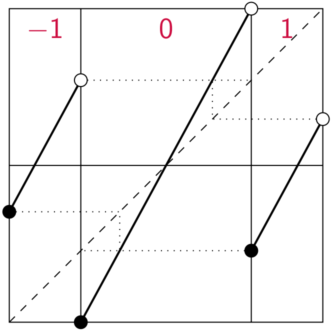
Let β be the golden mean, i.e., the real root > 1 of $x^2 - x - 1$, and $Tx = \beta x \pmod 1$. Then $A = \{0, 1\}$ and

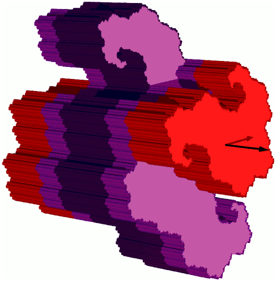
$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{v}_1 = \begin{pmatrix} \beta \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{\beta} \\ 1 \end{pmatrix}.$$



An example: the tribonacci number

Let β be the tribonacci number. Take $A = \{-1, 0, 1\}$, $X_{-1} = \left[-\frac{\beta}{\beta+1}, -\frac{1}{\beta+1}\right)$, $X_0 = \left[-\frac{1}{\beta+1}, \frac{1}{\beta+1}\right)$ and $X_1 = \left[\frac{1}{\beta+1}, \frac{\beta}{\beta+1}\right)$. Then T is a minimal weight transformation.





Symbolic coverings

Let $A = \{a_1, \ldots, a_m\} \subset \mathbb{Z}$. Recall the definition of the map $\psi : A^{\mathbb{Z}} \to \mathbb{R}^d$, but consider it on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$:

$$\psi(w \cdot u) = \sum_{n=1}^{\infty} \frac{u_n}{\beta^n}\mathbf{v}_1 - \sum_{j=2}^{d}\sum_{n=0}^{\infty} w_{-n}\beta_j^n \mathbf{v}_j \pmod{\mathbb{Z}^d}.$$

$$\begin{array}{ccc} A^{\mathbb{Z}} & \xrightarrow{\ \sigma\ } & A^{\mathbb{Z}} \\ \psi \downarrow & & \downarrow \psi \\ \mathbb{T}^d & \xrightarrow{\ M\ } & \mathbb{T}^d \end{array}$$

On $A^{\mathbb{Z}}$, ψ is a very many-to-one map. We would like to say more for $\psi|_{\mathcal{S}}$.

Finite-to-one covering map

Rauzy, 1982 For the Pisot number given by the polynomial $x^3 - x^2 - x - 1$ (tribonacci number), the map is a.e. one-to-one for the β-shift $\overline{\mathcal{S}}$ given by the map $x \mapsto \beta x \pmod 1$.

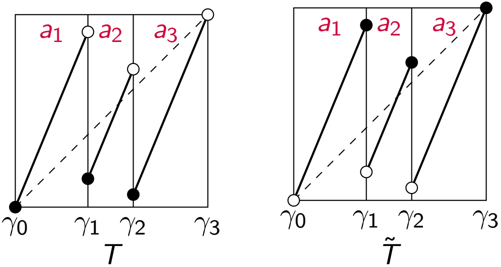
Kenyon and Vershik, 1998 Algebraic construction of a sofic subshift $V \subset \tilde{A}^{\mathbb{Z}}$ that gives an a.e. finite-to-one covering.

Schmidt, 2000 For every Pisot number β the set $\overline{\mathcal{S}}$, given by the map $x \mapsto \beta x \pmod 1$, provides an a.e. finite-to-one map.

Many others ...

An additional condition

Recall the transformation $\tilde{T}$:



For γ_i, let n_i be the minimal k such that $T^k\gamma_i = \tilde{T}^k\gamma_i$ with $n_i = \infty$ if this doesn't happen.

An additional condition

Suppose that $A = \{a_1, \ldots, a_m\}$. Define the set $\mathcal{V}$ by

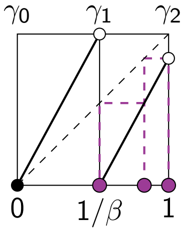
$$\mathcal{V} = \bigcup_{i=0}^{m} \{\gamma_i\} \cup \bigcup_{1 \le k < n_i, \gamma_i \in X, i \ne 0} \{T^k \gamma_i, \tilde{T}^k \gamma_i\}.$$

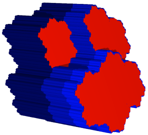
The extra assumtion we make is that the set $\mathcal{V}$ is finite. This happens in 2 cases.

- If the points γ_i have ultimately periodic orbits.
- If the orbits of the points γ_i come together after some steps.

An example: periodic endpoints

$$\mathcal{V} = \bigcup_{i=0}^{m+1} \{\gamma_i\} \cup \bigcup_{1 \le k < n_i, \gamma_i \in X, i \neq 0} \{T^k \gamma_i, \tilde{T}^k \gamma_i\}$$





- $\bigcup_{i=0}^{2} \{\gamma_i\} = \{0, \frac{1}{\beta}, 1\}$.
- $\{\gamma_i \in X \mid i \neq 0\} = \{1/\beta\}$.
- $T^k(\frac{1}{\beta}) = 0$ for all $k \ge 1$. $\tilde{T}(\frac{1}{\beta}) = 1$, $\tilde{T}^2(\frac{1}{\beta}) = \beta - 1$, $\tilde{T}^3(\frac{1}{\beta}) = \frac{1}{\beta}$. So, $n_1 = \infty$, but γ_1 is periodic for $\tilde{T}$.
- $\mathcal{V} = \{0, \frac{1}{\beta}, \beta - 1, 1\}$ is a finite set.
- The associated subshift is of finite type here, sofic in general.

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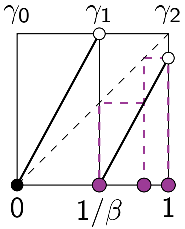

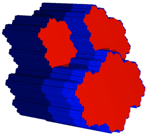

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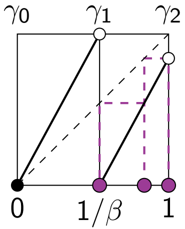

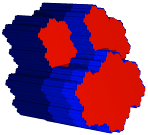

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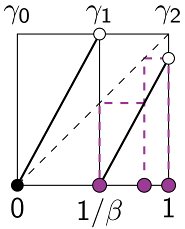

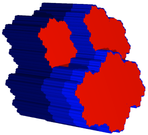

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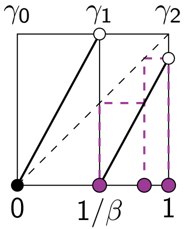

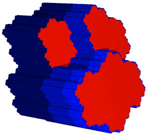

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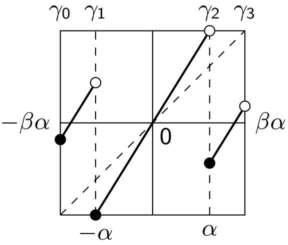

- $\bigcup_{i=0}^{3} \{\gamma_i\} = \{-\beta\alpha, -\alpha, \alpha, \beta\alpha\}$.
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- For all $\frac{1}{\beta^2} < \alpha < \frac{\beta}{\beta^2+1}$, $T^3\alpha = \tilde{T}^3\alpha$. So, $n_2 = 3$. By symmetry also $n_1 = 3$.
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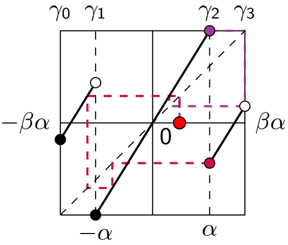

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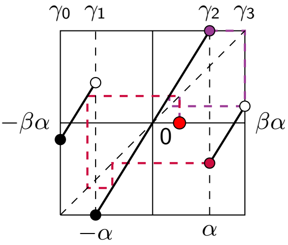

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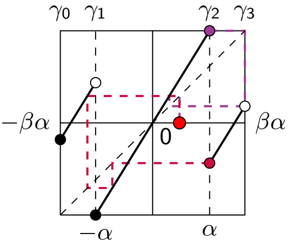

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A finite-to-one mapping

Theorem

If the set $\mathcal{V}$ is finite, then there is a constant $\kappa \geq 1$, such that the map $\psi : \overline{\mathcal{S}} \to \mathbb{T}^d$ is almost everywhere κ-to-one.

This includes cases in which $\mathcal{S}$ is not sofic.

If $\mathcal{V}$ is finite, then the density of the invariant measure $\mu = \lambda^d \circ \pi^{-1}$ of T is a sum of κ indicator functions.

Purely periodic expansions

Recall the definition of the map $\psi : \mathcal{S} \to \mathbb{R}^d$:

$$\psi(w \cdot u) = \sum_{n=1}^{\infty} \frac{u_n}{\beta^n} \mathbf{v}_1 - \sum_{j=2}^{d} \sum_{n=0}^{\infty} w_{-n} \beta_j^n \mathbf{v}_j .$$

For $x \in X$, we are interested in the set

$$\left\{ \sum_{j=2}^{d} \sum_{n=0}^{\infty} w_{-n} \beta_j^n \mathbf{v}_j \;\middle|\; \cdots w_{-1} w_0 \cdot b(x) \in \mathcal{S} \right\} \subset H.$$

Recall that H is the real contracting eigenspace for the matrix M.

Expansions and tiles

Recall that $\Gamma_j : \mathbb{Q}(\beta) \to \mathbb{Q}(\beta_j) : \beta \mapsto \beta_j$ and
$\Phi : \mathbb{Q}(\beta) \to H : x \mapsto \sum_{j=2}^{d} \Gamma_j(x)\mathbf{v}_j$.

Theorem

The origin $\mathbf{0} \in H$ belongs to a set

$$\Phi(x) + \left\{ \sum_{j=2}^{d} \sum_{n=0}^{\infty} w_{-n}\beta_j^n \mathbf{v}_j \;\middle|\; \cdots w_{-1} w_0 \cdot b(x) \in \mathcal{S} \right\}$$

for $x \in \mathbb{Z}[\beta] \cap X$ iff the expansion of x that is generated by T is purely periodic.

(For $x \mapsto \beta x \pmod 1$, Akiyama 1999 and Praggastis 1999)

One-to-one covering?

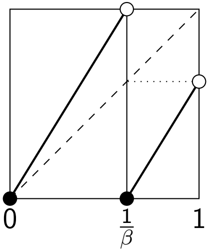
For certain specific cases it is known that the map $\psi : \overline{\mathcal{S}} \to \mathbb{T}^d$ is a.e. one-to-one for the map $x \mapsto \beta x \pmod 1$.

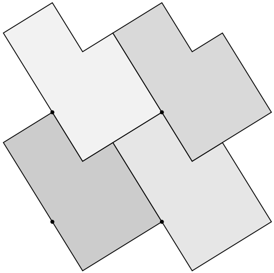
Pisot conjecture (Schmidt 2000, Akiyama 2002 and Sidorov 2003)

If β is a Pisot number and $Tx = \beta x \pmod 1$, then $\psi : \overline{\mathcal{S}} \to \mathbb{T}^d$ is almost everywhere one-to-one.

An example: the golden mean)

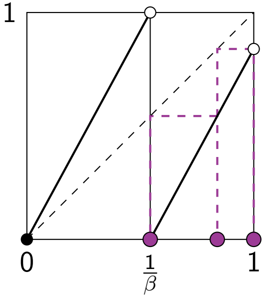
Let β be the golden mean and $Tx = \beta x \pmod 1$.

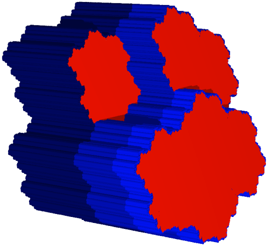




An example: the Rauzy tiling (Rauzy, 1982)

Let β be the tribonacci number and $Tx = \beta x \pmod 1$.





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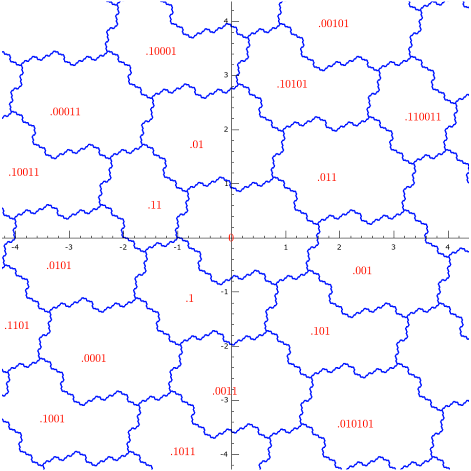
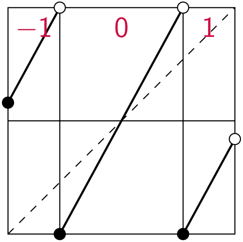
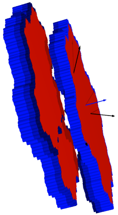


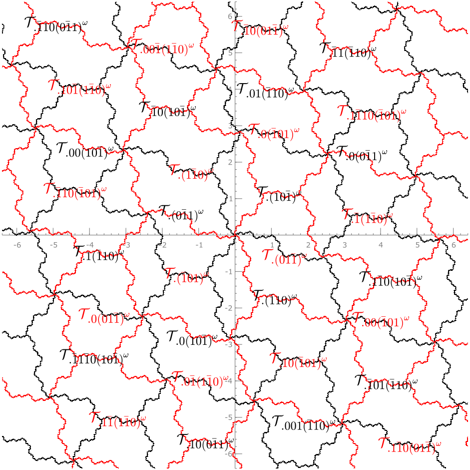
A two-to-one map: the tribonacci number

Let β be the tribonacci number. Take $A = \{-1, 0, 1\}$,
$X_{-1} = \left[-\frac{1}{2}, -\frac{1}{2\beta}\right)$, $X_0 = \left[-\frac{1}{2\beta}, \frac{1}{2\beta}\right)$ and $X_1 = \left[\frac{1}{2\beta}, \frac{1}{2}\right)$.





A double tiling: the tribonacci number



A double tiling: the tribonacci number

The map ψ is a.e. two-to-one if there is a ball in $\mathbb{R}^d$, such that for each $\mathbf{y}$ in this ball we have

$$\mathbf{y} = \mathbf{x} + \psi(w \cdot u) = \mathbf{x}' + \psi(w', u')$$

for two different copies of $\psi(\overline{\mathcal{S}})$.

We fixed specific $\mathbf{x}$, $\mathbf{x}'$, u and u' and transformed each w into a 'good' w'.

A double tiling: the tribonacci number

The transducer that transforms a sequence w into w':