Periodic Points and Entropy

Doug Lind

(*Joint work with Klaus Schmidt and Evgeny Verbitzkiy*)

Periodic Points: Definitions and results

- Let α be an action of $\mathbb{Z}^d$ by automorphisms of a compact abelian group X
- For every finite-index subgroup Γ of $\mathbb{Z}^d$ define $\mathrm{Fix}_\Gamma(\alpha)$ to be the subgroup of points in X fixed by every element of Γ
- Let $\langle\Gamma\rangle$ be the norm of the smallest nonzero element of Γ
- $\mathrm{Fix}_\Gamma^0(\alpha)$ is the connected component of the identity in $\mathrm{Fix}_\Gamma(\alpha)$
- Count the number of connected components $P_\Gamma(\alpha)$ of $\mathrm{Fix}_\Gamma(\alpha)$ by $|\mathrm{Fix}_\Gamma(\alpha)/\mathrm{Fix}_\Gamma^0(\alpha)|$
- Define $p^-(\alpha) = \liminf_{\langle\Gamma\rangle\to\infty} \frac{1}{|\mathbb{Z}^d/\Gamma|} \log P_\Gamma(\alpha)$ and $p^+(\alpha) = \limsup_{\langle\Gamma\rangle\to\infty} \frac{1}{|\mathbb{Z}^d/\Gamma|} \log P_\Gamma(\alpha)$
- Let $B(\varepsilon)$ be the ball of radius ε in X and μ be Haar measure on X
- Define the entropy $h(\alpha) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log \mu\left(\bigcap_{i=1}^{n} \alpha^{-i}(B(\varepsilon))\right)$
- Assume that $h(\alpha) < \infty$, that the dual group of X is finitely generated under the automorphism dual to α
- Obviously $p^-(\alpha) \le p^+(\alpha) \le h(\alpha)$
- Fact: Under our assumptions, $p^+(\alpha) = h(\alpha)$ [L-Schmidt, 1996]
- For toral automorphisms the equality of $p^-(\alpha)$ and $p^+(\alpha)$ is equivalent to a deep theorem of Gelfond
- We can use homoclinic points to provide an "easy" proof of a slightly weaker version of Gelfond's result

Who can possibly understand a slide like this?

\begin{curmudgeon}

Beamer is destroying Math Talks!!


Beamer
Math Talks

\end{curmudgeon}

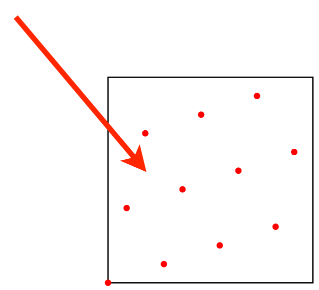
Classic Example

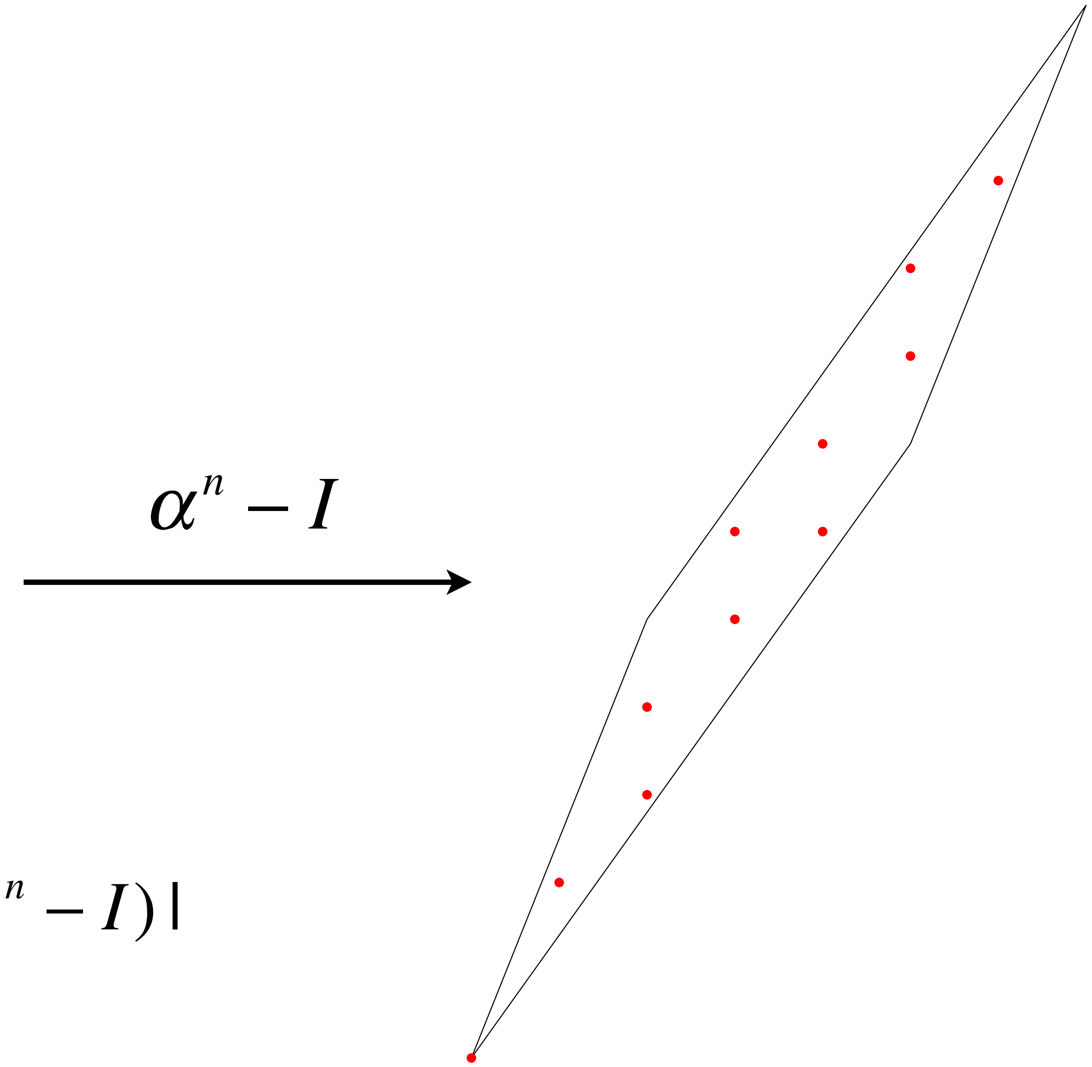
$$\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{on} \quad X = \mathbb{T}^2, \quad \mathbb{T} = \mathbb{R}/\mathbb{Z}$$

$$\begin{aligned} P_{n\mathbb{Z}}(\alpha) &= \left\{ t \in \mathbb{T}^2 : \alpha^k(t) = t \text{ for all } k \in n\mathbb{Z} \right\} \\ &= \left\{ t \in \mathbb{T}^2 : \alpha^n(t) = t \right\} \\ &= \ker(\alpha^n - I) \end{aligned}$$

$$p(\alpha) = \lim_{n \to \infty} \frac{1}{|\mathbb{Z}/n\mathbb{Z}|} \log |P_{n\mathbb{Z}}(\alpha)|$$

$$\ker(\alpha^n - I) = P_{n\mathbb{Z}}(\alpha)$$



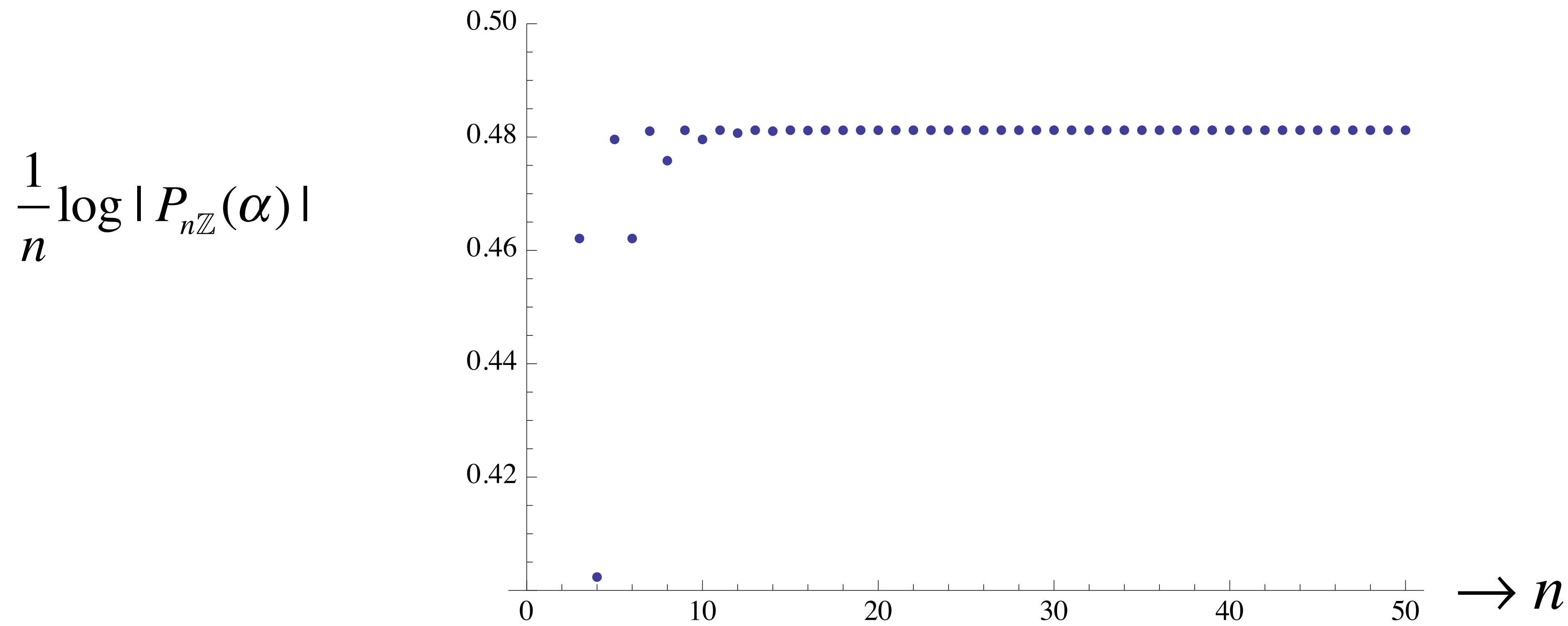


$$|P_{n\mathbb{Z}}(\alpha)| = \text{ area } = |\det(\alpha^n - I)|$$

α has eigenvalues $\lambda = \dfrac{1+\sqrt{5}}{2}$ and $\mu = \dfrac{1-\sqrt{5}}{2}$

$$\det(\alpha^n - I) = (\lambda^n - 1)(\mu^n - 1)$$

$$\frac{1}{n}\log|P_{n\mathbb{Z}}(\alpha)| = \frac{1}{n}\log|\lambda^n - 1| + \frac{1}{n}\log|\mu^n - 1| \to \log\lambda$$



$$\text{Entropy} = h(\alpha) = \lim_{n\to\infty} -\frac{1}{n}\log\ \text{meas}\left(\bigcap_{j=0}^{n-1}\alpha^{-j}\left(B(\varepsilon)\right)\right) = \log\lambda$$

$$p(\alpha) = h(\alpha)$$

Growth rate of periodic points equals entropy

Caution:

This doesn't work smoothly

Theorem (Kaloshin, Ph.D. 2001) : For any $2 \leq r < \infty$ there is an open set $U \subset \mathrm{Diff}^r(M)$ such that for "generic" $f \in U$ the periodic point growth for f is superexponential. Here "generic" means residual.

What could possibly go wrong?

Too many periodic points

$$\alpha = I$$

Solution: Count connected components

$$| P_{n\mathbb{Z}}(\alpha) / P^0_{n\mathbb{Z}}(\alpha) |$$

$\dim P^0_{n\mathbb{Z}}(\alpha) = \#$ of nth roots of unity that are eigenvalues of α

Not enough periodic points

$$X = \hat{\mathbb{Q}} \qquad \hat{X} = \mathbb{Q}$$

$$\alpha = \left(\times \frac{3}{2} \right)^{\wedge} \qquad \hat{\alpha} = \times \frac{3}{2}$$

$$P_{n\mathbb{Z}}(\alpha) \subset X \qquad \mathbb{Q} / \left(\left(\frac{3}{2} \right)^{n} - 1 \right) \mathbb{Q} = \{0\}$$

No nonzero periodic points!

$$p(\alpha) = 0 \qquad h(\alpha) = \log 3$$

Solution: require dual group to be finitely generated under the dual automorphism

Not quite enough periodic points

$$X = \widehat{\mathbb{Z}[1/3]} \qquad \hat{X} = \mathbb{Z}[1/3]$$

$$\alpha = (\times 2)\hat{} \qquad \hat{\alpha} = \times 2$$

$$P_{n\mathbb{Z}}(\alpha) \subset X \qquad \mathbb{Z}[1/3] / \left(2^n - 1\right)\mathbb{Z}[1/3]$$

Need to know the 3-divisibility of $2^n - 1$

To compute this we invoke the following powerful theorem from number theory:

$$2 = 3 - 1$$

Then $|2^n - 1|_3 = 1$ if n is odd, and $\frac{1}{3}|n|_3$ if n is even

This is small compared with 2^n and so $p(\alpha) = h(\alpha) = \log 2$

Infinite entropy

$$\alpha = \text{ shift on } \mathbb{T}^{\mathbb{Z}}$$

$$p(\alpha) = 0 \qquad h(\alpha) = \infty$$

Solution: require finite entropy

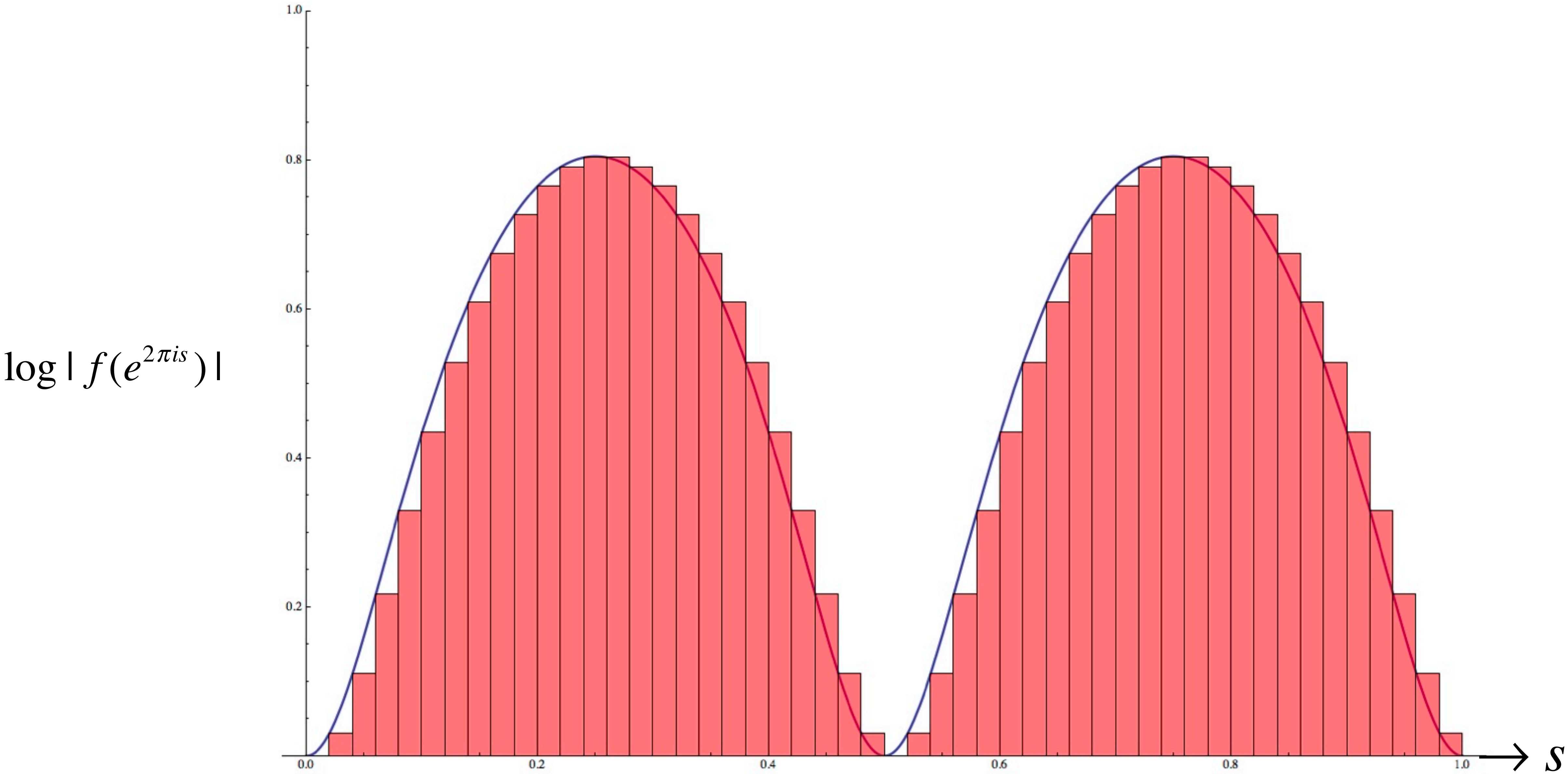
Diophantine Problems

$$f(x) = x^2 - x - 1 = (x-\lambda)(x-\mu) = \text{ char poly of } \alpha$$

$$(\lambda^n - 1)(\mu^n - 1) = \prod_{\omega^n=1} (\lambda - \omega)(\mu - \omega) = \prod_{\omega^n=1} (\omega - \lambda)(\omega - \mu) = \prod_{\omega^n=1} f(\omega)$$

$$\frac{1}{n}\log|(\lambda^n - 1)(\mu^n - 1)| = \frac{1}{n}\sum_{\omega^n=1} \log|f(\omega)| \overset{\text{R.S.}}{\approx} \int_{\mathbb{S}} \log|f| = \text{ Mahler measure of } f$$

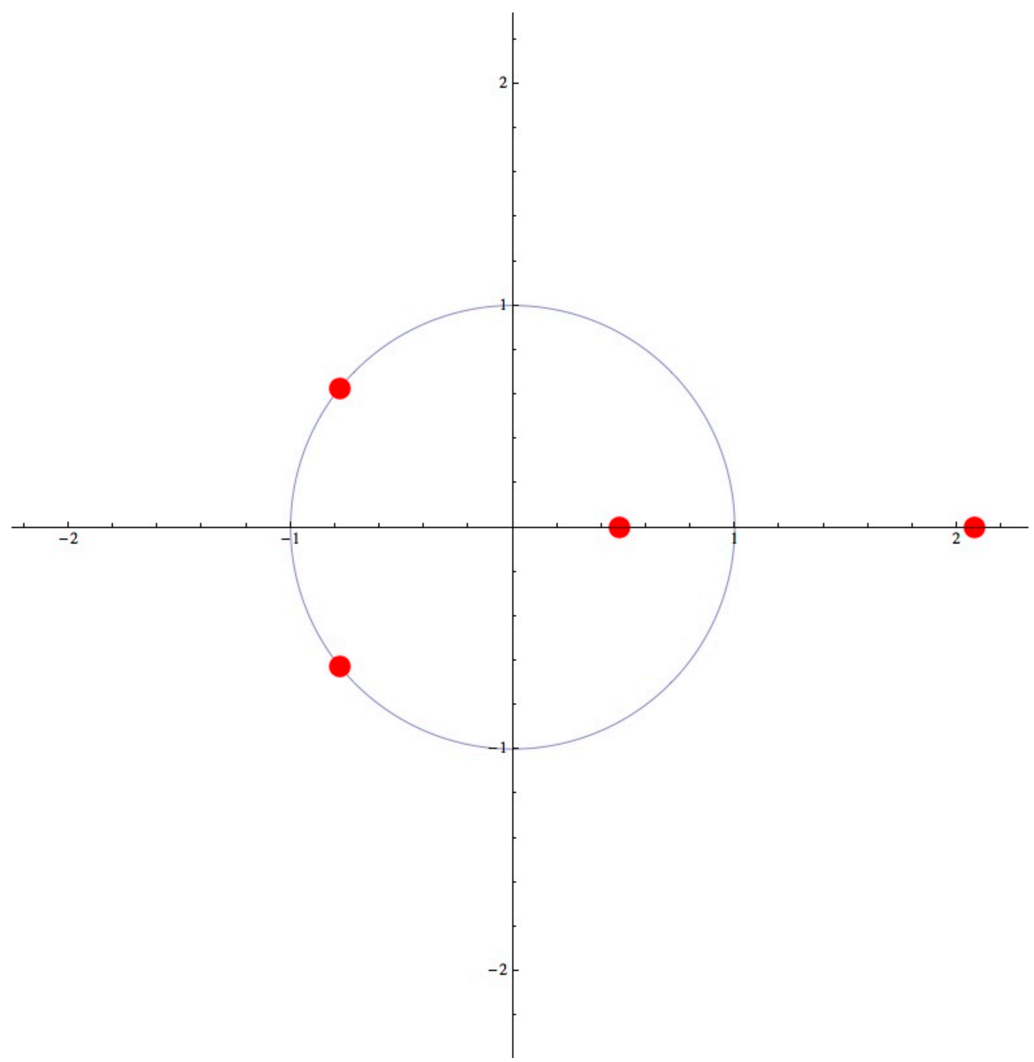
$$\text{where } \mathbb{S} = \text{ unit circle in } \mathbb{C} = e^{2\pi i \mathbb{T}}$$

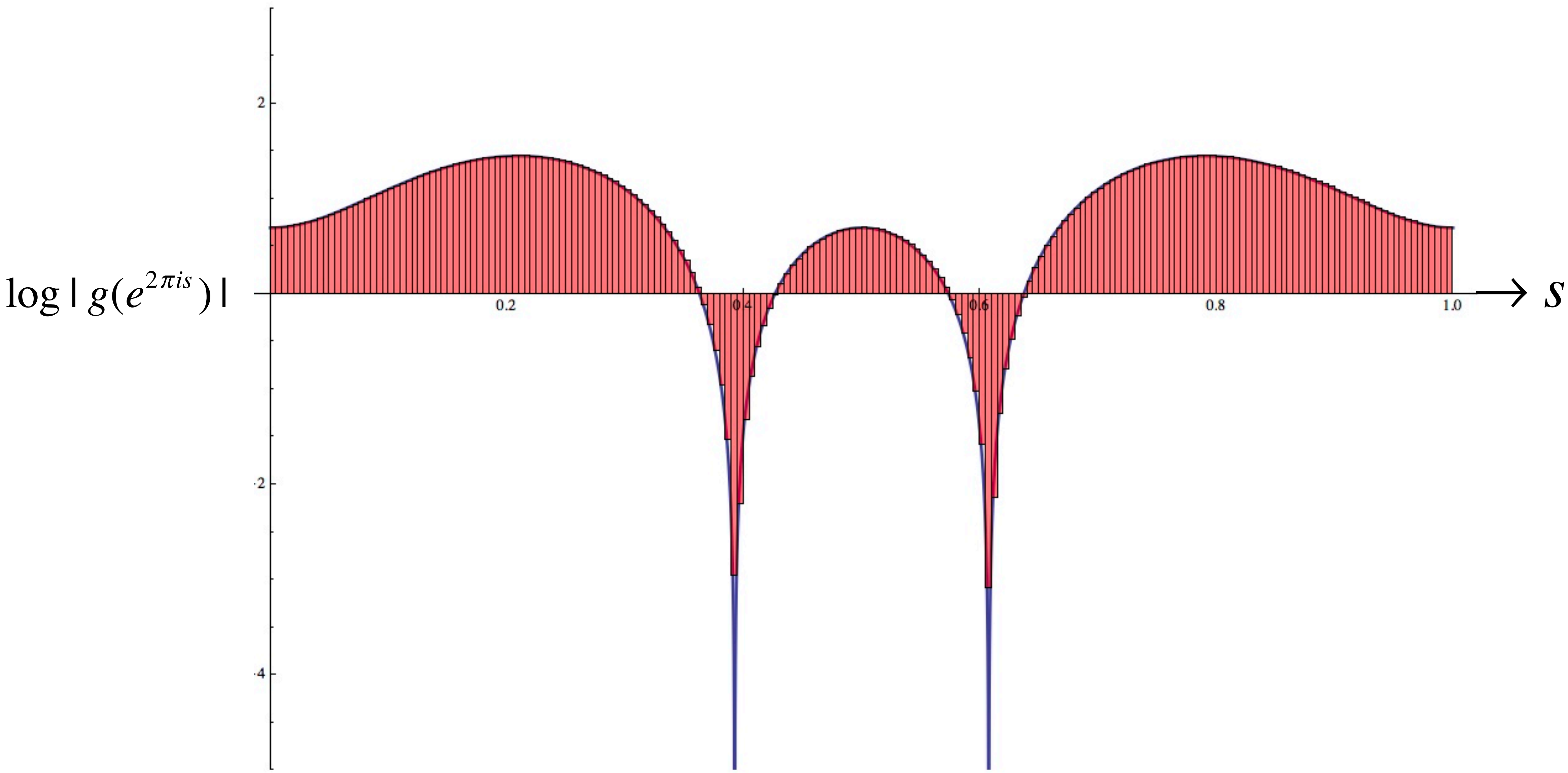


This works great because $f(e^{2\pi is})$ never vanishes

But what if it does?

$$g(x) = x^4 - x^3 - 2x^2 - x + 1$$





Do the Riemann sums for $\log|g|$ converge to $\int_{\mathbb{S}} \log|g|$?

Let $\xi \in \mathbb{S}$ be a root of g.

If ω is an nth root of unity, can $|\xi - \omega|$ be incredibly small?

Quantitatively, convergence of the Riemann sums is exactly equivalent to:

For every $\varepsilon > 0$ the inequality

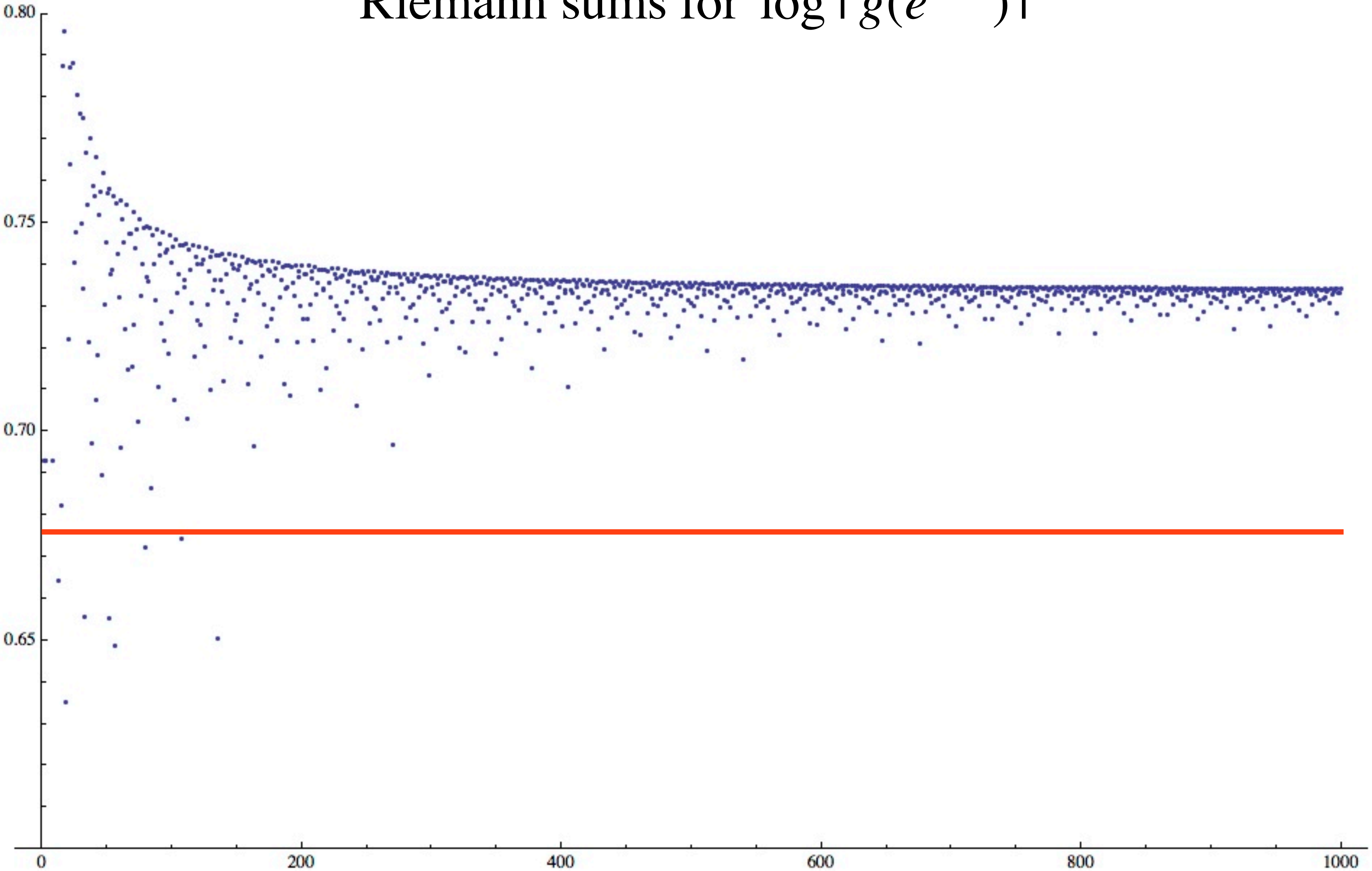
$$|\xi^n - 1| < e^{-\varepsilon n}$$

has only finitely many solutions

Simple to prove: $|\xi^n - 1| \geq e^{-(h/2)n}$

Use: $\prod_{g(\lambda)=0} (\lambda^n - 1) \in \mathbb{Z} \setminus \{0\}$

Riemann sums for $\log|g(e^{2\pi is})|$



Gelfond (1932): If $\xi \in \mathbb{S}$ is an algebraic number and $\varepsilon > 0$, then

$$|\xi^n - 1| < e^{-\varepsilon n}$$

has only finitely many solutions in n.

This is deep, one part of a much larger set of results that proves, for example, that $2^{\sqrt{2}}$ is transcendental

Theorem (L-Schmidt): Let α be an automorphism of a compact abelian group X, and make the necessary assumptions we discussed (finite entropy, finite generation). Then the limit growth rate of the periodic components exists and equals entropy.

The automorphism machine

$f(x) \in \mathbb{Z}[x^{\pm 1}] \longrightarrow \alpha_f$ an automorphism of a compact abelian group X_f

$f(x) = x^2 - x - 1 \longrightarrow X_f = \{t \in \mathbb{T}^{\mathbb{Z}} : t_{n+2} - t_{n+1} - t_n = 0 \text{ for all } n\}$

$\alpha_f =$ left shift

$$X_f \ni \cdots t_{-2} \quad t_{-1} \quad \boxed{t_0 \quad t_1} \quad t_2 \cdots \xrightarrow{\alpha_f} \cdots t_{-1} \quad t_0 \quad \boxed{t_1 \quad t_2} \quad t_3 \cdots$$

$$\cong \updownarrow$$

$$\mathbb{T}^2 \ni \begin{bmatrix} t_0 \\ t_1 \end{bmatrix} \xrightarrow{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} t_0 \\ t_1 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

$f^*(x) = f(x^{-1}) = x^{-2} - x^{-1} - 1 \qquad t = (t_n) \in \mathbb{T}^{\mathbb{Z}} \leftrightarrow \sum_{-\infty}^{\infty} t_n x^n$

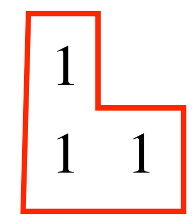
$t \in X_f \quad \text{iff} \quad t * f^*(x) = 0$

The automorphism machine (two-variable version)

$f(x,y) \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}] \longrightarrow$ $\mathbb{Z}^2$-action α_f on a compact abelian group X_f

$$\mathbb{T}^{\mathbb{Z}^2} \supset X_f = \left\{ t = \sum t_{m,n} x^m y^n : t * f^*(x,y) = 0 \right\} \qquad \alpha_f = \langle \text{left shift, down shift} \rangle$$

$$f(x,y) = 1 + x + y$$



Periodic points

Γ : finite-index subgroup of $\mathbb{Z}^2$ $\qquad \langle \Gamma \rangle = \min\{ \| \mathbf{n} \| : \mathbf{n} \in \Gamma \setminus \{\mathbf{0}\} \}$

$$P_\Gamma(\alpha_f) := \{ t \in X_f : \alpha_f^{\mathbf{n}}(t) = t \text{ for all } \mathbf{n} \in \Gamma \}$$

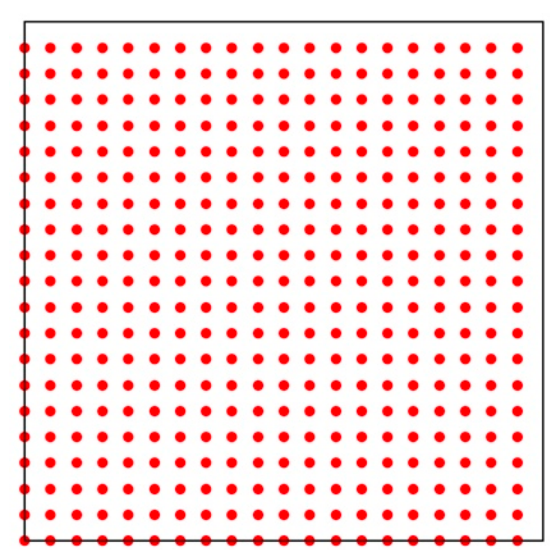
Main Goal

$$\lim_{\langle\Gamma\rangle\to\infty}\frac{1}{|\mathbb{Z}^2/\Gamma|}\log|P_\Gamma(\alpha_f)/P_\Gamma^0(\alpha_f)| = h(\alpha_f)$$

Connection to Riemann sums

Γ ⟷ $\Omega_\Gamma = (\mathbb{Z}^2/\Gamma)\hat{} \subset \mathbb{S}^2$, the "$\Gamma^{\text{th}}$ roots of unity"

$N\mathbb{Z}\oplus N\mathbb{Z}$ ⟷



$\Omega_{N\mathbb{Z}\oplus N\mathbb{Z}}$

$$\frac{1}{|\mathbb{Z}^2/\Gamma|}\log|P_\Gamma(\alpha_f)/P^0_\Gamma(\alpha_f)| = \frac{1}{|\Omega_\Gamma|}\sum_{\omega\in\Omega_\Gamma}\log_0|f(\boldsymbol{\omega})|$$

$$\overset{\text{R.S.}}{\approx} \int_{\mathbb{S}^2}\log|f| = h(\alpha_f)$$

$$\log_0 t = \begin{cases}\log t & \text{if } t>0\\ 0 & \text{if } t=0\end{cases}$$

$U(f) =$ unitary variety of f $= \{(\xi,\eta)\in\mathbb{S}^2 : f(\xi,\eta)=0\}$

If $U(f)=\varnothing$, then $\log|f|$ is continuous on $\mathbb{S}^2$, and everything is hunky-dorey

$U(f)=\varnothing \iff \alpha_f$ is expansive

$$U(f) \neq \varnothing \text{ ???}$$

$$f(x,y) = 2 - x - y \qquad U(f) = \{(1,1)\}$$

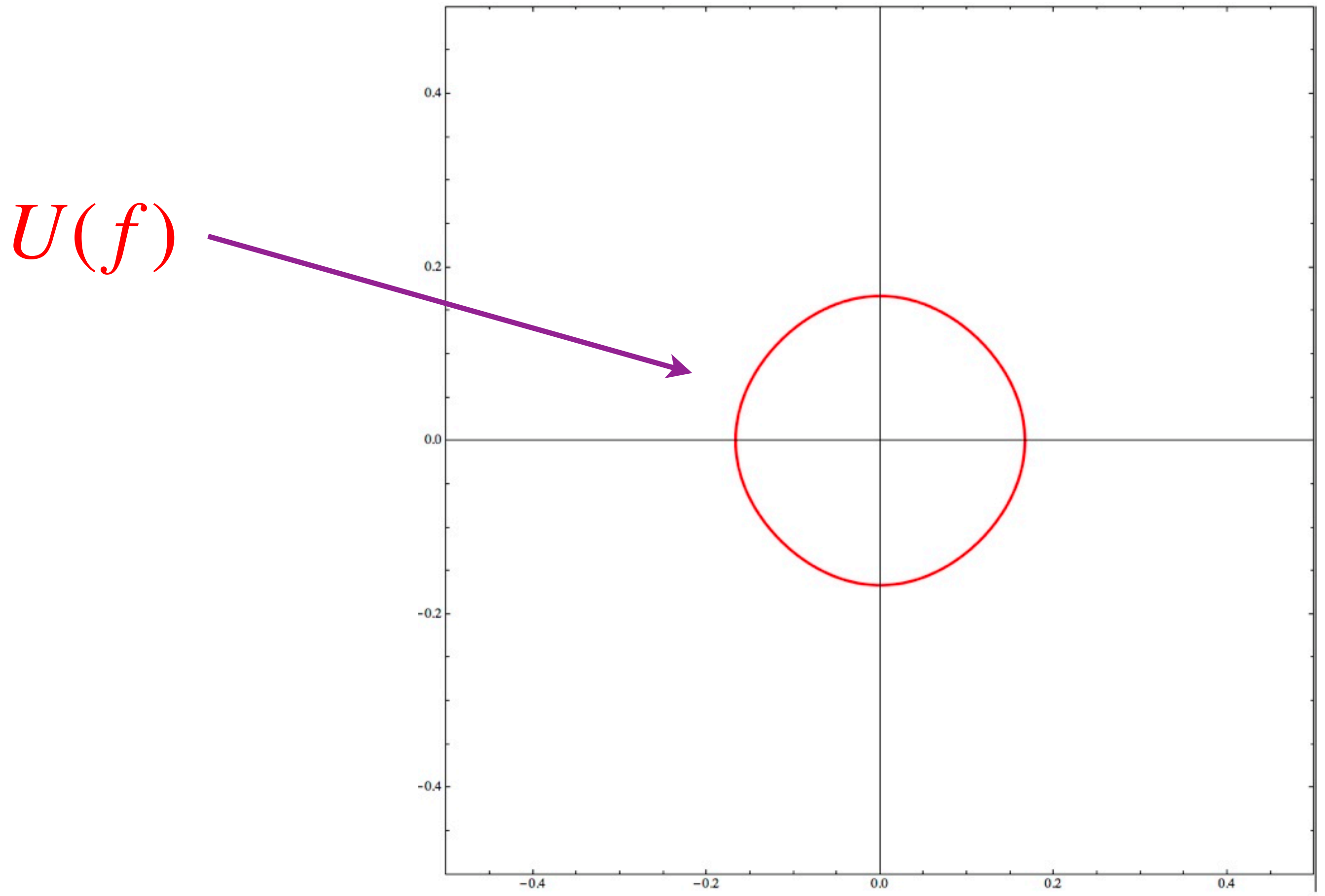
$$f(x,y) = 1 + x + y \qquad U(f) = \{(\omega,\omega^2),(\omega^2,\omega)\} \qquad \omega = e^{2\pi i/3}$$

$$f(x,y) = 2 - x^2 + y - xy \qquad U(f) = \{(\xi,\eta),(\overline{\xi},\overline{\eta})\}$$

$$\xi = \frac{1-\sqrt{57}}{8} + i\left(\frac{3+\sqrt{57}}{32}\right)^{1/2}$$

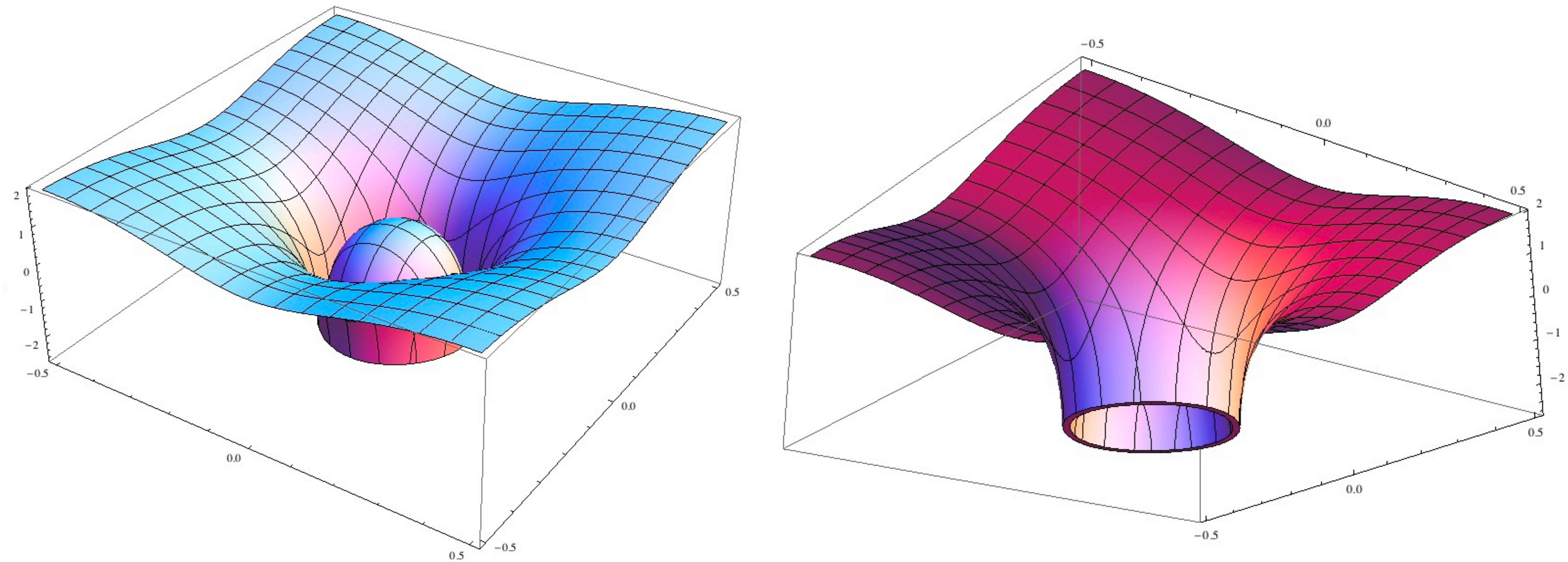
$$\eta = \frac{-1}{56+8\sqrt{57}}\left[34+6\sqrt{57}+i\left(11\sqrt{6+2\sqrt{57}}+\sqrt{342+114\sqrt{57}}\right)\right]$$

$$f(x,y) = 3 - x - x^{-1} - y - y^{-1}$$



$$t = \pm \frac{1}{2\pi} \cos^{-1}\left(\frac{3}{2} - \cos 2\pi s \right)$$

Two views of $\log | f |$

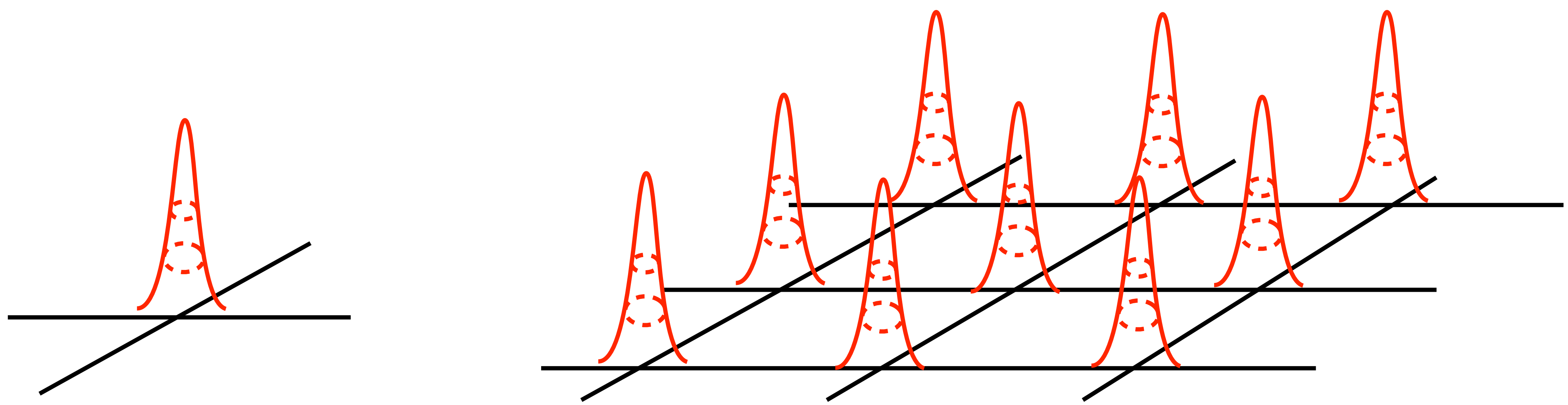


Do the Riemann sums over finite subgroups converge to the integral?

Homoclinic points

$t = (t_{\mathbf{n}}) \in X_f$ is homoclinic for α_f if $t_{\mathbf{n}} \to 0$ as $\|\mathbf{n}\| \to \infty$

$t = (t_{\mathbf{n}}) \in X_f$ is a summable homoclinic point if $\sum_{\mathbf{n}} |t_{\mathbf{n}}| < \infty$



If $(z_{\mathbf{n}})$ is any bounded Γ-periodic array of integers

then $\sum_{\mathbf{n}} z_{\mathbf{n}} \alpha_f^{\mathbf{n}}(t)$ is a well-defined Γ-periodic point in X_f

Where do homoclinic points come from?

$$\begin{array}{ccc} w \in \ell^{\infty}(\mathbb{Z}^2, \mathbb{R}) & \longrightarrow & w * f^* \in \ell^{\infty}(\mathbb{Z}^2, \mathbb{Z}) \\ \uparrow & & \downarrow \\ t \in X_f \subset \mathbb{T}^{\mathbb{Z}^2} & \longrightarrow & t * f^* = 0 \end{array}$$

$$w * f^* = \delta_{\mathbf{0}}$$

$$\hat{w} \cdot \widehat{f^*} = 1$$

$$\hat{w} = \frac{1}{\widehat{f^*}}$$

$$w = \left(\frac{1}{\widehat{f^*}} \right)^{\smile}$$

So the coordinates of w are just the Fourier coefficients of $1 / \widehat{f^*}$

$$f(x,y) = 2 - x - y \qquad\qquad f^*(x,y) = 2 - x^{-1} - y^{-1}$$

$$\frac{1}{\widehat{f^*}} = \frac{1}{2 - e^{-2\pi i u} - e^{-2\pi i v}} = \frac{1}{2}\left(\frac{1}{1 - \frac{1}{2}\left(e^{-2\pi i u} + e^{-2\pi i v}\right)}\right) = \sum_{n=0}^{\infty} 2^{-n-1}(e^{-2\pi i u} + e^{-2\pi i v})^n$$

$$\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{32} & \frac{1}{16} & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 \\
 & \frac{4}{32} & \frac{3}{16} & \frac{2}{8} & \frac{1}{4} & 0 & 0 & 0 \\
 & & \frac{6}{32} & \frac{3}{16} & \frac{1}{8} & 0 & 0 & 0 \\
 & & & \frac{4}{32} & \frac{1}{16} & 0 & 0 & 0 \\
 & & & & \frac{1}{32} & 0 & 0 & 0
\end{array}$$

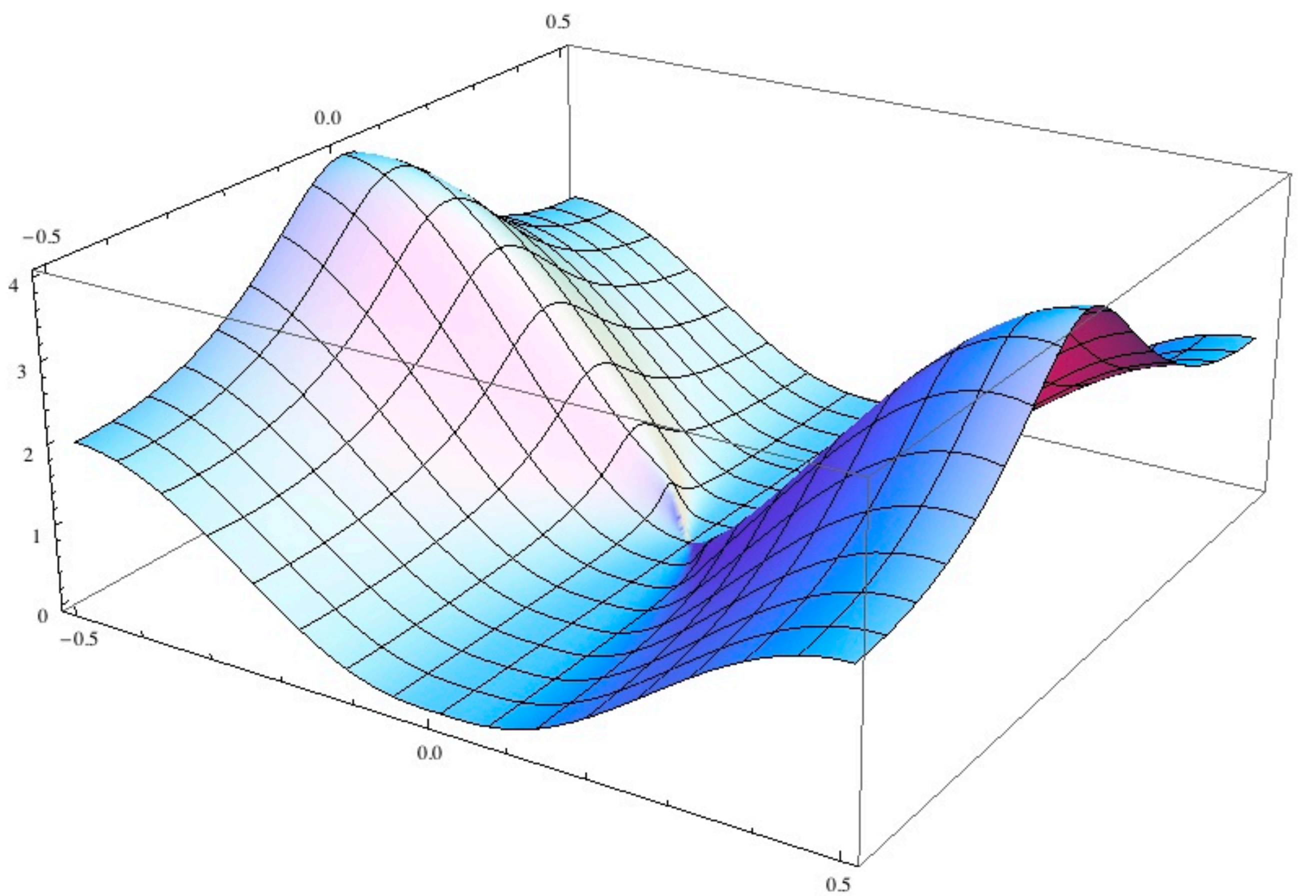
$$\begin{array}{|cc}
-1 & \\
2 & -1
\end{array}$$

$$\frac{1}{2^{2n+1}}\binom{2n}{n} \approx \frac{c}{\sqrt{n}}$$

Create a summable homoclinic point by killing off the singularity of $1/f(x,y)$

$$\frac{(x-1)^3}{2-x-y}$$

has absolutely convergent Fourier series



This idea handles the case $U(f)=\{(\xi_j,\eta_j):1\le j\le r\}$ is finite:

For each ξ_j find $g_j(x)\in\mathbb{Z}[x]$ with $g_j(\xi_j)=0$, and then $\dfrac{g_1(x)^{N_1}\cdots g_r(x)^{N_r}}{f(x,y)}$

will be smooth enough to have summable Fourier coefficients if $N_1,\ldots,N_r$ are big enough

However, this requires that each ξ_j is an algebraic number. Is it??

Logic to the rescue!

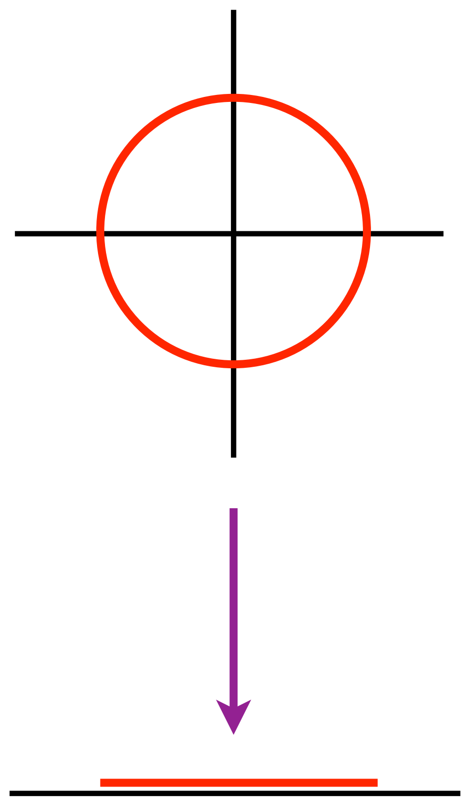
Algebraic set in $\mathbb{R}^n$: $\begin{cases} f_1(x_1,\ldots,x_n) = 0 \\ \vdots \\ f_r(x_1,\ldots,x_n) = 0 \end{cases}$

Semialgebraic set in $\mathbb{R}^n$: $\begin{cases} f_1(x_1,\ldots,x_n) \rhd_1 0 \\ \vdots \\ f_r(x_1,\ldots,x_n) \rhd_r 0 \end{cases}$

where each $\rhd_j$ is either $=, <, >, \leq,$ or $\geq$

What happens to such sets under projections to $\mathbb{R}^k$?

projection(algebraic) $\neq$ algebraic:



$$\{(x,y) : x^2 + y^2 - 1 = 0\}$$

$$\{x : \exists y, x^2 + y^2 - 1 = 0\}$$
$$= \{x : x - 1 \leq 0 \text{ and } x + 1 \geq 0\}$$

Tarski-Seidenberg: Projection(semialgebraic) = semialgebraic

Also, if A is semialgbraic using polynomials with rational coefficients (or A is definable over $\mathbb{Q}$), then so is its projection.

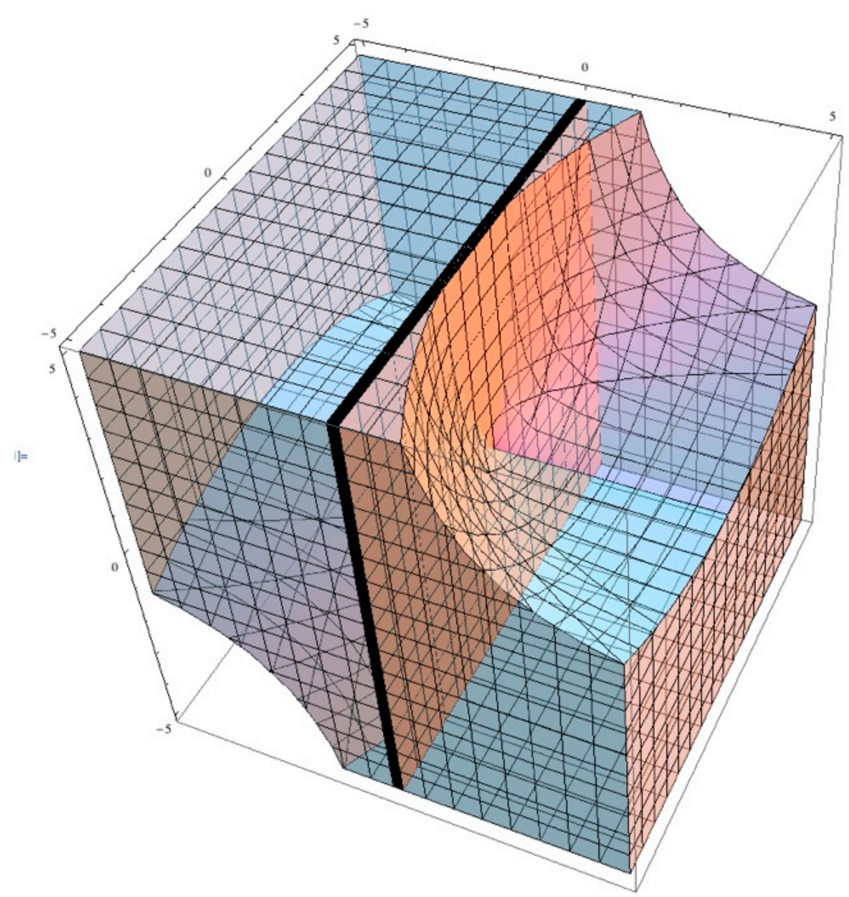
Quadratic formula

$$ax^2 + bx + c \in \mathbb{R}[a, b, c, x]$$

$$V = \{(a, b, c, x) : ax^2 + bx + c = 0\} \subset \mathbb{R}^4$$

$$\mathrm{proj}_{a,b,c}(V) = \{(a, b, c) : \exists x \in \mathbb{R}, ax^2 + bx + c = 0\}$$

$$(a \neq 0 \text{ and } b^2 - 4ac \geq 0) \text{ or } (a = 0 \text{ and } b \neq 0) \text{ or } (a = 0 \text{ and } b = 0 \text{ and } c = 0)$$



Mathematica does quantifier elimination with Reduce

In[14]:=
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Reduce[a x^2 + b x + c == 0, x, Reals] // TraditionalForm
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Out[14]//TraditionalForm=

$$(c<0 \bigwedge ((b<0 \bigwedge ((a=\frac{b^2}{4c} \bigwedge x=-\frac{1}{2}\sqrt{\frac{b^2-4ac}{a^2}}-\frac{b}{2a}) \bigvee (\frac{b^2}{4c}<a<0 \bigwedge (x=-\frac{1}{2}\sqrt{\frac{b^2-4ac}{a^2}}-\frac{b}{2a} \bigvee x=\frac{1}{2}\sqrt{\frac{b^2-4ac}{a^2}}-\frac{b}{2a})) \bigvee$$

$$(a=0 \bigwedge x=-\frac{c}{b}) \bigvee (a>0 \bigwedge (x=-\frac{1}{2}\sqrt{\frac{b^2-4ac}{a^2}}-\frac{b}{2a} \bigvee x=\frac{1}{2}\sqrt{\frac{b^2-4ac}{a^2}}-\frac{b}{2a})))) \bigvee (b=0 \bigwedge a>0 \bigwedge (x=-\sqrt{-\frac{c}{a}} \bigvee x=\sqrt{-\frac{c}{a}})) \bigvee$$

$$(b>0 \bigwedge ((a=\frac{b^2}{4c} \bigwedge x=-\frac{1}{2}\sqrt{\frac{b^2-4ac}{a^2}}-\frac{b}{2a}) \bigvee (\frac{b^2}{4c}<a<0 \bigwedge (x=-\frac{1}{2}\sqrt{\frac{b^2-4ac}{a^2}}-\frac{b}{2a} \bigvee x=\frac{1}{2}\sqrt{\frac{b^2-4ac}{a^2}}-\frac{b}{2a})) \bigvee$$

$$(a=0 \bigwedge x=-\frac{c}{b}) \bigvee (a>0 \bigwedge (x=-\frac{1}{2}\sqrt{\frac{b^2-4ac}{a^2}}-\frac{b}{2a} \bigvee x=\frac{1}{2}\sqrt{\frac{b^2-4ac}{a^2}}-\frac{b}{2a}))))) \bigvee$$

$$(c=0 \bigwedge ((b<0 \bigwedge ((a<0 \bigwedge (x=-\frac{1}{2}\sqrt{\frac{b^2}{a^2}}-\frac{b}{2a} \bigvee x=\frac{1}{2}\sqrt{\frac{b^2}{a^2}}-\frac{b}{2a})) \bigvee (a=0 \wedge x=0) \bigvee (a>0 \bigwedge (x=-\frac{1}{2}\sqrt{\frac{b^2}{a^2}}-\frac{b}{2a} \bigvee x=\frac{1}{2}\sqrt{\frac{b^2}{a^2}}-\frac{b}{2a})))) \bigvee$$

$$(b=0 \wedge ((a<0 \wedge x=0) \vee a=0 \vee (a>0 \wedge x=0))) \bigvee$$

$$(b>0 \bigwedge ((a<0 \bigwedge (x=-\frac{1}{2}\sqrt{\frac{b^2}{a^2}}-\frac{b}{2a} \bigvee x=\frac{1}{2}\sqrt{\frac{b^2}{a^2}}-\frac{b}{2a})) \bigvee (a=0 \wedge x=0) \bigvee (a>0 \bigwedge (x=-\frac{1}{2}\sqrt{\frac{b^2}{a^2}}-\frac{b}{2a} \bigvee x=\frac{1}{2}\sqrt{\frac{b^2}{a^2}}-\frac{b}{2a}))))) \bigvee$$

$$(c>0 \bigwedge ((b<0 \bigwedge ((a<0 \bigwedge (x=-\frac{1}{2}\sqrt{\frac{b^2-4ac}{a^2}}-\frac{b}{2a} \bigvee x=\frac{1}{2}\sqrt{\frac{b^2-4ac}{a^2}}-\frac{b}{2a})) \bigvee (a=0 \bigwedge x=-\frac{c}{b}) \bigvee$$

$$(0<a<\frac{b^2}{4c} \bigwedge (x=-\frac{1}{2}\sqrt{\frac{b^2-4ac}{a^2}}-\frac{b}{2a} \bigvee x=\frac{1}{2}\sqrt{\frac{b^2-4ac}{a^2}}-\frac{b}{2a})) \bigvee (a=\frac{b^2}{4c} \bigwedge x=-\frac{1}{2}\sqrt{\frac{b^2-4ac}{a^2}}-\frac{b}{2a}))) \bigvee$$

$$(b=0 \bigwedge a<0 \bigwedge (x=-\sqrt{-\frac{c}{a}} \bigvee x=\sqrt{-\frac{c}{a}})) \bigvee (b>0 \bigwedge ((a<0 \bigwedge (x=-\frac{1}{2}\sqrt{\frac{b^2-4ac}{a^2}}-\frac{b}{2a} \bigvee x=\frac{1}{2}\sqrt{\frac{b^2-4ac}{a^2}}-\frac{b}{2a})) \bigvee (a=0 \bigwedge x=-\frac{c}{b}) \bigvee$$

$$(0<a<\frac{b^2}{4c} \bigwedge (x=-\frac{1}{2}\sqrt{\frac{b^2-4ac}{a^2}}-\frac{b}{2a} \bigvee x=\frac{1}{2}\sqrt{\frac{b^2-4ac}{a^2}}-\frac{b}{2a})) \bigvee (a=\frac{b^2}{4c} \bigwedge x=-\frac{1}{2}\sqrt{\frac{b^2-4ac}{a^2}}-\frac{b}{2a})))))$$

How does this help us?

$$\begin{cases} x_1^2 + y_1^2 - 1 = 0 \\ x_2^2 + y_2^2 - 1 = 0 \\ \operatorname{Re} f(x_1 + iy_1, x_2 + iy_2) = 0 \\ \operatorname{Im} f(x_1 + iy_1, x_2 + iy_2) = 0 \end{cases} \quad \text{over } \mathbb{Q}[x_1, y_1, x_2, y_2]$$

Tarski-Seidenberg $\Rightarrow$ x_j, y_j are definable over $\mathbb{Q}$

$\Rightarrow$ x_j, y_j are algebraic numbers

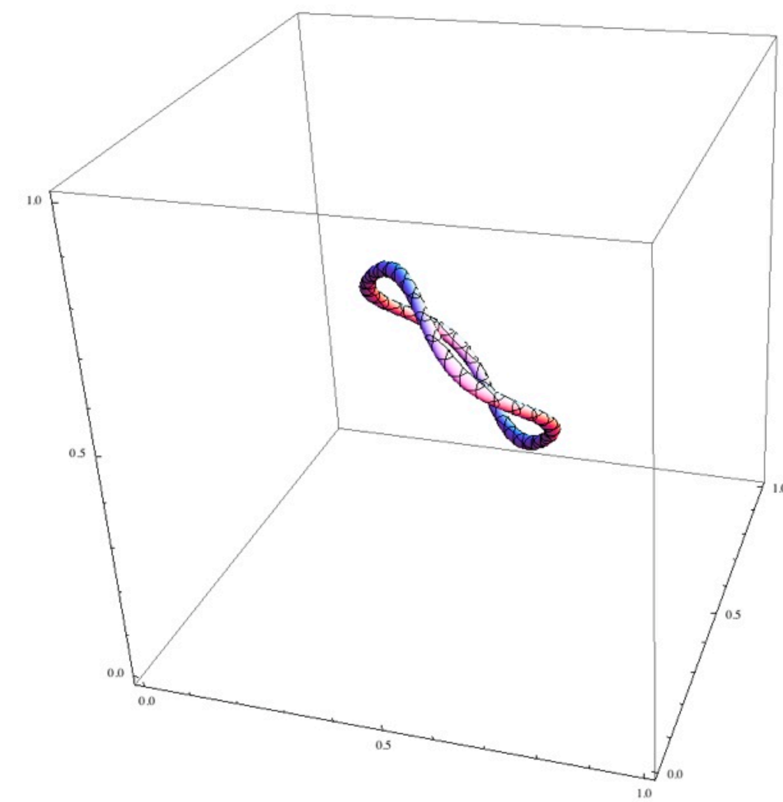
$\Rightarrow$ $x_j + iy_j \in \mathbb{S}$ is algebraic

and we're back in business

What if $U(f)$ is infinite? Can we find a g, not a multiple of f, with $U(g) = U(f)$?

If so, then $\dfrac{g^N}{f}$ will be smooth enough for big N to give summable homoclinic points,

and we would be back in business.



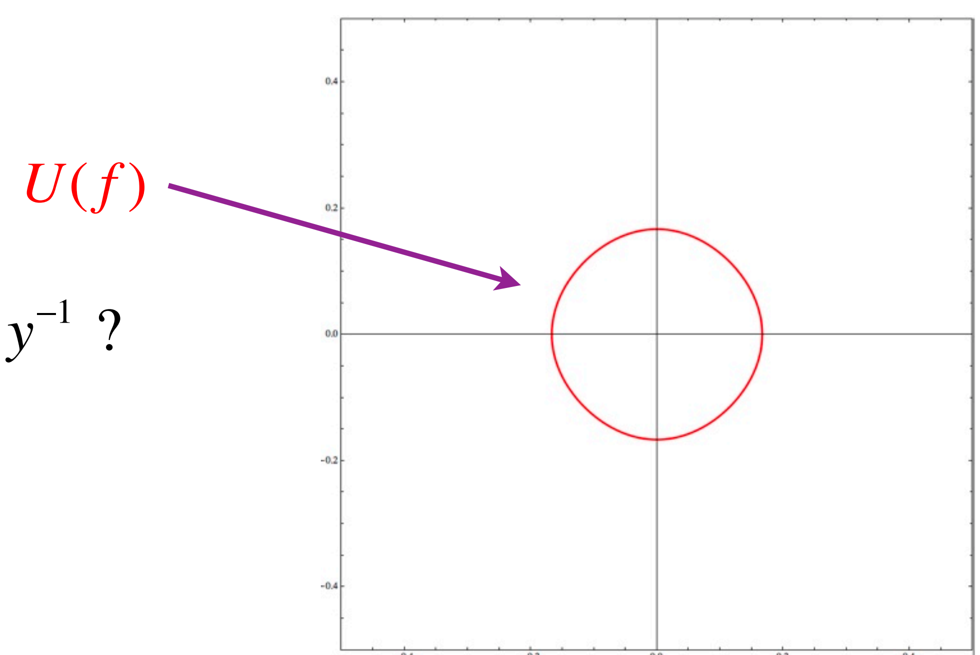
$$f(x,y,z) = 2 + x + y + z$$

$$g(x,y,z) = f^*(x,y,z) = 2 + x^{-1} + y^{-1} + z^{-1}$$

on $\mathbb{S}^3$, $\overline{f} = f^*$, and so $U(g) = U(f^*) = U(f)$

But what about

$$f(x,y) = 3 - x - x^{-1} - y - y^{-1} \ ?$$



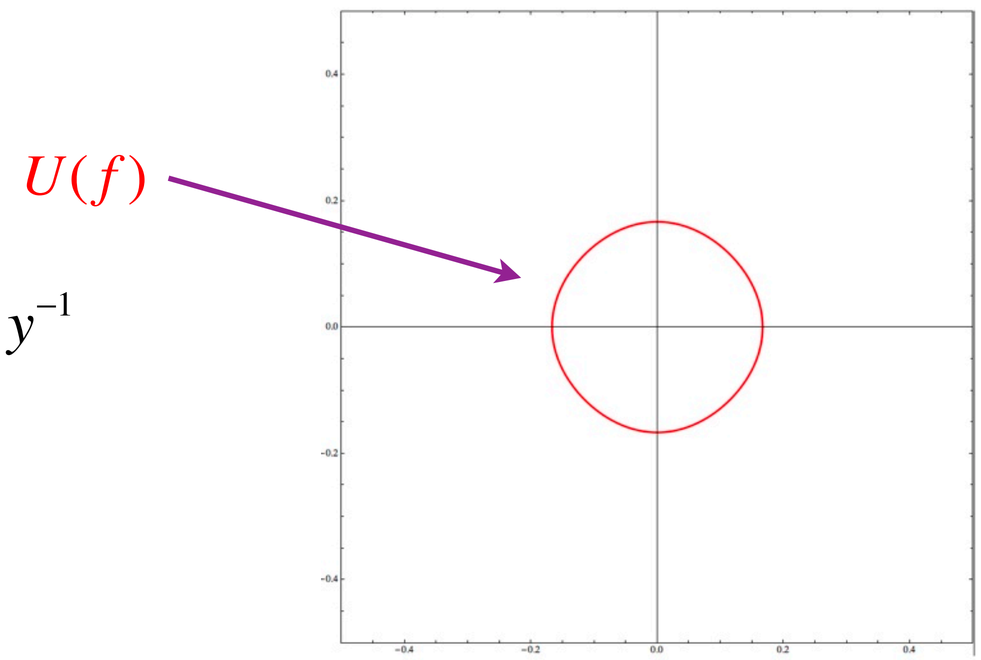
Let $f(x_1,\ldots,x_d) \in \mathbb{Z}[x_1^{\pm 1},\ldots,x_d^{\pm 1}]$. Then there is a g, not a multiple of f, with $U(g) = U(f)$ if and only if $\dim U(f) \le d-2$. In this case we're in business.

Another way to say this is that $U(f)$ is Zariski dense in its complex variety iff $\dim U(f) = d-1$.

This is essentially proved in a recent paper on several complex variables for use in interpolation. Tom Scanlon at Berkeley has shown us how to prove this using Tarski-Seidenberg using the cell decomposition of semialgebraic sets.

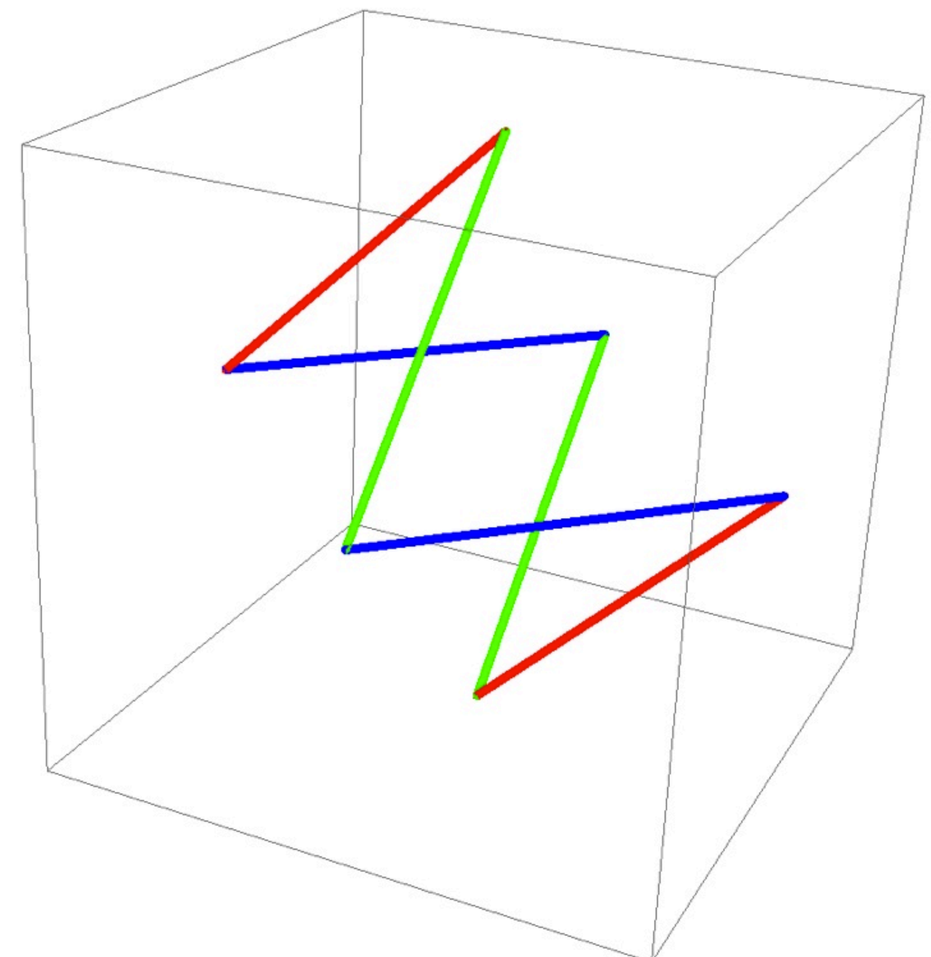
But : we're completely out of luck when $\dim U(f) = d-1$, and actually have very little idea how to handle it!

$$f(x,y) = 3 - x - x^{-1} - y - y^{-1}$$



Can there be infinitely many unit roots in $U(f)$?

Yes! Let $f(x,y,z) = 1 + x + y + z$:



But then a wonderful theorem comes to our rescue:

Theorem: If $U(f)$ contains infinitely many roots of unity, then they must all lie on the union of finitely many cosets of rational subtori.

Tom Scanlon's survey *Counting Special Points: Logic, Diophantine Geometry and Transcendence Theory*

Giving back to diophantine analysis

$$\xi \in \mathbb{S} \text{ algebraic}, \ g(\xi) = 0 \text{ for some } g(x) \in \mathbb{Z}[x]$$

$$f(x,y) = g(x)g^*(x) + g(y)g^*(y) = |g(x)|^2 + |g(y)|^2 \text{ for } (x,y) \in \mathbb{S}^2$$

Then (ξ,ξ) is an isolated point of $U(f)$, and can use $\dfrac{g(x)^N}{f(x,y)}$ to prove the Riemann sums for $\log|f|$ converge as $\langle\Gamma\rangle \to \infty$. Using a particular sequence of lattices, get:

Theorem: Let $\phi(n) \to \infty$ (think $\phi(n) = \log\log\log\log\log\log\log\log\log n$).
Given an algebraic number $\xi \in \mathbb{S}$ and $\varepsilon > 0$, the inequality

$$|\xi^n - 1| < e^{-\varepsilon n \phi(n)}$$

has only finitely many solutions in n.

When $U(f)$ is has codimension ≥ 2, we get diophantine results about how close torsion points can be to $U(f)$ using homoclinic points.

Open Questions

- Does specification hold when $\dim U(f) = d - 1$?
  (Known when $d = 1$ or and when $\dim U(f) \leq d - 2$)

- If $h(\alpha_f) > 0$, do the Haar measures on the periodic sets $P_\Gamma(\alpha_f)$ converge exponentially fast to Haar measure on X_f?

- If $h(\alpha_f) > 0$, does α_f mix sufficiently "smooth" functions exponentially fast?

- Does the geometry of $U(f)$ have dynamical significance?