Rational weak mixing in infinite measure spaces

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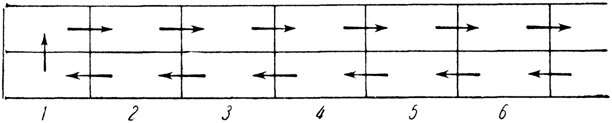
1. Hopf's example

$T = R \circ B : X = \mathbb{R}_+ \times [0,1] \equiv [0,1]^2 \times \mathbb{Z}_+ \to X$ where B is the Baker's transformation on each box $[0,1]^2 \times \{n\}$ and R is



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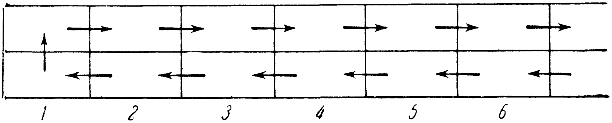

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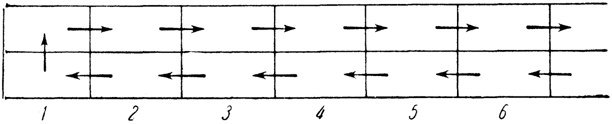

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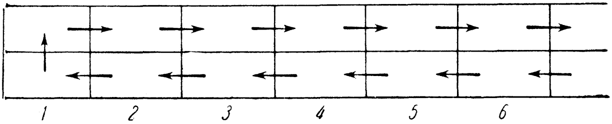

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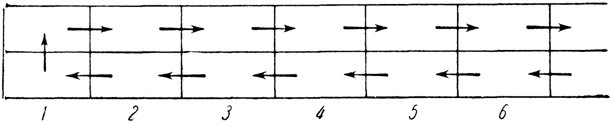

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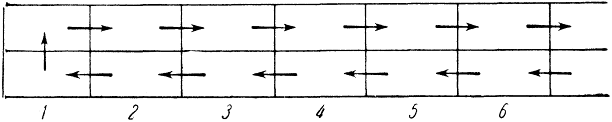

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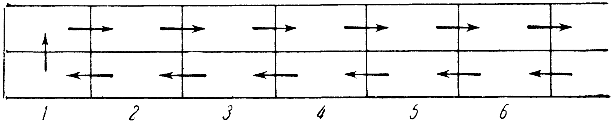

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- Summed (✠) OK $\forall\ A, B$ bdd. meas. via Markov property.

2. Weakly wandering sets

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Question about Hopf example $\exists$? exhaustive weakly wandering set of finite measure?

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So RWM of Hopf's example & box in $R(T) \implies$ (☕).

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¶4 Suppose $(X, \mathcal{B}, m, T)$ WRE, MPT & $\exists$ a countable generating partition $\alpha \subset R(T)$ and $\Omega \in \mathcal{C}_\alpha$ such that

$$\frac{m(A \cap T^{-n}B)}{u_n} \xrightarrow[n\to\infty]{u\text{-density}} m(A)m(B) \quad \forall\ A,\ B \in \mathcal{C}_\alpha$$

where $u = u(\Omega)$, then $(X, \mathcal{B}, m, T)$ is RWM.

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☺ Smoothness of ap. rec. $u = (u_0, u_1, \dots)$ with lifetime dist. $f \in \mathcal{P}(\mathbb{N})$ if e.g. $\exists\ N \ge 1,\ \sum_{n=N}^{\infty} \frac{1}{V(n)^2} < \infty$ $(V(t) := \sum_{1 \le n \le t} n^2 f_n)$ & $\sum_{k=1}^{n} f([k, \infty)) = o(\sqrt{n})$.

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$$s_n \xrightarrow[n\to\infty]{u-\texttt{density}} L \ \ \texttt{if } \exists\ K \subset \mathbb{N} \ \ u\texttt{-small such that } s_n \underset{n\to\infty,\ n\notin K}{\longrightarrow} L;$$

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¶6 Smoothing Lemma: $u \in \mathfrak{W}$, $x = (x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$ bdd. below:

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Use these to prove T RWM $\implies$ $T \times S$ ergodic $\forall$ ergodic PPT S & RWM $\forall$ weakly mixing PPT S.

9. Proof sketch of ¶4

¶4 $(X, \mathcal{B}, m, T)$ $\texttt{WRE}$, $\texttt{MPT}$ & $\exists$ cble gen. ptn $\alpha \subset R(T)$ & $\Omega \in \mathcal{C}_\alpha$ such that

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then $(X, \mathcal{B}, m, T)$ is $\texttt{RWM}$.

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Above ex's H-K mix. $\forall\ \gamma \in (\frac{1}{2}, 1)$. Refs: Thaler; Melbourne & Terhesiu.

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Spectral and rational weak mixing

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