Coexistence of Zero and Nonzero Lyapunov Exponents

Jianyu Chen
Pennsylvania State University

July 13, 2011

Outline

Nonuniform hyperbolicity

Let $\mathcal{M}$ be a compact smooth Riemannian manifold and $f \in \mathrm{Diff}(\mathcal{M})$. f is said to be *nonuniformly hyperbolic* on an invariant subset $\mathcal{R} \subset \mathcal{M}$ if for all $x \in \mathcal{R}$,

- $T_x\mathcal{M} = E^s(x) \oplus E^u(x)$ and $d_x f E^\sigma(x) = E^\sigma(f(x))$, $\sigma = s, u$;
- there exist numbers $0 < \lambda < 1 < \mu$, $\varepsilon > 0$ and Borel functions $C, K : \mathcal{R} \to \mathbb{R}^+$ such that

$$\|d_x f^n v\| \le C(x)\lambda^n e^{\varepsilon n}\|v\|, \ v \in E^s(x), \ n > 0,$$

$$\|d_x f^n v\| \le C(x)\mu^n e^{\varepsilon |n|}\|v\|, \ v \in E^u(x), \ n < 0,$$

$$\angle(E^s(x), E^u(x)) \ge K(x).$$

- $C(f^n(x)) \le C(x)e^{\varepsilon|n|}$ and $K(f^n(x)) \ge K(x)e^{-\varepsilon|n|}$, $n \in \mathbb{Z}$.

Hyperbolicity via Lyapunov exponents

Now consider $f \in \mathrm{Diff}(\mathcal{M}, \mu)$, i.e., f preserves a smooth measure μ ($\mu \sim \mathrm{vol}$).
We say f is *nonuniformly hyperbolic* w.r.t. μ if $\mu(\mathcal{M} \backslash \mathcal{R}) = 0$.

The *Lyapunov exponent* of f is defined by

$$\lambda(x, v) = \limsup_{n \to \infty} \frac{1}{n} \log \|df_x^n v\|, \quad x \in \mathcal{M}, v \in T_x\mathcal{M}.$$

f is nonuniformly hyperbolic $\Longleftrightarrow$ f has nonzero Lyapunov exponents μ-a.e.

Hyperbolicity via Lyapunov exponents

Now consider $f \in \mathrm{Diff}(\mathcal{M}, \mu)$, i.e., f preserves a smooth measure μ ($\mu \sim \mathrm{vol}$).
We say f is *nonuniformly hyperbolic* w.r.t. μ if $\mu(\mathcal{M} \backslash \mathcal{R}) = 0$.

The *Lyapunov exponent* of f is defined by

$$\lambda(x, v) = \limsup_{n \to \infty} \frac{1}{n} \log \|df_x^n v\|, \quad x \in \mathcal{M}, v \in T_x\mathcal{M}.$$

f is nonuniformly hyperbolic $\Longleftrightarrow$ f has nonzero Lyapunov exponents μ-a.e.

Hyperbolicity via Lyapunov exponents

Now consider $f \in \mathrm{Diff}(\mathcal{M}, \mu)$, i.e., f preserves a smooth measure μ ($\mu \sim \mathrm{vol}$).
We say f is *nonuniformly hyperbolic* w.r.t. μ if $\mu(\mathcal{M} \backslash \mathcal{R}) = 0$.

The *Lyapunov exponent* of f is defined by

$$\lambda(x, v) = \limsup_{n \to \infty} \frac{1}{n} \log \|df_x^n v\|, \quad x \in \mathcal{M}, v \in T_x\mathcal{M}.$$

f is nonuniformly hyperbolic $\Longleftrightarrow$ f has nonzero Lyapunov exponents μ-a.e.

Existence and Genericity

Question: Do nonuniformly hyperbolic diffeomorphisms exist on any manifold? If so, can it be generic in $\mathrm{Diff}(\mathcal{M}, \mu)$?

Existence: Yes.

- (Katok, 1979) Every compact surface admits a Bernoulli diffeomorhism with nonzero Lyapunov exponents a.e.;
- (Dolgopyat-Pesin, 2002) Every compact manifold (dim $\mathcal{M} \geq 2$) carries a hyperbolic Bernoulli diffeomorphism;
- (Hu-Pesin-Talitskaya, 2004) Every compact manifold (dim $\mathcal{M} \geq 3$) carries a hyperbolic Bernoulli flow.

Existence and Genericity

Question: Do nonuniformly hyperbolic diffeomorphisms exist on any manifold? If so, can it be generic in $\mathrm{Diff}(\mathcal{M}, \mu)$?

Existence: Yes.

- (Katok, 1979) Every compact surface admits a Bernoulli diffeomorhism with nonzero Lyapunov exponents a.e.;
- (Dolgopyat-Pesin, 2002) Every compact manifold (dim $\mathcal{M} \geq 2$) carries a hyperbolic Bernoulli diffeomorphism;
- (Hu-Pesin-Talitskaya, 2004) Every compact manifold (dim $\mathcal{M} \geq 3$) carries a hyperbolic Bernoulli flow.

Genericity ($\dim \mathcal{M} \geq 3$): Negative due to the following discrete version of KAM theory in the volume-preserving category:

Theorem (Cheng-Sun, Hermann, Xia, Yoccoz (1990's))

For any compact manifold $\mathcal{M}$ and any sufficiently large r there is an open set $U \subset \mathrm{Diff}^r(\mathcal{M}, \mathrm{vol})$ such that every $f \in U$ possesses a Cantor set of codim-1 invariant tori of positive volume. Moreover, f is C^1 conjugate to a Diophantine translation on each torus.

Invariant tori (with zero Lyapunov exponents) can not be destroyed by small perturbations.

Genericity ($\dim \mathcal{M} \geq 3$): Negative due to the following discrete version of KAM theory in the volume-preserving category:

Theorem (Cheng-Sun, Hermann, Xia, Yoccoz (1990's))

For any compact manifold $\mathcal{M}$ and any sufficiently large r there is an open set $U \subset \mathrm{Diff}^r(\mathcal{M}, \mathrm{vol})$ such that every $f \in U$ possesses a Cantor set of codim-1 invariant tori of positive volume. Moreover, f is C^1 conjugate to a Diophantine translation on each torus.

Invariant tori (with zero Lyapunov exponents) can not be destroyed by small perturbations.

Genericity ($\dim \mathcal{M} \geq 3$): Negative due to the following discrete version of KAM theory in the volume-preserving category:

Theorem (Cheng-Sun, Hermann, Xia, Yoccoz (1990's))

For any compact manifold $\mathcal{M}$ and any sufficiently large r there is an open set $U \subset \mathrm{Diff}^r(\mathcal{M}, \mathrm{vol})$ such that every $f \in U$ possesses a Cantor set of codim-1 invariant tori of positive volume. Moreover, f is C^1 conjugate to a Diophantine translation on each torus.

Invariant tori (with zero Lyapunov exponents) can not be destroyed by small perturbations.

Coexistence of zero and nonzero Lyapunov exponents

What is the dynamical behavior outside those invariant tori?
Is it possible to be nonuniformly hyperbolic?

- (Bunimovich, 2001) Coexistence of "elliptic islands" and "chaotic sea" (hyperbolic) was shown in billiard dynamics on a mushroom table.
- (Przytycki, 1982; Liverani, 2004) Birth of an elliptic island in chaotic sea for a one-parameter family of diffeomorphisms of $\mathbb{T}^2$.

Coexistence of zero and nonzero Lyapunov exponents

What is the dynamical behavior outside those invariant tori?
Is it possible to be nonuniformly hyperbolic?

- (Bunimovich, 2001) Coexistence of "elliptic islands" and "chaotic sea" (hyperbolic) was shown in billiard dynamics on a mushroom table.
- (Przytycki, 1982; Liverani, 2004) Birth of an elliptic island in chaotic sea for a one-parameter family of diffeomorphisms of $\mathbb{T}^2$.

We construct such examples

- in smooth dynamics;
- with a set of "elliptic islands" of positive measure.

Theorem (Hu-Pesin-Talitskaya)

Given $\alpha > 0$, there exist a compact manifold $\mathcal{M}^5$ and $P \in \mathrm{Diff}^\infty(\mathcal{M}, \mu)$ such that

(1) $\|P - Id\|_{C^1} \leq \alpha$ *and P is homotopic to Id;*

(2) *there is an open dense subset $\mathcal{G} \subset \mathcal{M}$ such that $P|\mathcal{G}$ has nonzero Lyapunov exponents μ-a.e. and is Bernoulli;*

(3) *the complement $\mathcal{G}^c = \mathcal{M} \setminus \mathcal{G}$ has positive volume and $P|\mathcal{G}^c = Id$.*

We construct such examples

- in smooth dynamics;
- with a set of "elliptic islands" of positive measure.

Theorem (Hu-Pesin-Talitskaya)

Given $\alpha > 0$, there exist a compact manifold $\mathcal{M}^5$ and $P \in \mathrm{Diff}^\infty(\mathcal{M}, \mu)$ such that

(1) $\|P - Id\|_{C^1} \leq \alpha$ *and P is homotopic to Id;*

(2) *there is an open dense subset $\mathcal{G} \subset \mathcal{M}$ such that $P|\mathcal{G}$ has nonzero Lyapunov exponents μ-a.e. and is Bernoulli;*

(3) *the complement $\mathcal{G}^c = \mathcal{M} \backslash \mathcal{G}$ has positive volume and $P|\mathcal{G}^c = Id$.*

In view of Pesin's ergodic decomposition theorem, i.e., a nonuniformly hyperbolic system has at most countably many ergodic components, we construct

Theorem (Chen)

Given $\alpha > 0$, there exist a compact manifold $\mathcal{M}^4$ and $P \in \mathrm{Diff}^\infty(\mathcal{M}, \mu)$ such that

(1) $\|P - Id\|_{C^1} \leq \alpha$ *and P is homotopic to Id;*

(2) *there is an open dense subset $\mathcal{G} \subset \mathcal{M}$ consisting of countably infinite many open connected components $\mathcal{G}_1, \mathcal{G}_2, \ldots$. For each k, $P|\mathcal{G}_k$ has nonzero Lyapunov exponents μ-a.e. and is Bernoulli;*

(3) *the complement $\mathcal{G}^c = \mathcal{M} \backslash \mathcal{G}$ has positive volume and $P|\mathcal{G}^c = Id$.*

Construction of $\mathcal{M}^5$

Pick an Anosov automorphism A of the 2-torus $X = \mathbb{T}^2$. Consider the action of suspension flow S^t over A with constant roof function 1 on the suspension manifold $\mathcal{N}$

$$\mathcal{N} = X \times [0, 1]/\sim,$$

where "$\sim$" is the identification $(x, 1) \sim (Ax, 0)$.

Set $Y = \mathbb{T}^2$. Choose a Cantor set $C \subset Y$ of positive but not full measure, and let $G = \mathcal{M} \backslash C$ be open connected.

Finally take $\mathcal{M} = \mathcal{N} \times Y$ and $\mathcal{G} = \mathcal{N} \times G$.

Construction of $\mathcal{M}^5$

Pick an Anosov automorphism A of the 2-torus $X = \mathbb{T}^2$. Consider the action of suspension flow S^t over A with constant roof function 1 on the suspension manifold $\mathcal{N}$

$$\mathcal{N} = X \times [0, 1]/\sim,$$

where "$\sim$" is the identification $(x, 1) \sim (Ax, 0)$.

Set $Y = \mathbb{T}^2$. Choose a Cantor set $C \subset Y$ of positive but not full measure, and let $G = \mathcal{M} \backslash C$ be open connected.

Finally take $\mathcal{M} = \mathcal{N} \times Y$ and $\mathcal{G} = \mathcal{N} \times G$.

Step 1: Slow down

Consider the partially hyperbolic flow

$$S^t \times Id_Y : \mathcal{M} = \mathcal{N} \times Y \to \mathcal{M}$$
$$(n, y) \mapsto (S^t(n), y)$$

Choose a C^∞ bump function $\kappa : Y \to \mathbb{R}^+ \cup \{0\}$ such that $\kappa \equiv 0$ on C, $\kappa > 0$ on $G = Y \backslash C$ and $\|\kappa\|_{C^1}$ is small.

Replace the speed by $\kappa(y)$ on each fiber $\mathcal{N} \times \{y\}$, and take the time-1 map of the new flow:

$$T(n, y) = (S^{\kappa(y)}(n), y).$$

Step 1: Slow down

Consider the partially hyperbolic flow

$$\begin{aligned} S^t \times Id_Y : \mathcal{M} = \quad \mathcal{N} \times Y \quad &\to \mathcal{M} \\ (n, y) \quad &\mapsto (S^t(n), y) \end{aligned}$$

Choose a C^∞ bump function $\kappa : Y \to \mathbb{R}^+ \cup \{0\}$ such that $\kappa \equiv 0$ on C, $\kappa > 0$ on $G = Y \backslash C$ and $\|\kappa\|_{C^1}$ is small.

Replace the speed by $\kappa(y)$ on each fiber $\mathcal{N} \times \{y\}$, and take the time-1 map of the new flow:

$$T(n, y) = (S^{\kappa(y)}(n), y).$$

One can verify that

- $T|\mathcal{G}^c = Id$;
- T is homotopic to Id and C^1-close to Id provided $\|\kappa\|_{C^1}$ is small;
- $T|\mathcal{G}$ is pointwisely partially hyperbolic with 1-dim stable, 1-dim unstable and 3-dim central;
- $T|\mathcal{A}$ is uniformly partially hyperbolic for any compact invariant subset $\mathcal{A} \subset \mathcal{G}$.

The we only need to do perturbations on $T|\mathcal{G}$.

Step 2: Removing zero exponents

1. Produce positive exponent in t-direction by a "surgery": take a small ball $B \subset \mathcal{G}$ with coordinate $(n, y) = (u, s, t, a, b)$, and construct $g_1 \in \mathrm{Diff}^\infty(\mathcal{G}, \mu)$ such that g_1 is a rotation along ut-plane inside B and $g_1 = Id$ outside B. Set $Q_1 = T \circ g_1$. There is a closed invariant subset $\mathcal{A} \subset \mathcal{G}$ of positive volume such that $Q_1|\mathcal{A}$ has positive Lyapunov exponents along ut-directions.

2. The invariant set $\mathcal{A}$ can be of extremely bad shape. we need to use Rokhlin-Halmos tower to construct a new diffeomorphism Q with positive Lyapunov exponent in y-directions on a set of positive volume.

Step 2: Removing zero exponents

1. Produce positive exponent in t-direction by a "surgery": take a small ball $B \subset \mathcal{G}$ with coordinate $(n, y) = (u, s, t, a, b)$, and construct $g_1 \in \mathrm{Diff}^\infty(\mathcal{G}, \mu)$ such that g_1 is a rotation along ut-plane inside B and $g_1 = Id$ outside B. Set $Q_1 = T \circ g_1$. There is a closed invariant subset $\mathcal{A} \subset \mathcal{G}$ of positive volume such that $Q_1|\mathcal{A}$ has positive Lyapunov exponents along ut-directions.

2. The invariant set $\mathcal{A}$ can be of extremely bad shape. we need to use Rokhlin-Halmos tower to construct a new diffeomorphism Q with positive Lyapunov exponent in y-directions on a set of positive volume.

Step 3: Obtain accessibility

Theorem (Burns-Dolgopyat-Pesin, 2002)

Let $f \in \mathrm{Diff}^2(\mathcal{G}, \mu)$ be pointwise partially hyperbolic such that

(1) *f has strongly stable and unstable (δ, q)-foliations W^s and W^u where δ and q are continuous functions on $\mathcal{G}$, and W^s and W^u are absolutely continuous;*

(2) *f has positive central exponents on a set of positive volume;*

(3) *f has the accessibility property via W^s and W^u;*

Then f has positive central exponents almost everywhere. $f|\mathcal{G}$ is ergodic and indeed, Bernoulli.

Q and its small perturbations will satisfy (1) and (2).

Step 3: Obtain accessibility

Theorem (Burns-Dolgopyat-Pesin, 2002)

Let $f \in \mathrm{Diff}^2(\mathcal{G}, \mu)$ be pointwise partially hyperbolic such that

(1) *f has strongly stable and unstable (δ, q)-foliations W^s and W^u where δ and q are continuous functions on $\mathcal{G}$, and W^s and W^u are absolutely continuous;*

(2) *f has positive central exponents on a set of positive volume;*

(3) *f has the accessibility property via W^s and W^u;*

Then f has positive central exponents almost everywhere. $f|\mathcal{G}$ is ergodic and indeed, Bernoulli.

Q and its small perturbations will satisfy (1) and (2).

Two points $p, q \in \mathcal{G}$ are *accessible* if there exists a collection of points $z_1, \ldots, z_n \in \mathcal{G}$ such that $p = z_1$, $q = z_n$ and $z_k \in W^i(z_{k-1})$ for $i = s$ or u and $k = 2, \ldots, n$. The accessibility class of q under $f \in \mathrm{Diff}^\infty(\mathcal{G}, \mu)$ is denoted by $\mathcal{A}_f(q)$.

Decompose $\mathcal{G}$ as $\mathcal{G} = \biguplus_{i=0}^{\infty} \mathcal{G}_i$, where $\mathcal{G}_i$ is a nested sequence of compact sets, and pick $q_i \in \mathcal{G}_i$. We shall perturb Q to P as follows:

$$Q \longrightarrow P_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \ldots \longrightarrow P$$

such that $P_n \neq P_{n-1}$ only on G_n, and $\mathcal{A}_{P_n}(q_n) \supset \mathcal{G}_n$. Therefore, $\mathcal{A}_{P_n}(q_0) \supset \biguplus_{i=0}^{n} \mathcal{G}_i$.

Moreover, we can guarantee that P_n is stably accessible, then taking $n \to \infty$, we get $\mathcal{A}_P(q_0) \supset \mathcal{G}$.

Two points $p, q \in \mathcal{G}$ are *accessible* if there exists a collection of points $z_1, \dots, z_n \in \mathcal{G}$ such that $p = z_1$, $q = z_n$ and $z_k \in W^i(z_{k-1})$ for $i = s$ or u and $k = 2, \dots, n$. The accessibility class of q under $f \in \mathrm{Diff}^\infty(\mathcal{G}, \mu)$ is denoted by $\mathcal{A}_f(q)$.

Decompose $\mathcal{G}$ as $\mathcal{G} = \biguplus_{i=0}^\infty \mathcal{G}_i$, where $\mathcal{G}_i$ is a nested sequence of compact sets, and pick $q_i \in \mathcal{G}_i$. We shall perturb Q to P as follows:

$$Q \longrightarrow P_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \dots \longrightarrow P$$

such that $P_n \neq P_{n-1}$ only on G_n, and $\mathcal{A}_{P_n}(q_n) \supset \mathcal{G}_n$. Therefore, $\mathcal{A}_{P_n}(q_0) \supset \biguplus_{i=0}^{n} \mathcal{G}_i$.

Moreover, we can guarantee that P_n is stably accessible, then taking $n \to \infty$, we get $\mathcal{A}_P(q_0) \supset \mathcal{G}$.

Two points $p, q \in \mathcal{G}$ are *accessible* if there exists a collection of points $z_1, \ldots, z_n \in \mathcal{G}$ such that $p = z_1$, $q = z_n$ and $z_k \in W^i(z_{k-1})$ for $i = s$ or u and $k = 2, \ldots, n$. The accessibility class of q under $f \in \mathrm{Diff}^\infty(\mathcal{G}, \mu)$ is denoted by $\mathcal{A}_f(q)$.

Decompose $\mathcal{G}$ as $\mathcal{G} = \biguplus_{i=0}^{\infty} \mathcal{G}_i$, where $\mathcal{G}_i$ is a nested sequence of compact sets, and pick $q_i \in \mathcal{G}_i$. We shall perturb Q to P as follows:

$$Q \longrightarrow P_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \ldots \longrightarrow P$$

such that $P_n \neq P_{n-1}$ only on G_n, and $\mathcal{A}_{P_n}(q_n) \supset \mathcal{G}_n$. Therefore, $\mathcal{A}_{P_n}(q_0) \supset \biguplus_{i=0}^{n} \mathcal{G}_i$.

Moreover, we can guarantee that P_n is stably accessible, then taking $n \to \infty$, we get $\mathcal{A}_P(q_0) \supset \mathcal{G}$.

Construction of $\mathcal{M}^4$

Take the same suspension manifold $\mathcal{N}$.

Set $Y = S^1 = [0,1]/\{0 \sim 1\}$. Construct a "fat" Cantor set $C \subset Y$ by consecutively removing disjoint open subintervals $I_1, I_2, \dots$ from Y, then set $\mathcal{I} = \biguplus_{n=1}^{\infty} I_n$ and $C = Y \backslash \mathcal{I}$. Moreover, let $\sum_{n=1}^{\infty} |I_n| < 1$ so that C is of positive Lebesgue measure.

Finally take $\mathcal{M} = \mathcal{N} \times Y$, $\mathcal{G} = \mathcal{N} \times \mathcal{I}$ and $\mathcal{G}_n = \mathcal{N} \times I_n$, $n = 1, 2, \dots$. Clearly $\{\mathcal{G}_n\}_{n=1}^{\infty}$ are open connected components of $\mathcal{G}$. Also the complement $\mathcal{G}^c = \mathcal{N} \times C$ is of positive Riemannian volume.

Reduction

After several simplifications, it suffices to do perturbation H_n on each $\mathcal{G}_n$.

Proposition

Set $\mathcal{Z} = \mathcal{N} \times (-2, 2)$ and $\hat{\mathcal{Z}} = \mathcal{N} \times [-3, 3]$. Given $\delta > 0$, $r \geq 1$, there exists $H \in \mathrm{Diff}^r(\hat{\mathcal{Z}}, \mu)$ such that

(1) $\|H - Id\|_{C^r} \leq \delta$;

(2) *H is homotopic to Id, and $H = Id$ on $\hat{\mathcal{Z}} \backslash \mathcal{Z}$;*

(3) *$H|\mathcal{Z}$ has nonzero Lyapunov exponents μ-a.e. and is ergodic, indeed Bernoulli.*

Reduction

After several simplifications, it suffices to do perturbation H_n on each $\mathcal{G}_n$.

Proposition

Set $\mathcal{Z} = \mathcal{N} \times (-2, 2)$ and $\hat{\mathcal{Z}} = \mathcal{N} \times [-3, 3]$. Given $\delta > 0$, $r \geq 1$, there exists $H \in \mathrm{Diff}^r(\hat{\mathcal{Z}}, \mu)$ such that

(1) $\|H - Id\|_{C^r} \leq \delta$;

(2) *H is homotopic to Id, and $H = Id$ on $\hat{\mathcal{Z}} \setminus \mathcal{Z}$;*

(3) *$H|\mathcal{Z}$ has nonzero Lyapunov exponents μ-a.e. and is ergodic, indeed Bernoulli.*

The End

Thank you very much