Diophantine condition of the interval exchange map

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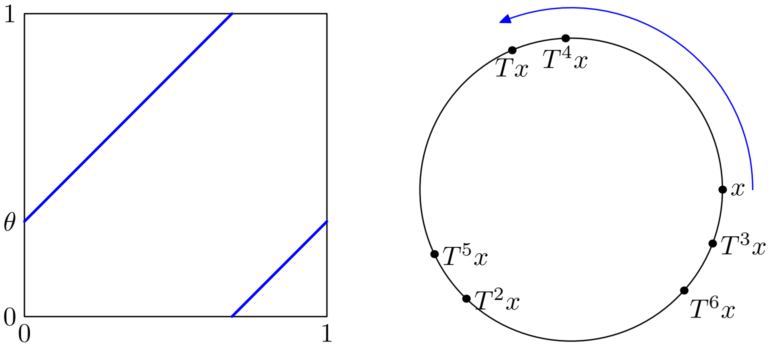
An irrational rotation

$$T : [0,1) \to [0,1), \quad T(x) = x + \theta \pmod 1.$$



Diophantine approximation

Theorem (Dirichlet, Hurwitz)

For any irrational θ there are infinitely many integers p, q such that

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

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The set of Roth type irrationals has full Lebesgue measure.

Roth type condition and recurrence speed

$T : [0,1) \to [0,1), \quad T(x) = x + \theta \pmod 1.$

recurrence time $\tau_r(x) = \min\{j \geq 1 : |T^j(x) - x| < r\}$.

Theorem

An irrational θ is of Roth type if and only if

$$\lim_{r \to 0^+} \frac{\log \tau_r(x)}{-\log r} = 1.$$

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$$|T^q x - x| < \delta \;\Leftrightarrow\; |q\theta - \exists p| < \delta \;\Leftrightarrow\; \left|\theta - \frac{p}{q}\right| < \frac{\delta}{q}.$$

Roth type condition and continued fraction

Continued fraction is the best method of Diophantine approximation.

$$\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots}}}}, \qquad \frac{p_n}{q_n} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}}.$$

An irrational θ is Roth type if and only if for all $\varepsilon > 0$ there is a constant C such that

$$a_{n+1} < C q_n^{\varepsilon} \quad \text{for all } n.$$

Roth type condition and uniform distribution

An irrational θ is of Roth type if and only if

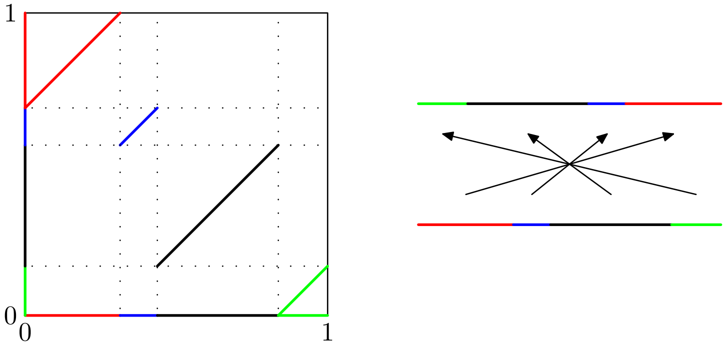
$$\{0, \theta, 2\theta, \ldots, n\theta\} \quad (\text{ in mod } 1 \text{ })$$

is uniformly distributed in $[0,1)$ in the sense that

$$(\text{minimal distance between any two points}) > \frac{C}{n^{1+\varepsilon}}.$$

An interval exchange map

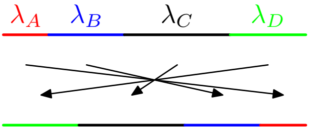
Generalization of the irrational rotation



The interval exchange map

An interval exchange map (i.e.m.) is determined by

- The combinatorial data: two bijections (π_0, π_1) from $\mathcal{A}$ (names for the intervals) onto $\{1, \dots, d\}$. ($d = \text{card}\ (\mathcal{A})$).
- The length data $(\lambda_\alpha)_{\alpha\in\mathcal{A}}$.



$$\mathcal{A} = \{A, B, C, D\}.$$

$$\pi_0(A) = 1,\ \pi_0(B) = 2,\ \pi_0(C) = 3,\ \pi_0(D) = 4,$$
$$\pi_1(A) = 4,\ \pi_1(B) = 3,\ \pi_1(C) = 2,\ \pi_1(D) = 1.$$

The Keane property

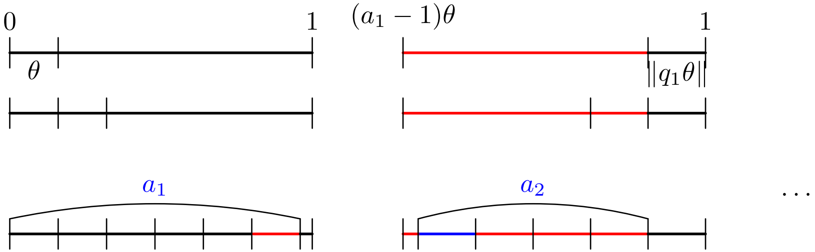
Consider only combinatorial data $(\mathcal{A}, \pi_t, \pi_b)$ which are *admissible*, meaning that for all $k = 1, 2, \ldots, d-1$, we have

$$\pi_0^{-1}(\{1, \ldots, k\}) \neq \pi_1^{-1}(\{1, \ldots, k\}) .$$

The Keane property is the appropriate notion of irrationality for i.e.m. since, as Keane himself proved,

- An i.e.m. with Keane's property is minimal (i.e. all orbits are dense);
- If the length data are rationally independent (and the combinatorial data are admissible) then T has Keane's property.

Continued fraction algorithm



$$\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}}$$

Consider the induced map.

Invariant measure for the continued fraction algorithm

Farey map for the irrational rotation.

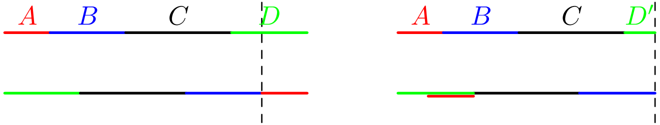
$$f(x) = \begin{cases} \dfrac{x}{1-x}, & 0 < x < \frac{1}{2}, \\ \dfrac{1-x}{x}, & \dfrac{1}{2} < x < 1 \end{cases}$$

with invariant measure $\dfrac{dx}{x}$.

Gauss map is a acceleration :

$$x \mapsto \left\{ \frac{1}{x} \right\} \text{ with invariant measure } \frac{dx}{1+x}$$

Generalization of continued fractions to i.e.m. (Rauzy, Veech, Zorich)



type 0

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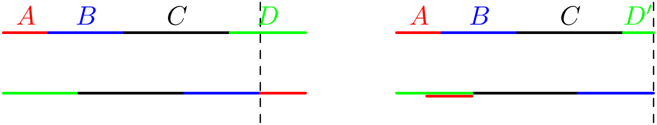

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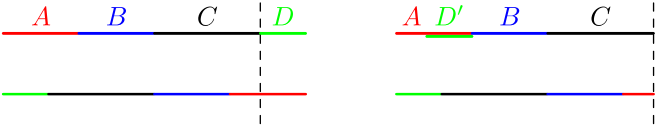


type 1

Permutation data

(π_0, π_1) : an admissible pair, $\alpha_0, \alpha_1 \in \mathcal{A}$, $\pi_0(\alpha_0) = \pi_1(\alpha_1) = d$;
Define two new admissible pairs $\mathcal{R}_0(\pi_0, \pi_1)$, $\mathcal{R}_1(\pi_0, \pi_1)$:

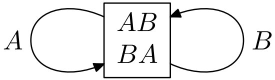
$$\mathcal{R}_0(\pi_0, \pi_1) = (\pi_0, \hat{\pi}_1), \quad \mathcal{R}_1(\pi_0, \pi_1) = (\hat{\pi}_0, \pi_1),$$

$$\hat{\pi}_0(\alpha) = \begin{cases} \pi_0(\alpha) & \text{if } \pi_0(\alpha) \le \pi_0(\alpha_1), \\ \pi_0(\alpha) + 1 & \text{if } \pi_0(\alpha_1) < \pi_0(\alpha) < d, \\ \pi_0(\alpha_1) + 1 & \text{if } \alpha = \alpha_0, \ (\pi_0(\alpha_0) = d). \end{cases}$$

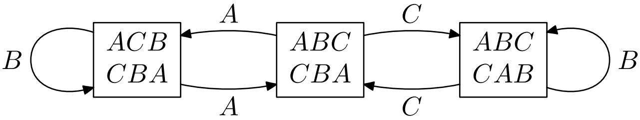
$$\hat{\pi}_1(\alpha) = \begin{cases} \pi_1(\alpha) & \text{if } \pi_1(\alpha) \le \pi_1(\alpha_0), \\ \pi_1(\alpha) + 1 & \text{if } \pi_1(\alpha_0) < \pi_1(\alpha) < d, \\ \pi_1(\alpha_0) + 1 & \text{if } \alpha = \alpha_1, \ (\pi_1(\alpha_1) = d); \end{cases}$$

Rauzy diagram

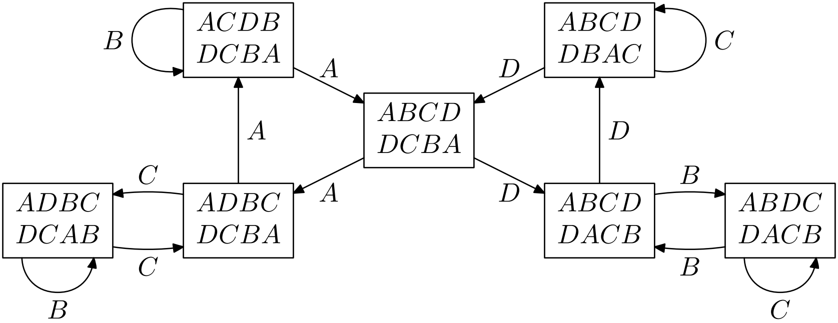
Each vertex (π_0, π_1) being the origin of two arrows joining (π_0, π_1) to $\mathcal{R}_0(\pi_0, \pi_1)$, $\mathcal{R}_1(\pi_0, \pi_1)$.



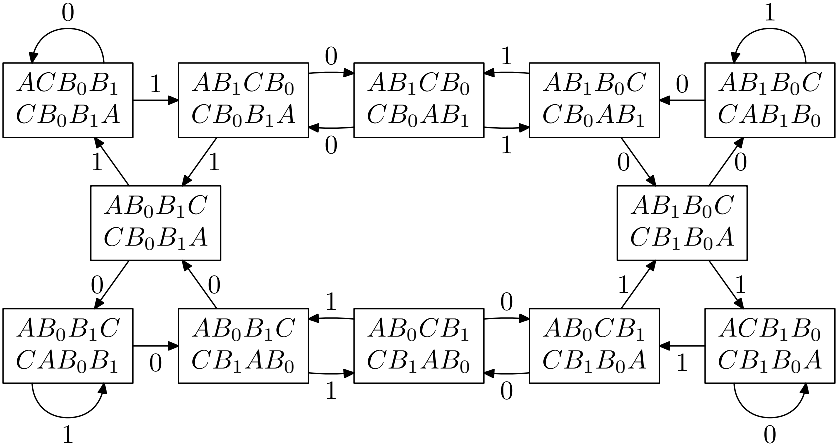
Rauzy diagram $d = 2$



Rauzy diagram $d = 3$



Rauzy diagram $d = 4$ first case



Rauzy diagram $d = 4$ second case

Length data

Define a new i.e.m. $\mathcal{V}(T)$ by the admissible pair $\mathcal{R}_\varepsilon(\pi_0, \pi_1)$ and the lengths $(\hat{\lambda}_\alpha)_{\alpha\in\mathcal{A}}$ given by

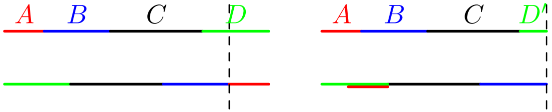
$$\begin{cases} \hat{\lambda}_\alpha = \lambda_\alpha & \text{if } \alpha \neq \alpha_\varepsilon, \\ \hat{\lambda}_{\alpha_\varepsilon} = \lambda_{\alpha_\varepsilon} - \lambda_{\alpha_{1-\varepsilon}} & \text{otherwise,} \end{cases}$$

i.e. the length data of T are obtained from those of $\mathcal{V}(T)$ as

$$\lambda = V(T)\hat{\lambda}$$

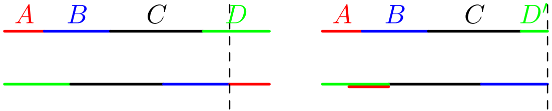
where the matrix $V(T)$ has all diagonal entries equal to 1 and all off-diagonal entries equal to 0 except the one corresponding to $(\alpha_\varepsilon, \alpha_{1-\varepsilon})$ which is also equal to 1.

Continued fraction matrix



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Continued fraction matrix



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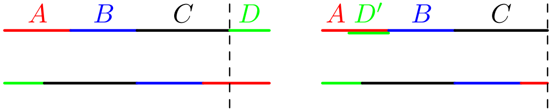

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For irrational rotations

$$V(T) = V(\mathcal{V}(T)) = \cdots = V(\mathcal{V}^{a_1-2}(T)) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$V(\mathcal{V}^{a_1-1}(T)) = V(\mathcal{V}^{a_1}(T)) = \cdots = V(\mathcal{V}^{a_1+a_2-2}(T)) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$V(\mathcal{V}^{a_1+a_2-1}(T)) = \cdots = V(\mathcal{V}^{a_1+a_2+a_3-2}(T)) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\vdots$$

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{a_1-1} \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}_{a_2} \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{a_3} \cdots\cdots$$

Invariant measure and Zorich's acceleration

Rauzy-Veech operation has a σ-finite invariant measure on the length data.

For the irrational rotation it is Farey map:

$$f(x) = \begin{cases} \frac{x}{1-x}, & 0 < x < \frac{1}{2}, \\ \frac{1-x}{x}, & \frac{1}{2} < x < 1 \end{cases}$$ with invariant measure $\dfrac{dx}{x}$.

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Zorich introduced an acceleration algorithm with finite invariant measure, corresponding to Gauss map in rotation (Gauss map: $x \mapsto \left\{\dfrac{1}{x}\right\}$ with invariant measure $\dfrac{dx}{1+x}$)

Accelerated algorithm

Zorich's acceleration:
n_{k+1} is taken as the largest integer $n > n_k$ such that one name in $\mathcal{A}$ are taken by arrows associated to iterations of $\mathcal{V}$ from $T(n_k)$ to $\mathcal{V}^n(T)$.

e.g. $AA,\ B,\ D,\ CCCC,\ B,\ AAAA,\ DDDD, \ldots$

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e.g. $AA,\ B,\ D,\ CCCC,\ B,\ AAAA,\ DDDD, \ldots$

Marmi-Moussa-Yoccoz's acceleration:
m_{k+1} is taken as the largest integer $n > m_k$ such that not all names in $\mathcal{A}$ are taken by arrows

e.g. $AABD,\ CCCCBAAAA,\ DDDD \cdots$

- $Y(n)$: continued fraction matrices for a given T.
- $\lambda(n)$: length data after nth iteration.
- $T^{(n)}$: induced map of T on $[0, \lambda^*(n))$, $\lambda^*(n) = \sum_\alpha \lambda_\alpha(n)$.
- $Q(n) = Y(1)Y(2)\cdots Y(n)$.
- $Z(k) = Y(n_{k-1}+1)Y(n_{k-1}+2)\cdots Y(n_k)$
- $A(k) = Y(m_{k-1}+1)Y(m_{k-1}+2)\cdots Y(m_k)$

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Diophantine condition:

For any $\varepsilon > 0$ there exist $C_\varepsilon > 0$ such that

$$\|A(k+1)\| \le C_\varepsilon \|Q(m_k)\|^\varepsilon \quad \text{or} \quad \|Z(k+1)\| \le C_\varepsilon \|Q(n_k)\|^\varepsilon.$$

Distance between discontinuities

$\Delta(T)$: minimal distance between the discontinuity points of T

We have another Diophantine condition:

For any $\varepsilon > 0$ there exist $D_\varepsilon > 0$ such that for all $m \geq 1$ we have

$$\Delta(T^m) \geq \frac{D_\varepsilon}{m^{1+\varepsilon}}.$$

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It has been unclear the relation between Diophantine condition by the size of Q matrices and condition using the distance between discontinuities.
(e.g. Marmi-Moussa-Yoccoz JAMS, 2006)

Diopahntine conditions

(A) For any $\varepsilon > 0$ there exist $C_\varepsilon > 0$ such that

$$\|A(k+1)\| \leq C_\varepsilon \|Q(m_k)\|^\varepsilon.$$

(Z) For any $\varepsilon > 0$ there exist $C_\varepsilon > 0$ such that

$$\|Z(k+1)\| \leq C_\varepsilon \|Q(n_k)\|^\varepsilon.$$

(D) For any $\varepsilon > 0$ there exist $C_\varepsilon > 0$ such that $\Delta(T^n) \geq \dfrac{C_\varepsilon}{n^{1+\varepsilon}}$.

(R) $$\lim_{r \to 0} \frac{\log \tau_r(x)}{-\log r} = 1 \text{ for almost every } x.$$

(U) $$\lim_{r \to 0} \frac{\log \tau_r(x)}{-\log r} = 1 \text{ uniformly.}$$

Relation between the Diophantine conditions

- In 2-interval exchange or the irrational rotation, all 5 conditions are equivalent.

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- In 2-interval exchange or the irrational rotation, all 5 conditions are equivalent.
- In 3-interval exchange, (A) and (D) are equivalent and (Z), (U) and (R) are equivalent.
- In general interval exchanges we have

$$(A) \Leftrightarrow (D) \begin{array}{c}\Rightarrow \\ \not\Leftarrow\end{array} (U) \begin{array}{c}\Rightarrow \\ \not\Leftarrow\end{array} (Z) \begin{array}{c}?? \\ \not\Leftarrow\end{array} (R)$$

$(U) \Rightarrow (R)$ by the definition

Irrational rotations

$$Z(1) = \begin{pmatrix} 1 & a_1 - 1 \\ 0 & 1 \end{pmatrix}, \; Z(k) = A(k) = \begin{cases} \begin{pmatrix} 1 & a_k \\ 0 & 1 \end{pmatrix}, & \text{odd } k, \\ \begin{pmatrix} 1 & 0 \\ a_k & 1 \end{pmatrix}, & \text{even } k. \end{cases}$$

By multiplying them

$$Q(n_k) = Q(m_k) = \begin{cases} \begin{pmatrix} q_{k-1} - p_{k-1} & q_k - p_k \\ p_{k-1} & p_k \end{pmatrix}, & \text{for odd } k, \\ \begin{pmatrix} q_k - p_k & q_{k-1} - p_{k-1} \\ p_k & p_{k-1} \end{pmatrix}, & \text{for even } k. \end{cases}$$

Therefore,

$$\|Z(k+1)\| = \|A(k+1)\| = a_{k+1} + 2,$$
$$\|Q(n_k)\| = \|Q(m_k)\| = q_k + q_{k-1}.$$

Condition (A) and (D) are equivalent to

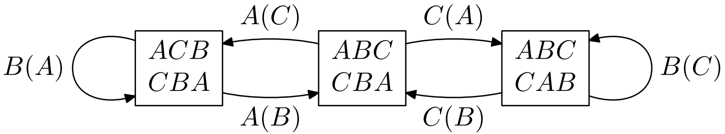
$$a_{k+1} + 2 \leq C_\varepsilon (q_k + q_{k-1})^\varepsilon < 2^\varepsilon C_\varepsilon q_k^\varepsilon.$$

The Roth type condition for the irrational number.

3-interval exchange maps

Let T be a 3-interval exchange map with $(\lambda_A, \lambda_B, \lambda_C)$.
Assume

$$\pi_0(A) = 1, \pi_0(B) = 2, \pi_0(C) = 3,$$
$$\pi_1(C) = 3, \pi_1(B) = 2, \pi_1(A) = 1.$$

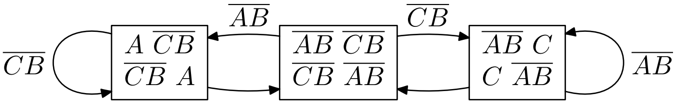


Define an irrational rotation $\bar{T}$ on $\bar{I} = [0, 1 + \lambda_B)$ by

$$\bar{T}(x) = \begin{cases} x + \lambda_B + \lambda_C & \text{if } x + \lambda_B + \lambda_C \in \bar{I}, \\ x - \lambda_B - \lambda_A & \text{if } x + \lambda_B + \lambda_C \notin \bar{I}. \end{cases}$$

T is the induced map of $\bar{T}$ on $[0, 1) = [0, \lambda_A + \lambda_B + \lambda_C)$.

Let $\bar{T}$ be a 2-interval exchange with length data $(\lambda_{\bar{A}}, \lambda_{\bar{B}})$, $\lambda_{\bar{A}} = \lambda_A + \lambda_B$ and $\lambda_{\bar{C}} = \lambda_A + \lambda_C$.



Let $\alpha = \dfrac{\lambda_B + \lambda_C}{1 + \lambda_B}$ be the rotation angle of $\bar{T}$ and a_n and $\dfrac{p_n}{q_n}$ be the partial quotient and partial convergent of α.

$\bar{T}$ satisfies condition (A) or (D): for any $\varepsilon > 0$ there exist $C_\varepsilon > 0$ such that

$$a_{n+1} \le C_\varepsilon {q_n}^{\varepsilon}.$$

For the irrational rotation

$$\lim_{r \to 0} \frac{\log \bar{\tau}_r(x)}{-\log r} = 1,$$

if α is of Roth type.

Proposition

For a 3-interval exchange map T,

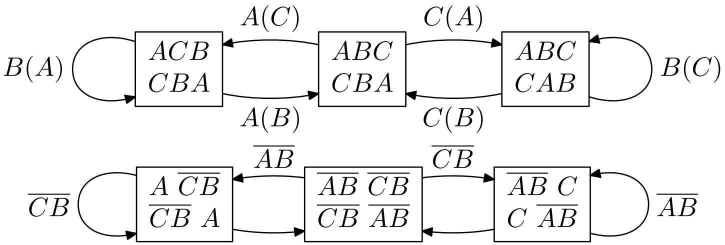
$$\lim_{r\to 0} \frac{\log \tau_r(x)}{-\log r} = 1$$

if and only if

$$\lim_{r\to 0} \frac{\log \bar{\tau}_r(x)}{-\log r} = 1,$$

where $\bar{\tau}_r$ is the first return time of $\bar{T}$.

T satisfies (R) or (U) if and only if $\bar{T}$ satisfies (A) (or (Z)).



No long sequence of A implies no long sequence of $\overline{AB}$ (center).

No long sequence of $\overline{AB}$ implies no long sequence $B^n C A^m$.

Proposition

The 3-interval exchange map T satisfies Condition (Z) if and only if the $\bar{T}$ satisfies (A) (or equivalently (Z)).

(A) implies (D)

- $\Delta((T^{(k)})^2) \leq \Delta(T^m)$ for $\min Q_\alpha(k-1) < m \leq \min Q_\alpha(k)$
- If $k' > k$ satisfies $\lambda^*(k') < \lambda_\alpha(k)$, $\pi_0^{(k)}(\alpha) = 1$, then

$$\min_\alpha \lambda_\alpha(k') \leq \Delta((T^{(k)})^2).$$

- $(A(k+1)A(k+2)\cdots A(k+r))_{\alpha\beta} > 0$, $r = \max(2d-3, 2)$
- (Marmi-Moussa-Yoccoz) If T satisfies Condition (A), then

$$\max_{\alpha\in\mathcal{A}} \lambda_\alpha(k) \leq C_\varepsilon \min_{\alpha\in\mathcal{A}} \lambda_\alpha(k)\|Q(k)\|^\varepsilon.$$

Theorem

If T satisfies condition (A), then it also satisfies Condition (D).

(D) implies (A)

- Suppose that T does not satisfy (A). Then for some $\delta > 0$ there are infinitely many k such that

$$\min_{\alpha \in \mathcal{A}} \lambda_\alpha(k) < \lambda^*(k)^{1+\delta}.$$

- If $\lambda_\alpha(k) < \lambda^*(k)^{1+\delta}$, then for some integer s, $1 \leq s < d$, we have

$$\Delta(T^{\lfloor 2/\lambda^*(k)^{1+s\delta/d} \rfloor}) < (d-1)\lambda^*(k)^{1+(s+1)\delta/d}.$$

Theorem

If T does not satisfy (A), then T does not satisfy (D).

The first return time of the interval exchange map

Let $\mathcal{P}_n$ be the partition of $I = [0, 1)$ consists of

$$T^i(I_\alpha(n)), \qquad 0 \leq i < Q_\alpha(n)$$

Define $R_n(x)$ by the first return time to the element of $\mathcal{P}_n$ which contains x.

Proposition

$$\min_{\beta\in\mathcal{A}} Q_\beta(n) \leq R_n(x) < 2 \max_{\beta\in\mathcal{A}} Q_\beta(n + m(d)) + \max_{\beta\in\mathcal{A}} Q_\beta(n).$$

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Condition (A) $\Rightarrow$ for all $\varepsilon > 0$ there is $C'_\varepsilon > 0$ such that

$$\max_{\alpha\in\mathcal{A}} \lambda_\alpha(n) \le C'_\varepsilon \|Q(n)\|^\varepsilon \min_{\alpha\in\mathcal{A}} \lambda_\alpha(n).$$

(A) and (D) imply (U)

Theorem

For an interval exchange map with condition (A)

$$\lim_{n\to\infty}\frac{\log R_n(x)}{-\log|P_n(x)|}\leq\limsup_{r\to 0^+}\frac{\log\tau_r(x)}{-\log r}=1 \text{ uniformly.}$$

Lemma

If $\tau_r(x)=n$ for some x, then we have $\Delta(T^{2n})<r$.

Theorem

Condition (D) implies Condition (U)

(U) implies (Z)

Assume there is a sequence k_i and $r > 0$ and C such that

$$\|Z(k_i + 1)\| \geq C\|Q(m_{k_i})\|^{\rho}. \tag{1}$$

Let α_i be the first name. Each $\beta \in \mathcal{A}_i$ appeared h_i or $h_i + 1$ times as the second name between $m_{k_i} + 1$ and m_{k_i+1}.

$$C\|Q_{\alpha_i}(m_{k_i})\|^{\rho} < C\|Q(m_{k_i})\|^{\rho} \leq \|Z(k_i + 1)\| < d + |\mathcal{A}_i| \cdot (h_i + 1)$$

$$|T(m_{k_i})(x) - x| = \sum_{\beta \in \mathcal{A}_i} \lambda_{\beta}(m_{k_i}) < \frac{\lambda_{\alpha_i}(m_{k_i})}{h_i} = r_i, \;\; x \in I_{\alpha_i}(m_{k_i})$$

$$\frac{\log \tau_{r_i}(x)}{-\log r_i} < \frac{\log \|Q_{\alpha_i}(m_{k_i})\|}{\log h_i - \log \lambda_{\alpha_i}(m_{k_i})} < \frac{\log \|Q_{\alpha_i}(m_{k_i})\|}{(1+\rho)\log \|Q_{\alpha_i}(m_{k_i})\| + \tilde{C}}.$$

Example with (R) without (Z)

Permutation data $\begin{pmatrix} A & B & D & C \\ D & A & C & B \end{pmatrix}$ and sequence of names of

$$C^{m_1} B \left(D^2 A^3 D\right)^{n_1} B \cdot C^{m_2} B \left(D^2 A^3 D\right)^{n_2} B \cdots$$

Let $\ell_k = \sum_{i=1}^{k} (m_i + 6n_i + 2), \quad \ell_0 = 0.$

$$Q(\ell_{k-1}, \ell_k) = \begin{pmatrix} F_{2n_k+1} & F_{2n_k+1} - 1 & F_{2n_k+1} - 1 & F_{2n_k+2} - 1 \\ 0 & 1 & 1 & 1 \\ 0 & m_k & m_k + 1 & m_k \\ F_{2n_k} & F_{2n_k} & F_{2n_k} & F_{2n_k+1} \end{pmatrix},$$

where $F_n = \frac{g^n - (-1/g)^n}{\sqrt{5}}$, $g = \frac{\sqrt{5}+1}{2}$ is the Fibonacci sequence.

Choose $m_k = F(2^{k+1})$ and $n_k = 2^k + k$. Then

$$\|Q(\ell_{k-1}, \ell_k)\| < g^{2^{k+1}+2k+4}, \qquad \|Q(\ell_k)\| < g^{2^{k+2}+k(k+1)+4k}$$

For large k

$$\|Q(\ell_k)\|^{1/2} < g^{2^{k+1}+k(k+1)/2+2k} \leq \frac{g^{2^{k+2}}}{\sqrt{5}} < \|Q(\ell_k, \ell_k + m_{k+1})\|$$

which implies that this i.e.m. does not satisfy Condition (Z).

We have

$$\lambda_A \gtrsim \lambda_D >> \lambda_C >> \lambda_B.$$

Only to show

$$\liminf_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \geq 1, \text{ a.e. } x.$$

- On $I_A(\ell_k)$ and $I_D(\ell_k)$, $T(\ell_k)$ is very close to the rotation by the golden mean. There is no "quick" return.
- On $I_C(\ell_k)$, $\tau_r(x)$ can be very small, but

$$\sum_k \mu\left(\bigcup_{i=0}^{Q_C(\ell_k)} T^i(I_C(\ell_k))\right) < \sum_k \frac{1}{g^k} < \infty$$

a.e. x belongs to $\bigcup_{i=0}^{Q_C(\ell_k)} T^i(I_C(\ell_k))$ finitely many k's.

Example with (Z) without (U)

Permutation data $\begin{pmatrix} A & B & D & C \\ D & A & C & B \end{pmatrix}$ and sequence of names

$$CB^3\left(D^2A^3D\right)^{2^1}B\cdot CB^3\left(D^2A^3D\right)^{2^2}B\cdots CB^3\left(D^2A^3D\right)^{2^k}B\cdots.$$

Let $\ell_k = \sum_{i=1}^{k}\left(5+6\cdot 2^i\right) = 5k+12\cdot(2^k-1), \quad \ell_0 = 0.$

$$Q(\ell_{k-1},\ell_k) = \begin{pmatrix} F_{2^{k+1}+1} & F_{2^{k+1}+1}-1 & F_{2^{k+1}+1}-1 & F_{2^{k+1}+2}-1 \\ F_{2^{k+1}} & F_{2^{k+1}}+1 & F_{2^{k+1}}+2 & F_{2^{k+1}+1}+1 \\ F_{2^{k+1}} & F_{2^{k+1}}+1 & F_{2^{k+1}}+3 & F_{2^{k+1}+1}+1 \\ F_{2^{k+1}} & F_{2^{k+1}} & F_{2^{k+1}} & F_{2^{k+1}+1} \end{pmatrix},$$

$$\|Q(\ell_{k-1},\ell_k)\| < g^{2^{k+1}+5}, \quad \lambda^*(\ell_k) < \frac{\lambda^*(\ell_{k-1})}{F_{2^{k+1}+3}}.$$

$T(\ell_k+3)$ has the same permutation data: $\begin{pmatrix} A & B & D & C \\ D & A & C & B \end{pmatrix}$.

$$Q(\ell_k+3,\ell_{k+1}) = \begin{pmatrix} F_{2^{k+2}+1} & F_{2^{k+2}+1}-1 & F_{2^{k+2}+1}-1 & F_{2^{k+2}+2}-1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ F_{2^{k+2}} & F_{2^{k+2}} & F_{2^{k+2}} & F_{2^{k+2}+1} \end{pmatrix}.$$

Put $r = \lambda_B(\ell_k+3)$. Then if $k \geq 4$, for $x \in I_C(\ell_k+3)$

$$\frac{\log \tau_r(x)}{-\log r} \leq \frac{\log Q_C(\ell_k+3)}{-\log \lambda_B(\ell_k+3)} < \frac{(2^{k+2}+5k)\log g + \log 2}{(2^{k+3}+k-4)\log g} < \frac{3}{4}.$$

Hence, $\dfrac{\log \tau_r(x)}{-\log r}$ does not converges to 1 uniformly.