On the real baker map

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Kneading dough

1. Rolling out a box $(a, b, c) \longrightarrow$ box with sides $\alpha a, \beta b, \frac{1}{\alpha\beta}c$ where $\alpha \geq 1$ and $\beta \geq 1$ are *coefficients of rolling*

2. Folding, or stacking $\longrightarrow$ box with dimensions $\alpha a, \frac{1}{2}\beta b, \frac{2}{\alpha\beta}c$

3. Flipping, or rotating $\longrightarrow$ box with dimensions $a' = \frac{1}{2}\beta b, b' = \alpha a, c' = \frac{2}{\alpha\beta}c$

4 kneading procedures

"roll, fold and flip" $K_{ff} = O \circ F \circ W$
"roll, stack and flip" $K_{sf} = O \circ S \circ W$
"roll, fold and rotate" $K_{fr} = R \circ F \circ W$,
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After two cycles of kneading with coefficients of rolling α_1, β_1 and $\alpha_2, \beta_2 \longrightarrow$ box with dimensions

$$\frac{\alpha_1\beta_2}{2}a, \quad \frac{\beta_1\alpha_2}{2}b, \quad \frac{4}{\alpha_1\alpha_2\beta_1\beta_2}c$$

Do we have anything to say to a sloppy baker who rolls the dough inconsistently, i.e., with varying rolling coefficients?

Relative positions $(x, y, z), \; (x', y', z') \in C := [0,1]^3$

$$u = ax, v = by, w = cz, \; K(u, v, w) = (a'x', b'y', c'z')$$

$$K_{ff}(ax, by, cz) = \begin{cases} (a'2y, b'x, c'(1 - \frac{1}{2}z)) & \text{if } 0 \le y \le \frac{1}{2} \\ (a'2(1-y), b'x, c'\frac{1}{2}z) & \text{if } \frac{1}{2} < y \le 1 \end{cases}$$

$$(x', y', z') = \begin{cases} (2y, x, 1 - \frac{1}{2}z) & \text{if } 0 \le y \le \frac{1}{2} \\ (2(1-y), x, \frac{1}{2}z) & \text{if } \frac{1}{2} < y \le 1 \end{cases}$$

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Rolling does not matter?

Only folding does?

$2N$ cycles $\longrightarrow$ box with dimensions
$$a' = a\frac{\alpha_1\beta_2\alpha_3\beta_4\ldots\alpha_{2N-1}\beta_{2N}}{2^N},\quad b' = b\frac{\beta_1\alpha_2\beta_3\alpha_4\ldots\beta_{2N-1}\alpha_{2N}}{2^N},\quad c' = c\frac{ab}{a'b'}.$$

product of coefficients of rolling must average out to 2^N,

or else the dough would aquire an odd shape.

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Kneading map(s)

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$$K_{ff}(x,y,z) = \left(2(1-y), x, \frac{1}{2}z\right), \qquad \text{if } \frac{1}{2} < y \le 1,$$

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geometric coding map $h : \Sigma \to C = [0,1]^3$

$x_i, y_i, z_i, i = 1, 2, \ldots,$ binary digits of coordinates $(x, y, z) \in C = [0,1]^3$

$$h(\ldots z_3, z_2, z_1 \,.\, y_1, x_1, y_2, x_2, \ldots) = (x, y, z)$$

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this dependence is permutation of all 2^{l+k} blocks

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Quantifying mixing

Definition

For a map $K : C \to C$, and a finite σ-algebra $\mathcal{B}$ of subsets of C, we say that a family $\mathcal{A}$ of subsets of C has *the decay rate* $r_0 < 1$*, with the resolution* $\mathcal{B}$, if for every $r > r_0$ and every $A \in \mathcal{A}$ there is a constant $d > 0$ such that

$$|\mu(K^n(A) \cap B) - \mu(A)\mu(B)| \le d\, r^n \quad \text{for every} \quad B \in \mathcal{B}, n = 1, 2, \ldots.$$

If $r_0 = 0$ then we say that the decay rate is *super-exponential.*

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for natural N, partition Π_N of C into 2^{3N} cubes, with sides equal to $\frac{1}{2^N}$

partition $\widehat{\Pi}_N$ of Σ is partition into cylinders of length $3N$

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$\mathcal{B}(\Pi_N), \mathcal{B}(\widehat{\Pi}_N)$,
σ-algebra of subsets generated by respective partition
for any $A \in \mathcal{B}(\Pi_M), B \in \mathcal{B}(\Pi_N)$

$$\mu(K^n(A) \cap B) = \mu(A)\mu(B) \quad \text{for} \quad n \geq 2M + N$$

Theorem

For the kneading map K and any natural N, M, the family $\mathcal{B}(\Pi_M)$ has the super-exponential decay rate with the resolution $\mathcal{B}(\Pi_N)$.

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wider family of subsets with common decay rate?

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A subset $A \subset C$ has *finite surface area* if there is a constant $s > 0$ such that the Lebesgue measure of the ϵ-neighborhood of its boundary ∂A does not exceed $s\,\epsilon$, for any $\epsilon > 0$.

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$A_2 \setminus A_1$ equal to union of all elements of partition Π_M with nonempty intersection with the boundary ∂A.

A has finite surface area, cubes in Π_N have diameter $2^{-M}\sqrt{3}$

$\Longrightarrow$ there is $s > 0$ such that

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Lipschitz functions (observables) $f, g : C \to \mathbb{R}$

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