Algebraic difference of random Cantor sets

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Outline

Motivation

- Algebraic difference of sets

Almost self-similar sets

- Larsson's family
- Self-similar sets with random translations

Difference of Mandelbrot percolation (with unequal probabilities)

- existence of an interval in the difference set
- The Lebesgue measure of the difference set
- Palis Conjecture does not hold in this case

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Introduction

$F_1, F_2 \subset \mathbb{R}$. The algebraic difference set

$$F_2 - F_1 := \{f_2 - f_1 : f_1 \in F_1,\ f_2 \in F_2\}.$$

Motivation to study it comes from e.g. :

- Dynamical systems, unfolding of homoclinic tangency (Palis, Takens)
- Diophantine approximation (Moreira, Yoccoz).

Palis conjectured: For dynamically defined Cantor sets: "Generically" Either

- $F_2 - F_1$ is small: $\mathcal{L}eb(F_2 - F_1) = 0$ or
- $F_2 - F_1$ is big: $F_2 - F_1$ contains some intervals.

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Summary

Palis conjecture holds:

- For self-similar sets with random translations.
- For Mandelbrot percolation.

Palis conjecture Does NOT hold:
For more general Mandelbrot percolation when we select the intervals with different probability .

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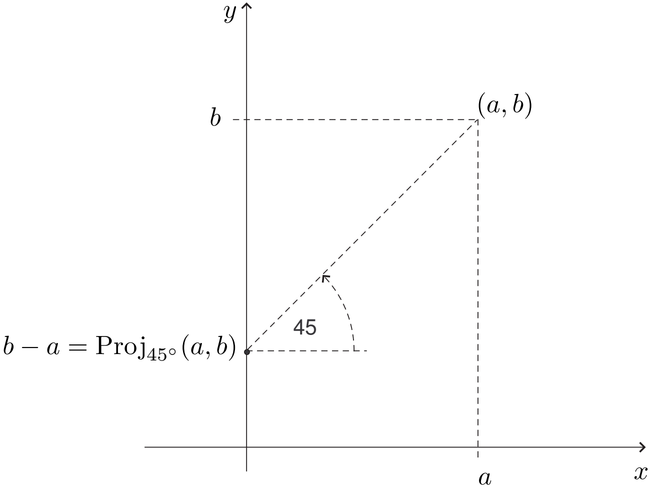

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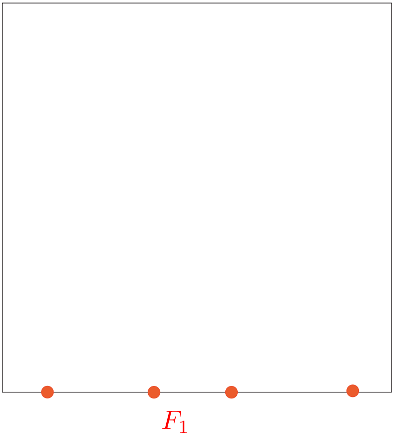

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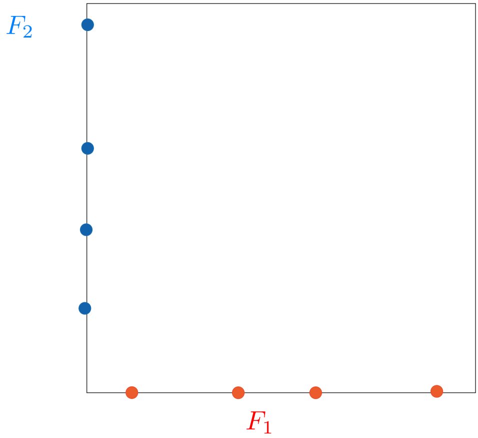

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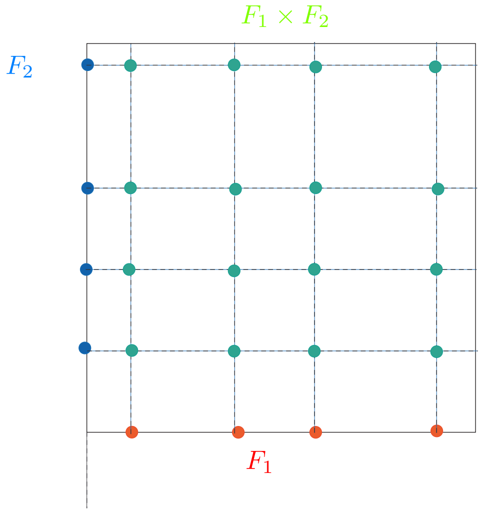

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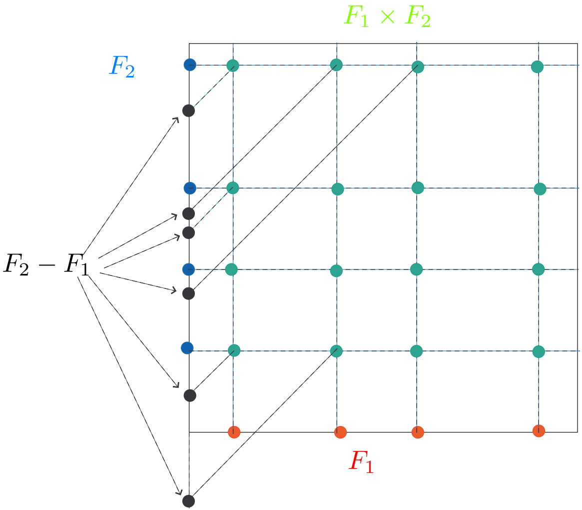

The algebraic difference from geometric point of view II

Definition

Let $A \subset \mathbb{R}^2$. We define $\text{Proj}_{45^\circ}(A)$ as the projection of A to the y axis along lines having a 45° angle with the x axis.

Then

$$F_2 - F_1 = \text{Proj}_{45^\circ}(F_1 \times F_2). \qquad (1)$$

So,

$$\dim_{\rm H} F_1, \dim_{\rm H} F_2 < \frac{1}{2} \Longrightarrow \dim_{\rm H}(F_2 - F_1) < 1$$

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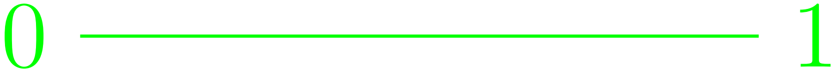

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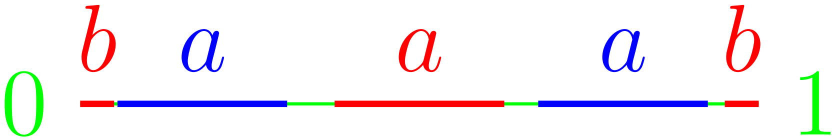

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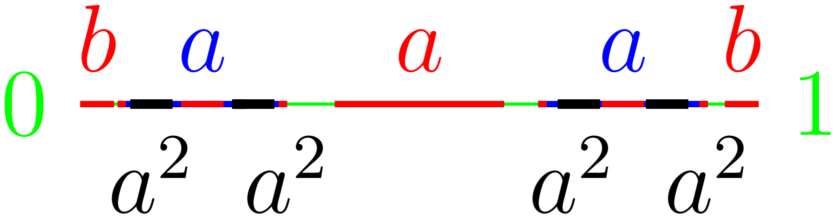

Larsson's random Cantor set is what we have after infinitely many steps. Larsson (a student of Karleson) in 1991 stated:

Theorem

Let C_1, C_2 be two independent realizations of the Larsssson's Cantor set. Then $C_1 - C_2$ contains interval almost surely.

The proof contained many interesting ideas and but was incorrect. The correct proof was given in

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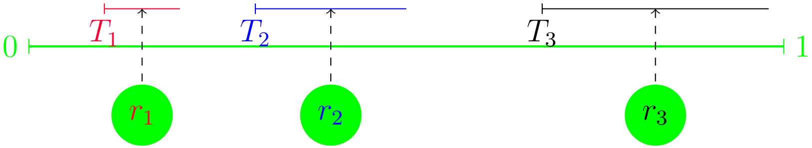

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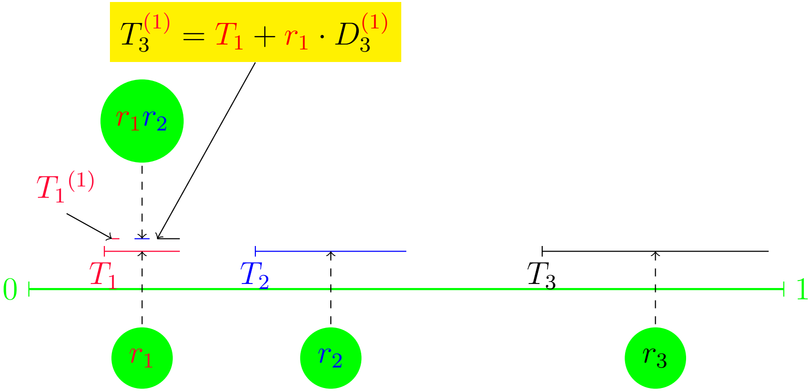

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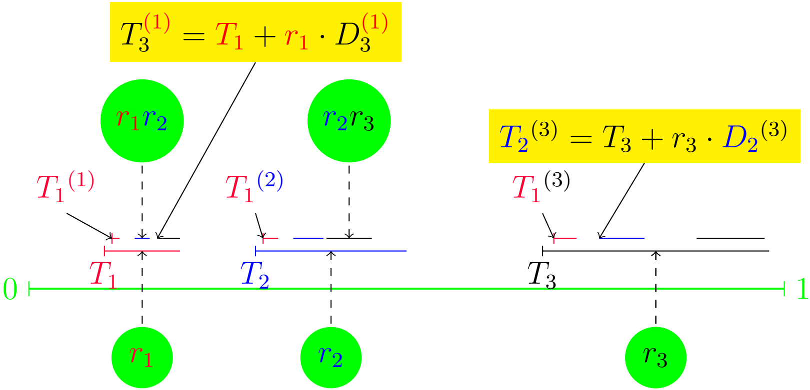

Contraction ratios $r_1, \dots r_m$ are fixed. Left endpoints $T_1, \dots, T_m$ are absolute continuous r.v. so that the random intervals $I_i := T_i + r_i$ are disjoint. $T_k^{(\ell)} = T_\ell + r_k \cdot D_k^{(\ell)}$, where $\left\{ D_k^{(\ell)} \right\}_{k=1}^m$ has the same distribution as $\{T_k\}_{k=1}^m$ and **independent of EVERY THING**. So we get

$$I_k^{(\ell)} = T_k^{(\ell)} + r_k \cdot r_\ell.$$

Similarly, we construct $I_k^{\mathbf{i}}$ for every $\mathbf{i} \in \{1, \dots m\}^n$ and $k = 1, \dots m$. The attractor Λ of the random IFS is

$$\Lambda = \bigcap_{n=0}^{\infty} \bigcup_{|\mathbf{i}|=n,k} I_k^{\mathbf{i}}$$

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Palis Conjecture holds in this case

Theorem (Dekking, S., Székely)

Let C_1, C_2 be two independent realizations. Let s be the similarity dimension of the system:

$$r_1^s + \cdots + r_m^s = 1$$

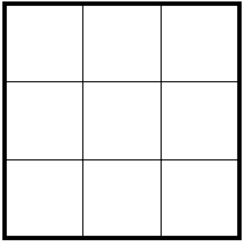
Assume that $s > 1$. Then

$C_2 - C_1$ *contains some intervals* .

(If $s < 1$ then $\dim_{\mathrm{H}}(C_2 - C_1) < 1$)

Mandelbrot percolation, introduced by Mandelbrot early 1970's:

We partition the unit square into M^2 congruent sub squares each of them are independently retained with probability p and discarded with probability $1-p$. We repeat the same process into those of these squares which were kept after the previous step. Then repeat this at infinitum.



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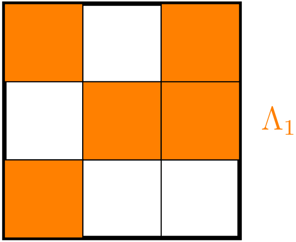

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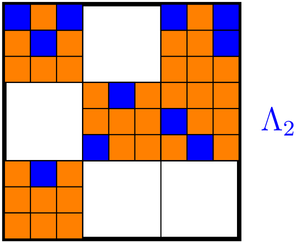

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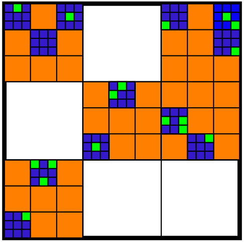
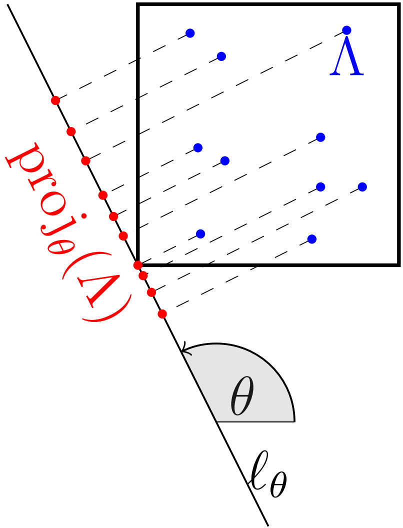
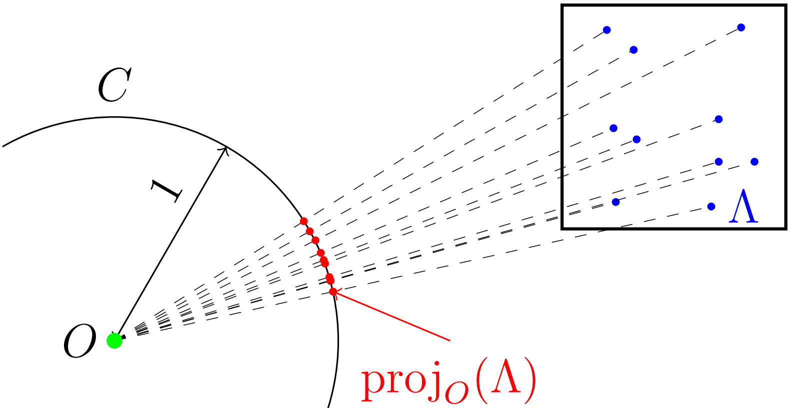


Orthogonal projection to ℓ_θ



Radial projection with center O



Theorem [M. Rams, S.]

We assume that $p > \frac{1}{M}$ (for having $\dim_{\mathrm{H}} \Lambda > 1$)

Then the following statements hold for almost all realization Λ of the Mandelbrot percolation conditioned on $\Lambda \neq \emptyset$:

$$\forall \theta \in [0, \pi], \ \mathrm{proj}_\theta(\Lambda) \text{ containes an interval }.$$

Further,

$$\forall O \in \mathbb{R}^2, \ \mathrm{proj}_O(\Lambda) \text{ containes an interval }.$$

That is: if $p > 1/M$ then all orthogonal projections and all radial projections contain some intervals almost surely, conditioned on $\Lambda \neq \emptyset$.

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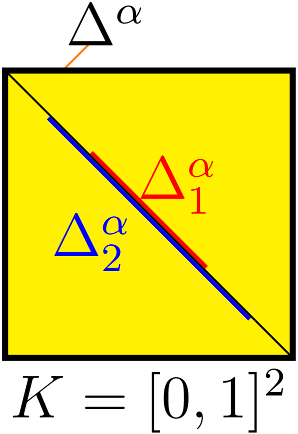

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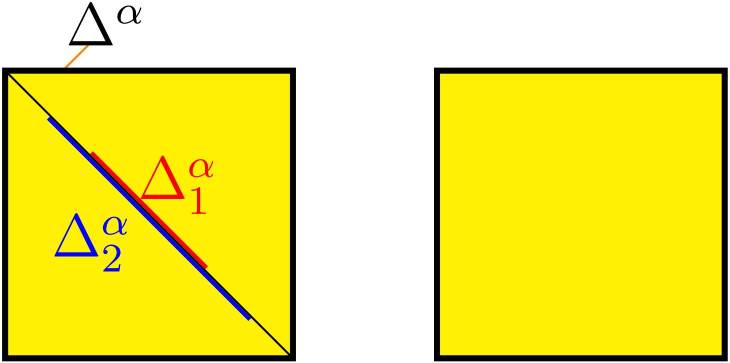

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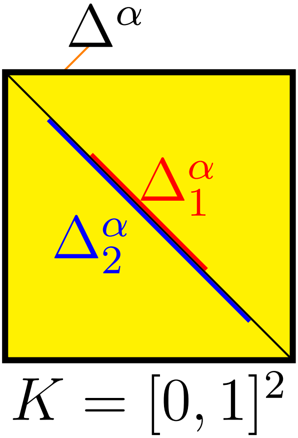

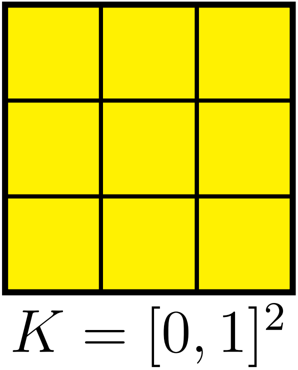

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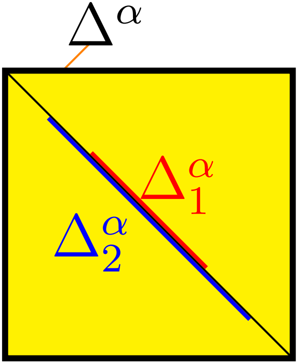

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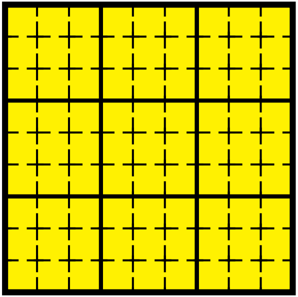

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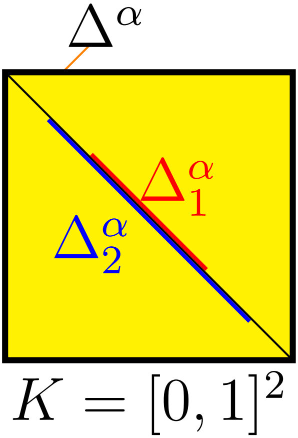

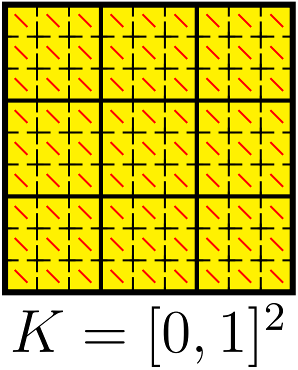

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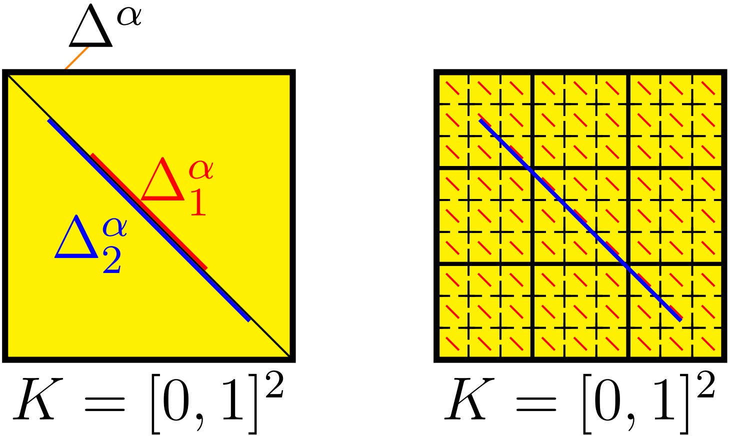

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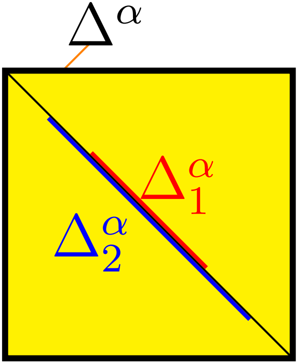

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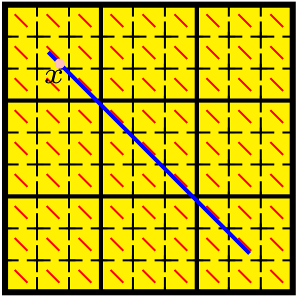

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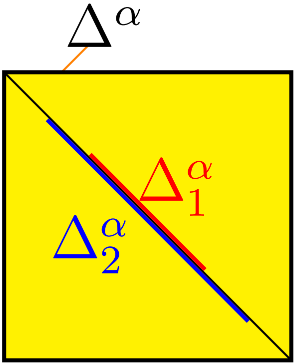

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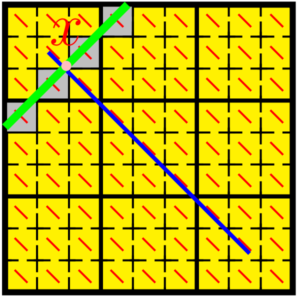

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Definition of $r = r(\alpha)$

Lemma (Very Important Lemma)

$$\forall \alpha \neq 0, 90, \quad \exists\, r, \Delta_1^\alpha, \Delta_2^\alpha$$

(all depends on α) such that

for every $x \in \Delta_2^\alpha$

$$\mathbb{E}\left[\#\left\{\text{ level } r, \Delta_1^\alpha \text{ whose } \alpha\text{projection covers } x\right\}\right] > 2.$$

Definition of r.v. $Y_1, \dots, Y_R$

Let r be as above. Let $\mathrm{left}_{\underline{i}_r,\underline{j}_r}, \mathrm{right}_{\underline{i}_r,\underline{j}_r}$ be the left and right end points of the red sub diagonal $\Delta^1_{\underline{i}_r,\underline{j}_r}$.

$$\left\{ \mathrm{left}_{\underline{i}_r,\underline{j}_r}, \mathrm{right}_{\underline{i}_r,\underline{j}_r} \right\}_{\underline{i}_r,\underline{j}_r}$$

gives a partition of Δ^2 into R intervals $I_1, \dots, I_R \subset \Delta^2$ (left closed right open say). Pick an arbitrary $x_k \in I_k$, $k = 1, \dots, R$. The r.v. Y_k is defined as the random number of those level r squares call them $K_{\underline{i}_r,\underline{j}_r}$ which are

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- for which $x_k \in \Pi_\alpha(\Delta^1_{\underline{i}_r,\underline{j}_r})$.

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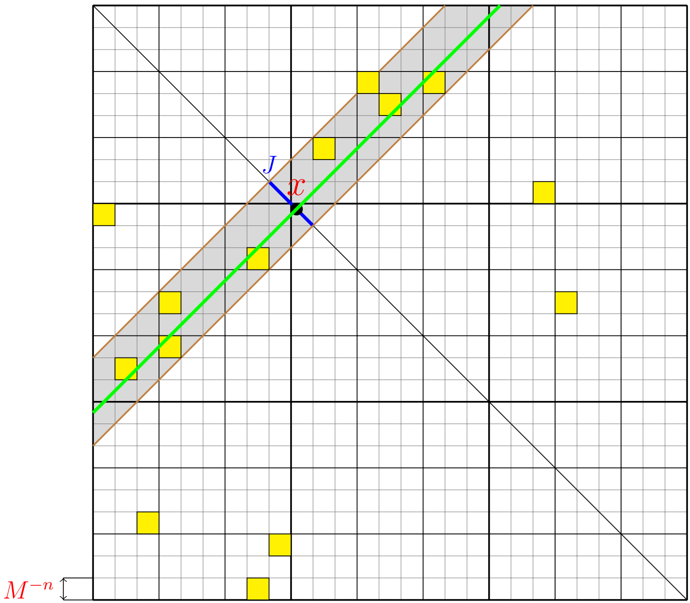

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By assumption

$$\dim_{\mathrm{H}} > 1.$$

So, the number of selected M^{-n} squares is $M^{q \cdot n}$, $q > 1$. So, $\exists J$ s.t. for $\forall x \in J$ the the green line intersects exponentially (say v^n) many selected (yellow) squares.

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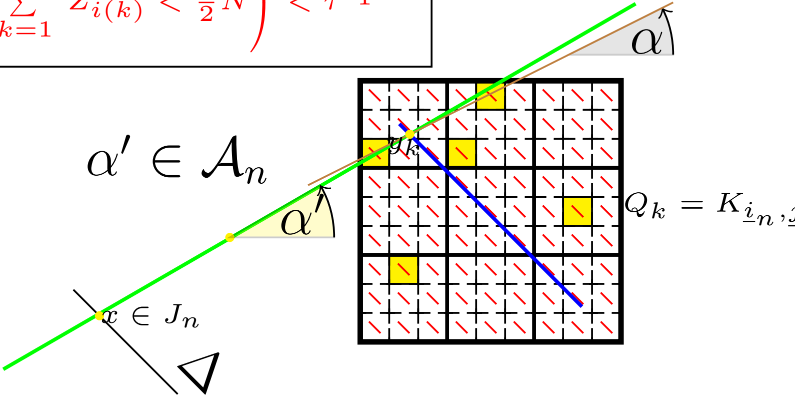

Outline

Motivation
- Algebraic difference of sets

Almost self-similar sets
- Larsson's family
- Self-similar sets with random translations

Difference of Mandelbrot percolation (with unequal probabilities)
- existence of an interval in the difference set
- The Lebesgue measure of the difference set
- Palis Conjecture does not hold in this case

The outline of the construction I

Given an integer $M \geq 2$ and a vector of probabilities

$$(p_o, p_1, \ldots, p_{M-1}) \in [0,1]^M.$$

Which is in general NOT a probability vector.
We divide the unit interval $I = [0,1]$ into the M subintervals $I_k = \left[\frac{k-1}{M}, \frac{k}{M}\right]$, $k = 0, \ldots, M-1$. We keep I_k with probability p_k. For all intervals kept, repeat this algorithm infinitely many times in each step independently from each other and from the past. Whatever remains it is our random Cantor set F.

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Let $Z_0 := 1$ and let Z_n be the (random) number of level n intervals selected.

Then

- $\{Z_n\}_{n \in \mathbb{N}}$ is a branching process.
- The expected value of Z_1:

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The next fact is well known. Falconer (1986) and Mauldin, Williams (1986)

Theorem

Assuming that $F \neq \emptyset$ we have:

$$\dim_{\mathrm{B}} F = \dim_{\mathrm{H}} F = \frac{\log \left(\sum_{i=0}^{M-1} p_i \right)}{\log M}. \qquad (2)$$

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$$\dim_{\mathrm{H}} F < \frac{1}{2} \Longrightarrow F_2 - F_1$$

does not contain any interval. Using (2):

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For $i \in \{0, \ldots, M-1\}$ let

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where p_i was the probability that we choose the interval i-th interval $I_i = \left[\frac{i-1}{M}, \frac{i}{M}\right]$.

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In the case of the Mandelbrot percolation all $p_i = p$ for some $0 \leq p \leq 1$. In this case for all i,

$$\gamma_i = Mp^2 > 1 \Longleftrightarrow p > \frac{1}{\sqrt{M}}.$$

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The case of $M = 2$

M. Dekking, B. Kuijvenhoven gave the following full characterization:

$$\text{If } M = 2 \text{ then } \gamma_0 = p_0^2 + p_1^2, \ \gamma_1 = 2p_0p_1$$

If $\gamma_0\gamma_1 > 1$ $F_2 - F_1$ contains some interval conditioned on $\{F_2 - F_1 \neq \emptyset\}$.

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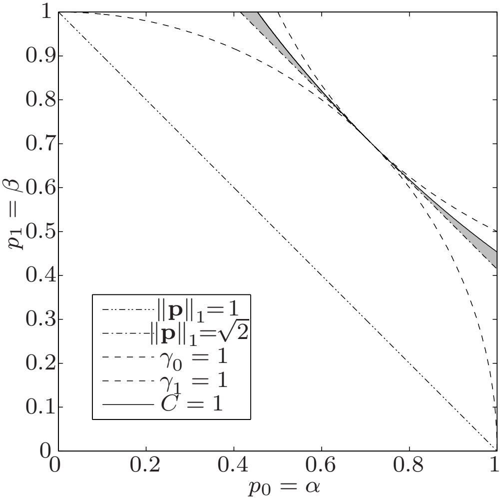

Gray region: no intervals in $F_1 - F_2$ but $\dim_{\mathrm{H}} F_1 + \dim_{\mathrm{H}} F_2 > 1$.

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$$\gamma_0 \geq \gamma_1 = \gamma_2.$$

That is the following holds almost surely:

$$\gamma_1 > 1 \implies \text{there is interval in } F_2 - F_1$$

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So, for $M = 3$ our theorem is essentially complete.

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We remind: $\gamma_i := \sum_{k=0}^{M-1} p_k p_{k+i \bmod M}$

Theorem (Mora, S., Solomyak)

We assume that $p_0, \dots, p_{M-1} > 0$ Moreover, we require that

$$\Gamma := \gamma_0 \cdots \gamma_{M-1} > 1. \qquad \text{(A2)}$$

Then conditional on $F_1, F_2 \neq \emptyset$, we have

$$\mathcal{L}eb(F_2 - F_1) > 0. \qquad (4)$$

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$$\mathcal{L}eb(F_2 - F_1) > 0. \qquad (4)$$

holds almost surely.

The Lebesgue measure of $F_2 - F_1$

We remind: $\gamma_i := \sum_{k=0}^{M-1} p_k p_{k+i \bmod M}$

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By formulae (2) which was

$$\dim_{\mathrm{B}} F = \dim_{\mathrm{H}} F = \frac{\log\left(\sum_{i=0}^{M-1} p_i\right)}{\log M}.$$

$F_2 - F_1 = \mathrm{proj}_{45^\circ}(F_1 \times F_2)$ **MAY** contain an interval only if

$$\frac{\sum_{i=0}^{M-1} p_i}{\sqrt{M}} > 1.$$

Easy calculation shows that

$$\frac{\sum_{i=0}^{M-1} \gamma_i}{M} = \left(\frac{\sum_{i=0}^{M-1} p_i}{\sqrt{M}}\right)^2$$

Positive Lebesgue measure with no intervals

Let $M = 3$ and

$$(p_0, p_1, p_2) = (0.52, 0.5, 0.72).$$

In this case we have

$$\gamma_0 = p_0^2 + p_1^2 + p_2^2 = 1.0388,$$

$$\gamma_1 = \gamma_2 = p_0 p_1 + p_1 p_2 + p_2 p_0 = 0.9944,$$

So, there is no interval.

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