Recurrence of the Ehrenfest Wind-tree model

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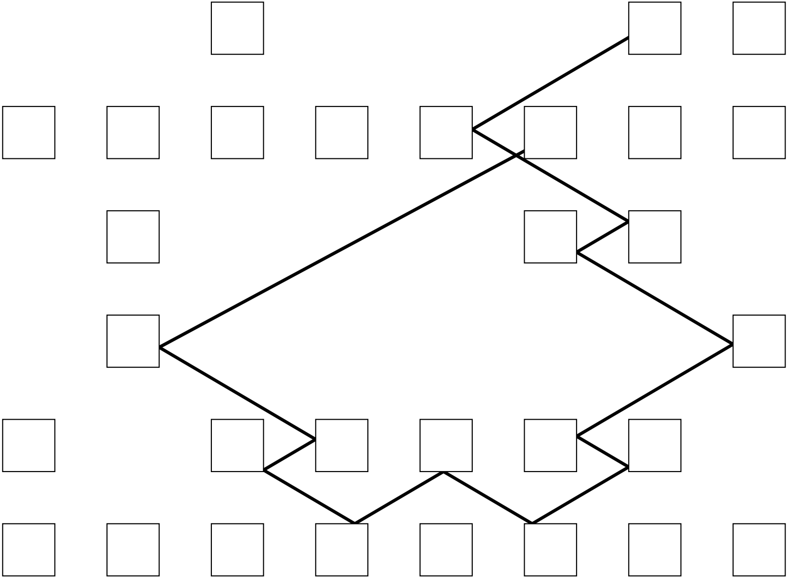
In the Ehrenfest wind-tree model, a point particle (the "wind") moves freely on the plane and collides with the usual law of geometric optics (angle of incidence = angle of reflection) with randomly placed parallel rectangular scatterers (the "trees").



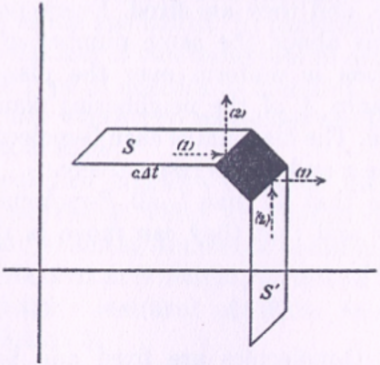


Figure 1

The only illustration in the Ehrenfests' article.

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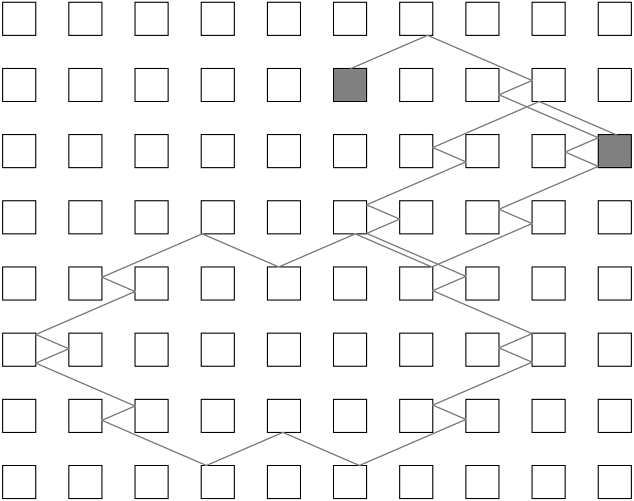
The phase volume is an infinite invariant measure.

One would like to know if it is almost surely recurrent for a certain direction, for almost every direction.

Full occupancy lattice case

Consider the $\mathbb{Z}^2$ lattice with a fixed a by b rectangle centered at each lattice point $(i, j) \in \mathbb{Z}^2$.



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For these special choices of parameters the application T^4, the fourth return to the boundary of rectangles, is a skew product over a rotation by angle α, i.e.
$T^4(s,(i,j)) = (s+\alpha,(i,j)+\phi(s))$ where s is arc length on the boundary of the rectangle.

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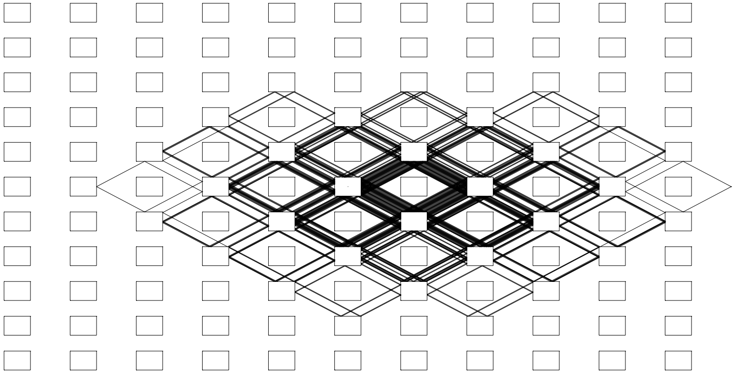
They also estimated the diffusion rate, the particle escapes at a speed of at most $\log t \log\log t$

For each (a, b) with a/b irrational this method works for a finite collection of directions θ.

For these special directions one can identify the ergodic decomposition (Conze and Gutkin 2010). For example, in the direction $\pi/4$ with a/b irrational, there are 2 ergodic components. In the first component the generic orbit hits all obstacles with center (i, j) with $i + j = 0 \mod 2$, while in the second $i + j = 1 \mod 2$.



Almost all direction results on the full occupancy case
Hubert, Lelièvre, T.

Define a dense subset of parameter values:

$$\mathcal{E} = \{ (a, b) = (p/q, r/s) \in \mathbb{Q} \times \mathbb{Q} : \\ (p, q) = (r, s) = 1, \quad 0 < p < q, \quad 0 < r < s, \\ p, r \text{ odd}, \quad q, s \text{ even} \}.$$

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Theorem (HLT)

If the rectangular obstacles have dimension $(a,b) \in \mathcal{E}$, then, for the billiard table $T_{a,b}$:

- *there is a subset P of $\mathbb{Q}$, dense in $\mathbb{R}$, such that every regular trajectory starting with direction in P is periodic;*
- *for almost every direction, the billiard flow is recurrent with respect to the natural phase volume.*

Let

$$\mathcal{E}' = \{ (a,b) = (p/q, r/s) \in \mathbb{Q} \times \mathbb{Q} : \\ (p,q) = (r,s) = 1, \quad 0 < p < q, \quad 0 < r < s, \\ p, r \text{ even}, \quad q, s \text{ odd} \},$$

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Theorem (HLT)

If the rectangular obstacles have dimension $(a,b) \in \mathcal{E}'$, then, for the billiard table $T_{a,b}$:

- *there is no direction $\alpha \in \mathbb{Q}$ such that all regular trajectories starting with direction α are periodic;*
- *there is a subset P of $\mathbb{Q}$, dense in $\mathbb{R}$, such that no trajectory starting with direction in P is periodic;*
- $\forall\, k \geq 1$, *for a.e.* θ, $\limsup_{t\to\infty} \dfrac{\mathrm{dist}(\phi_t^\theta x, x)}{\prod_{j=1}^k \log_j t} = \infty$ *almost surely.*

Generic full occupancy results

As a corollary, we obtain a result for a dense G_δ of parameters in $(0,1)^2$.

Corollary (HLT)

Consider any closed $\mathcal{Y} \subset (0,1)^2$ for which $\mathcal{E} \cap \mathcal{Y}$ and $\mathcal{E}' \cap \mathcal{Y}$ are dense in $\mathcal{Y}$. Then there is a residual set $G \subset \mathcal{Y}$ such that, for each $(a,b) \in G$,

- *the billiard flow on $T_{a,b}$ is recurrent;*
- *the set of periodic points is dense in the phase space of $T_{a,b}$;*
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Recently Avila and Hubert have improved the recurrence part of this result to all (a,b).

Theorem

(Delecroix) If a and b are rational or quadratic of the form $1/(1-a) = x + y\sqrt{D}$ *and* $1/(1-b) = (1-x) + y\sqrt{D}$ *with* x,y *rationals and* $D > 1$ *a square free integer, then there exists a set* $\Lambda \subset \mathbb{S}^1$ *of Hausdorff dimension at least* $1/2$ *such that for every* $\theta \in \Lambda$ *and all* x *with infinite forward orbit in the direction* θ, $\liminf_{t\to\infty} d(x, \phi_t^\theta(x)) = \infty$, *i.e the flow* ϕ_t^θ *is divergent.*

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Theorem

(Delecroix,Hubert, Lelievre) For almost every (a, b) including the (a, b) described above

$$\limsup_{t\to\infty} \frac{\log d(x, \phi_t^\theta x)}{\log t} = \frac{2}{3}.$$

for almost all θ and all x with infinite forward orbit.

Non-periodic models

Fix a finite or countable set of dimensions of obstacles $\mathcal{F} \subset (0,1)^2 \cup \{(0,0)\}$ such that $\mathcal{F} \cap (\mathcal{E} \cup G) \neq \emptyset$. Consider the set configurations $\mathcal{F}^{\mathbb{Z}^2}$ with the product topology on $\mathcal{F}^{\mathbb{Z}^2}$. A lattice site with an obstacle of dimension $(0,0)$ will be interpreted as a lattice site without obstacle.

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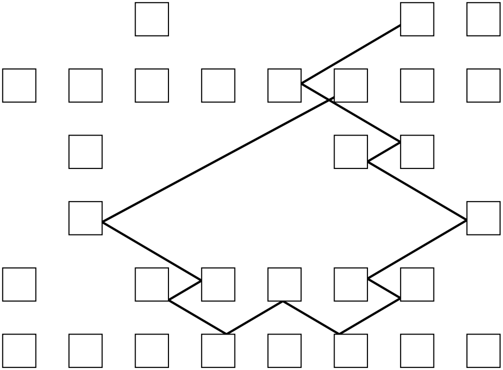

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For each ergodic shift invariant measure on the configuration space $\mathcal{F}^{\mathbb{Z}^2}$, the wind-tree models are almost surely recurrent or almost surely non recurrent.

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Theorem (T)

There is a dense G_δ subset G of configurations such that the billiard flow is recurrent for every billiard table in G with respect to the natural phase volume.

The proof also proves recurrence for a dense G_δ of Lorentz gases (without finite horizon).

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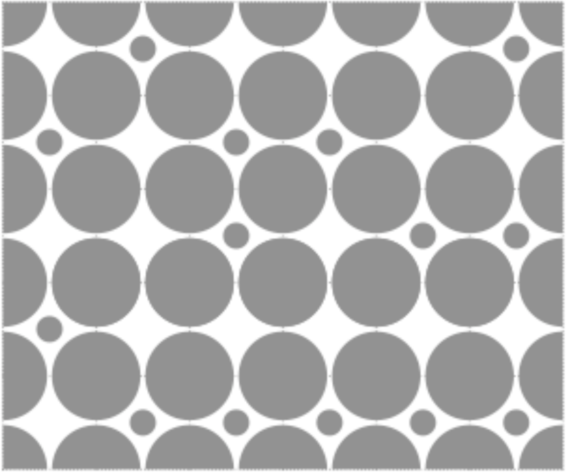
Erodicity (joint work with M. Lenci)

In the one dimensional case one can prove recurrence and ergodicity for almost every Lorentz tube (without finite horizon).

Ergodicity can also be shown for a dense set of aperiodic Lorentz gases (without finite horizon).

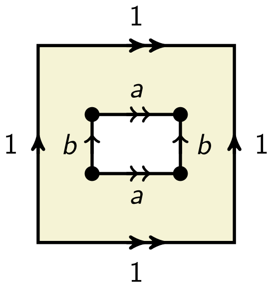


The ideas of the proof of Theorem 1.

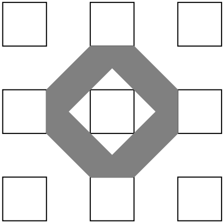
Theorem 1 (HLT) *If the rectangular obstacles have dimension* $(a, b) \in \mathcal{E}$*, then, for the billiard table* $T_{a,b}$*:*

- *there is a subset* P *of* $\mathbb{Q}$*, dense in* $\mathbb{R}$*, such that every regular trajectory starting with direction in* P *is periodic;*
- *for almost every direction, the billiard flow is recurrent with respect to the natural phase volume.*

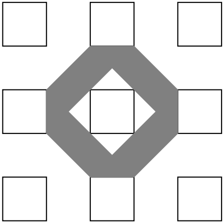
1) The dynamics can be understood by considering a compact genus 2 translation surface whose periodic orbit structure can be well understood.



Consider a direction θ which connects a center of a vertical side of an obstacle to a center of a horizontal side (or vice versa). A study of the symmetries imply that all orbits are periodic on the compact surface and on the infinite surface.



Obstacle size (1/2,1/2), slope 1.



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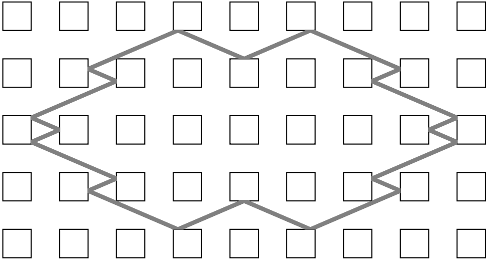
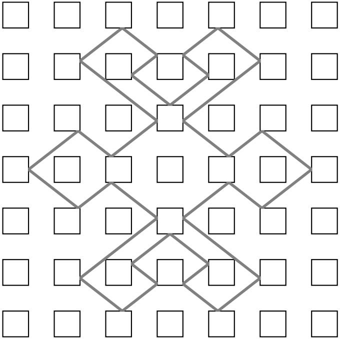
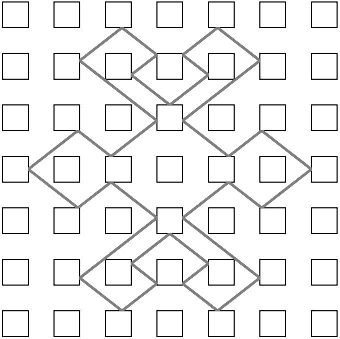


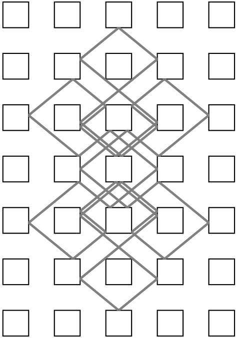
Slope 3/7



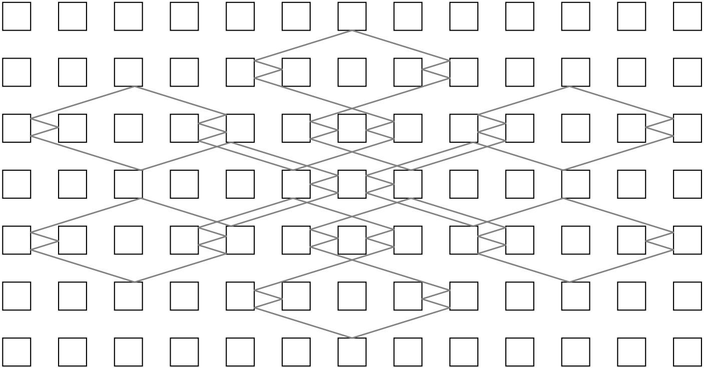
Slope 7/9



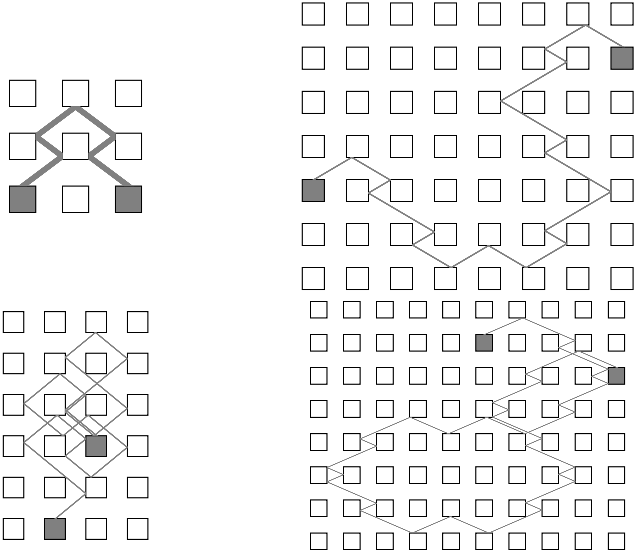
Slope 7/9



Slope 9/11



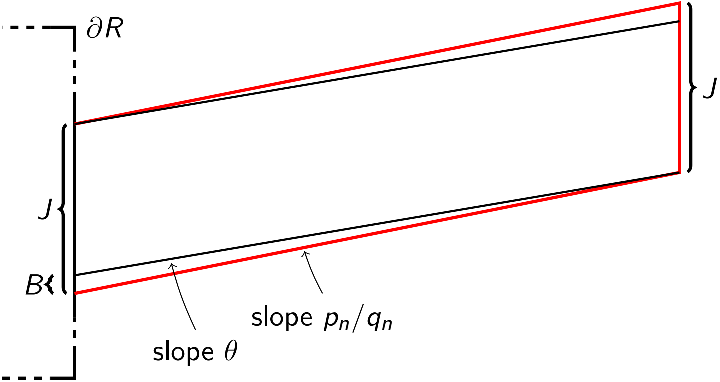
Slope 9/29



Escaping trajectories: slopes 3/4, 10/17, 14/17, 16/37, for obstacle size $1/2 \times 1/2$. The grayed obstacles are hit at the same location, so the trajectory repeats afterwards with a drift.

2) Using the action of the Veech group (group of affine diffeomorphisms of the surface) one proves that these directions are dense.

3) Finally using diophantine approximation we can prove that a.e. direction is recurrent.



Recurrence: trajectories starting in J with direction θ follow the cylinder with direction p_n/q_n unless they are in the small subinterval B. Thus those point in J not in B recur to J.

Ideas of the proof of Theorem 7.

Theorem 7 (T) *The is a dense G_δ subset G of configurations such that the billiard flow is recurrent for every billiard table in G with respect to the natural phase volume.*

The idea of the proof is simple. A table in G will have infinitely many large annuli for which the table has a fixeed obstacle at all lattice sites in the annuli. The widths of these annuli will increase sufficiently quickly to guarantee the recurrence.

P. Hubert, S. Lelièvre and S. Troubetzkoy *The Ehrenfest wind-tree model: periodic directions, recurrence, diffusion* Journal fuer die reine und angewandte Mathematik (Crelle's Journal)

S. Troubetzkoy *Typical recurrence for the Ehrenfest wind-tree model Journal of Statistical Physics* 141 (2010) 60-67.

V. Delecroix *DIvergent trajectories in the periodic wind-tree model*

V. Delecroix, P. Hubert, and S. Lelièvre *Diffusion for the wind-tree model*

J.P. Conze and E. Gutkin *On recurrence and ergodicity for geodesic flows on noncompact periodic polygonal surfaces*

M. Lenci and S. Troubetzkoy *Infinite-horizon Lorentz tubes and gases: recurrence and ergodic properties* Physica D (2011).