Polynomial Decay for Nonmarkov Maps

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Let $X \subset R^m$ be a compact set with positive Lebesgue measure ν. We assume $\nu X = 1$. Let d be the metric induced from R^m.

The transfer (Perron-Frobenius) operator $\mathscr{P} = \mathscr{P}_\nu : L^1(X, \nu) \to L^1(X, \nu)$ is defined by

$$\int \psi \circ T \phi d\nu = \int \psi \mathscr{P} \phi d\nu$$

$\forall \phi \in L^1(X, \nu)$, $\psi \in L^\infty(X, \nu)$.

Let $\widehat{X} \subset X$ be a subset of X such that $\cup_{n \geq 0} T^n \widehat{X} = X$.
First return map of T with respect to $\widehat{X} \subset X$ by $\widehat{T}(x) = T^{\tau(x)}(x)$, where $\tau(x) = \min\{i \geq 1 : T^i x \in \widehat{X}\}$ is the return time. Then we let $\widehat{\mathscr{P}} = \widehat{\mathscr{P}}_\nu$ be the transfer operator of $\widehat{T}$.

Put

$$R_n f = 1_{\widehat{X}} \cdot \mathscr{P}^n(f 1_{\{\tau=n\}}) \quad \text{and} \quad T_n f = 1_{\widehat{X}} \cdot \mathscr{P}^n(f 1_{\widehat{X}}) \qquad (0.1)$$

for any function f on $\widehat{X}$. For any $z \in C$, denote $R(z) = \sum_{n=1}^{\infty} z^n R_n$.
It is clear that $\widehat{\mathscr{P}} = R(1) = \sum_{n=1}^{\infty} R_n$.

Consider $L^1(\widehat{X}, \nu)$ as a subspace $L^1(X, \nu)$ consisting of functions supported on $\widehat{X}$.

Let $|\cdot|_{\mathcal{B}}$ a seminorm for functions in $L^1(\widehat{X}, \nu)$. Consider the set $\mathcal{B} = \mathcal{B}(\widehat{X}) = \{f \in L^1(\widehat{X}, \nu) : |f|_{\mathcal{B}} \leq \infty\}$and define a complet norm on $\mathcal{B}$ by

$$\|f\|_{\mathcal{B}} = |f|_{\mathcal{B}} + \|f\|_1$$

for any $f \in \mathcal{B}$, where $\|f\|_1$ is the L^1 norm.

Assumption B. (a) (Compactness) *The inclusion* $\mathcal{B} \hookrightarrow L^1$ *is compact.*

(b) (Boundness) *The inclusion* $\mathcal{B} \hookrightarrow L^\infty$ *is bounded;* $\|f\|_\infty \leq C_b\|f\|_{\mathcal{B}}$.

(c) (Algebra) $\mathcal{B}$ *is an algebra and* $\|fg\|_{\mathcal{B}} \leq C_a\|f\|_{\mathcal{B}}\|g\|_{\mathcal{B}}$.

Theorem A. *Let* $X \subset R^m$ *be compact subset with* $\nu X = 1$ *and* $\widehat{X} \subset X$ *is a compact subset of* X *with* $\cup_{n\geq 0}T^n\widehat{X} = X$. *Let* $T : M \to M$ *be a map whose first return map with respect to* $\widehat{X}$ *is* $\widehat{T} = T^\tau$, *and* $\mathcal{B}$ *be a Banach space satisfying Assumption B(a) to (d). We assume the following.*

(i) (Lasota-Yorke inequality) *There are constants* $\eta \in (0, 1)$, $D > 0$ *such that for any* $f \in \mathcal{B}$,

$$|\widehat{\mathscr{P}}f|_{\mathcal{B}} \leq \eta|f|_{\mathcal{B}} + D\|f\|_1. \tag{0.2}$$

(ii) (Spectral radius) *There exist* $B > 0, D_1 > 0$ *and* $\eta_1 \in (0, 1)$ *such that*

$$\|R(z)^n f\|_{\mathcal{B}} \leq |z^n|\big(B\eta_1^n\|f\|_{\mathcal{B}} + D_1\|f\|_1\big). \tag{0.3}$$

(iii) (Mixing) *The measure* μ *given by* $\mu(f) = \nu(hf)$ *has only one ergodic component, where* h *is a fixed point of* $\widehat{\mathscr{P}}$.

(iv) (Aperiodicity) *The function* $e^{it\tau}$ *given by the return time is aperiodic, that is, the only solutions for* $e^{it\tau} = f/f \circ T$ *almost everywhere with a measurable function* $f : \widehat{X} \to S$ *are* f *constant almost everywhere.*

If for any $n \geq 1$, R_n *satisfies* $\sum_{k=n+1}^{\infty} \|R_k\|_{\mathcal{B}} < O(n^{-\beta})$ *for some* $\beta > 1$, *then there exists* $C > 0$ *such that for any function* $f \in \mathcal{B}$, $g \in L^{\infty}(X, \nu)$ *with* $\operatorname{supp} f, \operatorname{supp} g \subset \widehat{X}$,

$$\left|\operatorname{Cov}(f, g \circ T^n) - \Big(\sum_{k=n+1}^{\infty} \mu(\tau > k)\Big) \int f d\mu \int g d\mu\right| \leq C F_\beta(n) \|g\|_\infty \|f\|_{\mathcal{B}}, \tag{0.4}$$

where $F_\beta(n) = 1/n^\beta$ *if* $\beta > 2$, $(\log n)/n^2$ *if* $\beta = 2$, *and* $1/n^{2\beta-2}$ *if* $2 > \beta > 1$.

Further assumptions on the Maps

We put conditions on the map $T : X \to X$ and its first return map $\widehat{T} : \widehat{X} \to \widehat{X}$. We denote by $B_\varepsilon(\Gamma)$ the ε neighborhood of a set $\Gamma \subset X$.

Assumption T. (a) (Piecewise smoothness) *There are countably many disjoint open sets* $U_1, U_2, \cdots$, *with* $\widehat{X} = \bigcup_{i=1}^{\infty} \overline{U_i}$ *such that for each* i, $\widehat{T}_i := \widehat{T}|_{U_i}$ *extends to a* $C^{1+\alpha}$ *diffeomorphism from* $\overline{U}_i$ *to its image, and* $\tau|_{U_i}$ *is constant.*

(b) (Finite images) $\{\widehat{T}U_i : i = 1, 2, \cdots\}$ *is finite, and* $\nu B_\varepsilon(\partial TU_i) = O(\varepsilon)$ $\forall i = 1, 2, \cdots$.

(c) (Expansion) *There exists* $s \in (0, 1)$ *such that* $d(\widehat{T}x, \widehat{T}y) \geq s^{-1}d(x, y)$ $\forall x, y \in \overline{U}_i$ $\forall i \geq 1$.

(d) (Topologically mixing) $T : X \to X$ *is topologically mixing.*

Also we put one more **assumption** on the Banach space $\mathcal{B}$.

A set U is said to be almost open $\bmod \nu$ or with respect to ν if for ν almost every point $x \in U$, there is an neighborhood $V(x)$ with $\nu(V(x) \setminus U) = 0$.

Assumption B. (d) (Denseness) *The image of the inclusion* $\mathcal{B} \hookrightarrow L^1(\widehat{X}, \mu)$ *is dense.*

(e) (Openess) *For any nonnegative function* $f \in \mathcal{B}$, *the set* $\{f > 0\}$ *is almost open with respect to* ν.

SKEW EXTENSIONS

Take a partition ξ of $\widehat{X}$. Consider the family of skew-products of the form

$$\widetilde{T_S} : \widehat{X} \times Y \to \widehat{X} \times Y \ , \ \widetilde{T}_S(x, y) = (\hat{T}x, S(\xi(x))(y)) \tag{0.5}$$

where $(Y, \mathcal{F}, \rho)$ is a Lebesgue probability space, $\mathrm{Aut}(Y)$ is the collection of its automorphisms, that is, invertible measure-preserving transformations, and $S : \xi \to \mathrm{Aut}(Y)$ is arbitrary.

For $\widetilde{f} \in L^1_{\nu\times\rho}$, define

$$|\widetilde{f}|_{\widetilde{\mathcal{B}}} = \int_Y |\widetilde{f}(\cdot, y)|_{\mathcal{B}} d\rho(y), \qquad \|\widetilde{f}\|_{\widetilde{\mathcal{B}}} = |\widetilde{f}|_{\widetilde{\mathcal{B}}} + \|\widetilde{f}\|_{L^1_{\nu\times\rho}}. \tag{0.6}$$

Then we let

$$\widetilde{\mathcal{B}} = \{\widetilde{f} \in L^1_{\nu\times\rho} : |\widetilde{f}|_{\widetilde{\mathcal{B}}} < \infty\}. \tag{0.7}$$

Theorem B. *Suppose $\widehat{T}$ satisfies Assumption T(a) to (d) and and $\mathcal{B}$ satisfies Assumption B(d) and (e), and $\mathscr{P}$ satisfies Lasota-Yorke inequality*

$$|(\widetilde{\mathscr{P}}\widetilde{f})|_{\widetilde{\mathcal{B}}} \le \widetilde{\eta}|\widetilde{f}|_{\widetilde{\mathcal{B}}} + \widetilde{D}\|\widetilde{f}\|_{L^1_{\nu\times\rho}} \tag{0.8}$$

for some $\widetilde{\eta} \in (0, 1)$ and $\widetilde{D} > 0$. Then any absolutely continuous invariant measure μ obtained from the Lasota-Yorke inequality (1.3) is ergodic and $e^{it\tau}$ is aperiodic. Therefore Conditon (iii) and (iv) in Theorem A follow.

1 Rates of Decay of Correlations

Theorem A is based on the results of Sarig. Notice that we do not assume the existence of an invariant measure (which will be by the way given by the Lasota-Yorke inequality).

Theorem. *Let T_n be bounded operators on a Banach space $\mathcal{B}$ such that $T(z) = I + \sum_{n\geq 1} z^n T_n$ converges in $\mathrm{Hom}(\mathcal{B}, \mathcal{B})$ for every $z \in D$, where D is the open unit disk. Assume that:*

(1) (Renewal equation) *for every $z \in D$, $T(z) = (I - R(z))^{-1}$, where $R(z) = \sum_{n\geq 1} z^n R_n$, $R_n \in \mathrm{Hom}(\mathcal{B}, \mathcal{B})$ and $\sum_{n\geq 1} \|R_n\| < +\infty$.*

(2) (Spectral gap) *1 is a simple isolated eigenvalue of $R(1)$.*

(3) (Aperiodicity) *for every $z \in \overline{D} - \{1\}$, $I - R(z)$ is invertible.*

Let P be the eigenprojection of $R(1)$ at 1. If $\sum_{k>n} \|R_k\| = O(1/n^\beta)$ for some $\beta > 1$ and $PR'(1)P \neq 0$, then for all n

$$T_n = \frac{1}{\lambda} P + \frac{1}{\lambda^2} \sum_{k=n+1}^{\infty} P_k + E_n, \tag{1.1}$$

where λ is given by $PR'(1)P = \lambda P$, $P_n = \sum_{k>n} PR_kP$ and $E_n \in \mathrm{Hom}(\mathcal{B}, \mathcal{B})$ satisfies $\|E_n\| = O(1/n^\beta)$ if $\beta > 2$, $O(\log n/n^2)$ if $b = 2$, and $O(1/n^{2\beta-2})$ if $2 > \beta > 1$.

In our case we apply the theorem by setting, as above,

$$R_n f = 1_{\widehat{X}} \cdot \mathscr{P}^n(f 1_{\{\tau = n\}}) \quad \text{and} \quad T_n f = 1_{\widehat{X}} \cdot \mathscr{P}^n(f 1_{\widehat{X}}) \tag{1.2}$$

Theorem A. *Let $X \subset R^m$ be compact subset with $\nu X = 1$ and $\widehat{X} \subset X$ is a compact subset of X with $\cup_{n\geq 0} T^n \widehat{X} = X$. Let $T : M \to M$ be a map whose first return map with respect to $\widehat{X}$ is $\widehat{T} = T^\tau$, and $\mathcal{B}$ be a Banach space satisfying Assumption B(a) to (d). We assume the following.*

(i) (Lasota-Yorke inequality) *There are constants $\eta \in (0,1)$, $D > 0$ such that for any $f \in \mathcal{B}$,*

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(ii) (Spectral radius) *There exist $B > 0, D_1 > 0$ and $\eta_1 \in (0,1)$ such that*

$$\|R(z)^n f\|_{\mathcal{B}} \leq |z^n| \big(B\eta_1^n \|f\|_{\mathcal{B}} + D_1 \|f\|_1\big). \tag{1.4}$$

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(iv) (Aperiodicity) *The function $e^{it\tau}$ given by the return time is aperiodic, that is, the only solutions for $e^{it\tau} = f/f \circ T$ almost everywhere with a measurable function $f : \widehat{X} \to S$ are f constant almost everywhere.*

If for any $n \geq 1$, R_n satisfies $\sum_{k=n+1}^{\infty} \|R_k\|_{\mathcal{B}} < O(n^{-\beta})$ for some $\beta > 1$, then there exists $C > 0$ such that for any function $f \in \mathcal{B}$, $g \in L^\infty(X, \nu)$ with $\operatorname{supp} f, \operatorname{supp} g \subset \widehat{X}$,

$$\Big|\mathrm{Cov}(f, g \circ T^n) - \big(\sum_{k=n+1}^{\infty} \mu(\tau > k)\big) \int f d\mu \int g d\mu\Big| \leq C F_\beta(n) \|g\|_\infty \|f\|_{\mathcal{B}} \tag{1.5}$$

where $F_\beta(n) = 1/n^\beta$ if $\beta > 2$, $(\log n)/n^2$ if $\beta = 2$, and $1/n^{2\beta-2}$ if $2 > \beta > 1$.

2 Aperiodicity

The proof of Theorem B is based on a result in [**?**].

A fibred system is a quintuple $(\widetilde{X}, \mathcal{A}, \nu, \widetilde{T}, \xi)$, where $(\widetilde{X}, \mathcal{A}, \nu, \widetilde{T})$ is a nonsingular transformation on a σ-finite measure space and $\xi \subset \mathcal{A}$ is a finite or countable partition (mod ν) such that:

(1) $\xi_\infty = \bigvee_{i=0}^{\infty} \widetilde{T}^{-i}\xi$ generates $\mathcal{A}$;

(2) every $A \in \xi$ has positive measure;

(3) for every $A \in \xi$, $\widetilde{T}|_A : A \to \widetilde{T}A$ is bimeasurable invertible with nonsingular inverse.

The transformation given in (0.5) is called the skew products over ξ. Put $\widetilde{\mathscr{P}} = \widetilde{\mathscr{P}}_{\nu\times\rho}$. A fibred system $(X, \mathcal{A}, \nu, T, \xi)$ with ν finite is called skew-product rigid if for every invariant function $h(x, y)$ of $\widetilde{\mathscr{P}}$ of an arbitrary skew product $\widetilde{T}_S$, the set $\{h(\cdot, y) > 0\}$ is almost open mod ν for almost every $y \in Y$.

A cylinder C of length n_0 is called a cylinder of full returns, if for almost all $x \in C$ there exist $n_k \nearrow \infty$ such that $\widehat{T}^{n_i+n_0}\xi_{n_i+n_0}(x) = C$.

In this case we say that $\widehat{T}^{n_0}(C)$ is a recurrent image set.

Theorem (ADSZ). *Let $(X, \mathcal{A}, \mu, T, \xi)$ be a skew-product rigid measure preserving fibred system whose image sets are almost open. Let G be a locally compact Abelian polish group. If $\gamma \circ \phi = \lambda f / f \circ T$ holds almost everywhere, where $\phi : X \to G$, ξ measurable, $\gamma \in \widehat{G}$, $\lambda \in S$, then f is constant on every recurrent image set.*

WE CAN PROVE

Skew product rigidity

Lemma 2.1. *For any $L^1(\mu \times \rho)$ function $\widetilde{h}$ on $\widehat{X} \times S$ that satisfies $\widetilde{\mathscr{P}}_{\mu \times \rho} \widetilde{h} = \widetilde{h}$, the set $\{\widetilde{h}(\cdot, y) > 0\}$ is almost open with respect to μ.*

Existence of a recurrence set

Lemma 2.2. *There is a recurrent image set J contained in $\widehat{X}$ with $\mu J > 0$.*

3 Systems on the multidimensional space

The main difficulty in higher dimensional space comes from unbounded distortion in the following sense: there are uncountably many points z such that for any neighborhood V of z, we can find $\hat{z} \in V$ with the ratio

$$|\det DT_1^{-n}(z)|/|\det DT_1^{-n}(\hat{z})|$$

unbounded as $n \to \infty$.

EXAMPLE

We let $M \subset R^2$ and near the fixed point $p = (0,0)$, the map T has the form

$$T(x,y) = \big(x(1+x^2+y^2),\ y(1+x^2+y^2)^2\big) \tag{3.1}$$

up to order $O(|z|^4)$, where $z = (x,y)$ and $|z| = \sqrt{x^2+y^2}$.

It is easy to see that

$$DT(x,y) = \begin{pmatrix} 1+3x^2+y^2+O(|z|^4) & 2xy+O(|z|^4) \\ 4xy+O(|z|^4) & 1+2x^2+6y^2+O(|z|^4) \end{pmatrix} \tag{3.2}$$

and

$$\det DT(x,y) = 1+5x^2+7y^2+O(|z|^4), \tag{3.3}$$

Take $z' = (x_0, 0)$ and denote $z'_n = T^{-n}z'$. One can show that $|z'_n| \sim \dfrac{1}{\sqrt{2n}}$ and $|\det DT^{-n}(z')| \leq \dfrac{D'}{n^{5/2}}$ for some $D' > 0$. On the other hand if we take $z'' = (0, y_0)$ and denote $z''_n = T^{-n}z''$, then $|z''_n| \sim \dfrac{1}{\sqrt{4n}}$ and $|\det DT^{-n}(z'')| \geq \dfrac{D''}{n^{7/4}}$ for some $D'' > 0$. So $\dfrac{|\det DT^{-n}(z'')|}{|\det DT^{-n}(z')|} \to \infty$ as $n \to \infty$.

We take a curve from z' to z'' that does not contain the origin. If for every z on the curve, there is a neighborhood V such that for all $\hat{z} \in V$, the ratio of the determinants is bounded for all $n > 0$, then the ratio $|\det DT^{-n}(z'')|/|\det DT^{-n}(z')|$ should be bounded. This contradicts the above fact. So we know that there are some points on the curve at which distortion is unbounded.

Let $X \subset R^m$, $m \geq 1$, be a compact subset with $\overline{\operatorname{int} X} = X$, d the Euclidean distance, and ν the Lebesgue measure on X with $\nu X = 1$.

Assume taht $T : X \to X$ is a map satisfying the following assumptions.

Assumption T″. (a) *(Piecewise smoothness) There are finitely many disjoint open sets $U_1, \cdots, U_K$ with piesewise smooth boundary such that $X = \bigcup_{i=1}^K \overline{U_i}$ and for each i, $T_i := T|_{U_i}$ can be extended to a $C^{1+\alpha}$ diffeomorphism $T_i : \tilde{U}_i \to B_{\varepsilon_1}(T_i U_i)$, where $\tilde{U}_i \supset U_i$, $\alpha \in (0,1)$ and $\varepsilon_1 > 0$.*

(b) *(Fixed point) There is a fixed point $p \in U_1$ and a neighborhood V of p such that $T^{-n}V \notin \partial U_j$ for any $j = 1, \ldots, K$ and for any $n \geq 0$.*

For any $\varepsilon_0 > 0$, denote

$$G_U(x, \varepsilon, \varepsilon_0) = 2 \sum_{j=1}^K \frac{\nu(T_j^{-1} B_\varepsilon(\partial T U_j) \cap B_{(1-s)\varepsilon_0}(x))}{\nu(B_{(1-s)\varepsilon_0}(x))};$$

Remark 3.1. *For smooth boundaries*

$$G_U(\varepsilon,\varepsilon_0) \le 2N_U Y \frac{\gamma_{m-1}}{\gamma_m} \frac{s\varepsilon}{(1-s)\varepsilon_0}\big(1+o(1)\big)$$

, where N_U is the maximal number of smooth components of the boundary of all U_i that meet in one point and γ_m is the volume of the unit ball in R^m.

For any $x \in U_i$, we define $s(x)$ as the inverse of the slowest expansion near x, that is,

$$s(x) = \min\big\{s : d(x,y) \le sd(Tx,Ty).$$

when x, y are close.

Take a neighborhood Q of p such that $TQ \subset U_1$, and denote $Q_0 = TQ \setminus Q$. Then let

$$s = s(Q) = \max\{s(x) : x \in X \backslash Q\}. \tag{3.4}$$

Let $\widehat{T} = \widetilde{T}_Q$ be the first return map with respect to $\widehat{X} = \widehat{X}_Q = X \setminus Q$. Then for any $x \in U_j$, we have $\widehat{T}(x) = T_j(x)$ if $T_j(x) \notin Q$, and $\widehat{T}(x) = T_1^i T_j(x)$ for some $i > 0$ if $T_j(x) \in Q$. Denote $\widehat{T}_{ij} = T_1^i T_j$ for $i \ge 0$. Further, we denote $U_{01} = U_1 \setminus Q$, $U_{0j} = U_j \setminus T_j^{-1} Q_0$ if $j > 1$, and $U_{ij} = \widehat{T}_{ij}^{-1} Q$ for $i > 0$.

For $0 < \varepsilon \le \varepsilon_0$, we denote

$$G_Q(x,\varepsilon,\varepsilon_0) = 2\sum_{j=1}^{K}\sum_{i=0}^{\infty} \frac{\nu(\hat{T}_{ij}^{-1} B_\varepsilon(\partial G_0) \cap B_{(1-s)\varepsilon_0}(x))}{\nu(B_{(1-s)\varepsilon_0}(x))},$$

and then denote

$$G(\varepsilon,\varepsilon_0) = \sup_{x\in X}\{G_U(x,\varepsilon,\varepsilon_0) + G_Q(x,\varepsilon,\varepsilon_0)\}. \tag{3.5}$$

Assumption T$''$. (c) *(Expansion) T satisfies $0 < s(x) < 1$ $\forall x \in X \setminus \{p\}$.*

Moreover, there exists an open region Q with $p \in Q \subset \overline{Q} \subset TQ \subset \overline{TQ} \subset U_1$ and a constants $\alpha \in (0,1)$, $\eta_0 \in (0,1)$, such that all ε_0 small,

$$s^\alpha + \lambda \le \eta_0 < 1,$$

where s is defined in (3.4) and

$$\lambda = 2 \sup_{\varepsilon \le \varepsilon_0} \frac{G(\varepsilon, \varepsilon_0)}{\varepsilon^\alpha} \varepsilon_0^\alpha.$$

(d) *(Distortion) For any $b > 0$, there exist $J > 0$ such that for any small ε_0 and $\varepsilon \in (0, \varepsilon_0)$, we can find $0 < N = N(\varepsilon) \le \infty$ with*

$$\frac{|\det DT_1^{-n}(y)|}{|\det DT_1^{-n}(x)|} \le 1 + J\varepsilon^\alpha \quad \forall y \in B_\varepsilon(x),\ x \in B_{\varepsilon_0}(Q_0),\ n \in (0, N],$$

and

$$\sum_{n=N}^{\infty} \sup_{y \in B_\varepsilon(x)} |\det DT_1^{-n}(y)| \le b\varepsilon^{m+\alpha} \quad \forall x \in B_{\varepsilon_4}(Q_0).$$

LOCAL BEHAVIORS

To estimate the decay rates, we also assume that there are constants $\gamma' > \gamma > 0$, $C_i, C_i' > 0$, $i = 0, 1, 2$, such that in a neighborhood of the indifferent fixed point $p = 0$,

$$|x|\big(1 - C_0'|x|^\gamma + O(|x|^{\gamma'})\big) \le |T_1^{-1}x| \le |x|\big(1 - C_0|x|^\gamma + O(|x|^{\gamma'})\big), \quad (3.6)$$

$$1 - C_1'|x|^\gamma \le \|DT_1^{-1}(x)\| \le 1 - C_1|x|^\gamma, \quad (3.7)$$

$$C_2'|x|^{\gamma-1} \le \|D^2T_1^{-1}(x)\| \le C_2|x|^{\gamma-1}. \quad (3.8)$$

FUNCTIONAL SPACES

Take $f \in L^1(\widehat{X}, \nu)$ function f; define the oscillation

$$\mathrm{osc}(f, \Omega) = \operatorname*{Esup}_{\Omega} f - \operatorname*{Einf}_{\Omega} f.$$

For $0 < \alpha < 1$ and $\varepsilon_0 > 0$, we define a seminorm of f as

$$|f|_{\mathcal{V}} = |f|_{\mathcal{V}^\alpha_{\varepsilon_0}} = \sup_{0<\epsilon\le\epsilon_0} \epsilon^{-\alpha} \int_{R^m} \mathrm{osc}(f, B_\epsilon(x)) d\nu(x), \quad (3.9)$$

and take the space of the functions as

$$\mathcal{V} = \mathcal{V}^\alpha_{\varepsilon_0} = \Big\{ f \in L^1(\widehat{X}, \nu) : |f|_{\mathcal{V}} < \infty \Big\}$$

and then equip $\mathcal{V}^\alpha_{\varepsilon_0}$ with the norm

$$\|\cdot\|_{\mathcal{V}} = \|\cdot\|_1 + |\cdot|_{\mathcal{V}}.$$

For an open set O, let $\mathcal{H} = \mathcal{H}^\alpha_{\varepsilon_1} = \mathcal{H}^\alpha_{\varepsilon_1}(O, H)$ be the set of Hölder functions f over O that satisfies $|f(x) - f(y)| \leq Hd(x, y)^\alpha$ for any $x, y \in O$ with $d(x, y) \leq \varepsilon_1$.

Let h be a fixed point of the transfer operator $\widehat{\mathscr{P}}$, which will be unique under the assumption of the theorem below. We define $\mathcal{B} = \mathcal{B}^\alpha_{\varepsilon_0,\varepsilon_1}$ by

$$\mathcal{B} = \left\{ f \in \mathcal{V}^\alpha_{\varepsilon_0} : \exists H > 0 \ s.t. \ (f/h)|_{V_I} \in \mathcal{H}^\alpha_{\varepsilon_1}(V_I, H) \ \forall I \in \mathcal{I} \right\},$$

and for any $f \in \mathcal{B}$, let

$$|f|_{\mathcal{H}} = |f|_{\mathcal{H}^\alpha_{\varepsilon_1}} = \inf\{H : (f/h)|_{V_I} \in \mathcal{H}^\alpha_{\varepsilon_1}(V_I, H) \ \forall I \in \mathcal{I}\}.$$

Let us assume that $h > 0$ on all V_{ij}, then we define the norm in $\mathcal{B}$ by

$$\| \cdot \|_{\mathcal{B}} = \| \cdot \|_1 + | \cdot |_{\mathcal{V}} + | \cdot |_{\mathcal{H}}. \tag{3.10}$$

Clearly, $\mathcal{B}^\alpha_{\varepsilon_0,\varepsilon_1} \subset \mathcal{V}^\alpha_{\varepsilon_0}$ and $\|f\|_{\mathcal{B}} \geq \|f\|_{\mathcal{V}}$ if $f \in \mathcal{B}$.

Let $s_n = \max |\det DT^n(T^{-n}(x)|^{-1} : x \in B_\varepsilon(Q_0), j = 2, \cdots K\}$.

Theorem D. *Let $\widehat{X}$, $\widehat{T}$ and $\mathcal{B}$ are defined as above. Suppose T satisfies Assumption T (a) to (d) and Assumption T'' (a) to (d). Then there exist $\varepsilon_0 \geq \varepsilon_1 > 0$ such that Assumption B(a) to (e) and Condition (i) to (iv) in Theorem A are satisfied and $\|R_n\| \leq O(s_n^{m/(m+\alpha)})$. Hence, if $s_n^{m/(m+\alpha)} \leq O(n^{-1/\beta})$ for some $\beta > 1$, then there exists $C > 0$ such that for any functions $f \in \mathcal{B}$, $g \in L^\infty(X, \nu)$ with $\operatorname{supp} f, \operatorname{supp} g \subset \widehat{X}$, (1.5) holds.*

In particular, if T satisfies (3.6) to (3.8) near p, then $\sum_{k=n+1}^{\infty} \mu(\tau > k)$ has the order $n^{-(m/\gamma-1)}$. In this case, if $s_n = O(n^{-\beta'})$ for some $\beta' > 1$ and

$$\beta' \cdot \frac{m}{m+\alpha} \geq \frac{m}{\gamma}, \tag{3.11}$$

then

$$\operatorname{Cov}(f, g \circ T^n) \approx \sum_{k=n+1}^{\infty} \mu(\tau > k) \int f d\mu \int g d\mu \ = O(1/n^{m/\gamma-1}). \tag{3.12}$$

3.1 Examples

Example 1. *Assume* $m = 3$, *and near the fixed point* $p = (0, 0, 0)$, *the map* T *has the form*

$$T(w) = \big(x(1+|w|^2+O(|w|^3)),\ y(1+|w|^2+O(|w|^3)), z(1+2|w|^2+O(|w|^3)\big)$$

where $w = (x, y, z)$ *and* $|w| = \sqrt{x^2 + y^2 + z^2}$.

Denote $w_n = T_1^{-n} w$. Clearly, $|w| + |w|^3 + O(|w|^4) \leq |T(w)| \leq |w| + 2|w|^3 + O(|w|^4)$. By standard arguments we know that

$$\frac{1}{\sqrt{4n}} + O\Big(\frac{1}{\sqrt{n^3}}\Big) \leq |w_n| \leq \frac{1}{\sqrt{2n}} + O\Big(\frac{1}{\sqrt{n^3}}\Big)$$

. Since it is in three dimensional space, it follows that $\nu(\tau > k) = O\Big(\frac{1}{k^{3/2}}\Big)$, and therefore $\sum_{k=n+1}^{\infty} \nu(\tau > k) = O\Big(\frac{1}{n^{1/2}}\Big)$.

It is easy to see that $DT(w)$ has the form

$$\begin{pmatrix} 1 + 3x^2 + y^2 + z^2 & 2xy & 2xz \\ 2xy & 1 + x^2 + 3y^2 + z^2 & 2yz \\ 4xz & 4yz & 1 + 2x^2 + 2y^2 + 6z^2 \end{pmatrix} + O(|w|^3)$$

and hence

$$\det DT(w) = 1 + 6x^2 + 6y^2 + 8z^2 + O(|w|^3).$$

We have $||R_n|| \sim \frac{1}{n^{9/4}}$ and a decay rates of order $O(1/\sqrt{n})$.

Example 2. *Assume $m = 2$, and near the fixed point $p = (0,0)$, the map T has the form*

$$T(z) = \big(x(1 + |z|^\gamma + O(|z|^{\gamma'})),\ y(1 + 2|z|^\gamma + O(|z|^{\gamma'}))\big)$$

where $z = (x, y)$, $|z| = \sqrt{x^2 + y^2}$, $\gamma \in (0,1)$ and $\gamma' > \gamma$.

Denote $z_n = T_1^{-n} z$. Since $|z| + |z|^{1+\gamma} + O(|z|^{\gamma'}) \le |T(z)| \le |z| + 2|z|^\gamma + O(|z|^{\gamma'})$,

$$\frac{1}{(2\gamma n)^{1/\gamma}} + O\Big(\frac{1}{n^\delta}\Big) \le |z_n| \le \frac{1}{(\gamma n)^{1/\gamma}} + O\Big(\frac{1}{n^\delta}\Big)$$

for some $\delta > 1/\gamma$ (see also Lemma 3.1 in [**?**]). It follows that $\nu(\tau > k) = O\Big(\frac{1}{k^{2/\gamma}}\Big)$, and therefore $\sum_{k=n+1}^{\infty} \nu(\tau > k) = O\Big(\frac{1}{n^{2/\gamma - 1}}\Big)$.

Also,

$$DT(z) = \begin{pmatrix} 1 + \dfrac{(1+\gamma)x^2 + y^2}{|z|^{2-\gamma}} & \dfrac{\gamma xy}{|z|^{2-\gamma}} \\ \dfrac{2\gamma xy}{|z|^{2-\gamma}} & 1 + \dfrac{2x^2 + 2(1+\gamma)y^2}{|z|^{2-\gamma}} \end{pmatrix} + O(|z|^{\gamma'})$$

and hence

$$\det DT(z) = 1 + \frac{(3+\gamma)x^2 + (3+2\gamma)y^2}{|z|^{2-\gamma}} + O(|z|^{\gamma'}).$$

We have $||R_n|| \sim \frac{1}{n^{(1+\frac{3}{\gamma})\frac{2}{3}}}$ and a decay rates of order $O(1/n^{\frac{2}{\gamma}-1})$.

Example 3. *Assume $m = 2$, and near the fixed point $p = (0, 0)$, the map T has the form*

$$T(x, y) = \big(x(1 + x^2 + y^2),\ y(1 + x^2 + y^2)^2\big) \tag{3.13}$$

up to order $O(|z|^4)$, where $z = (x, y)$ and $|z| = \sqrt{x^2 + y^2}$.

The map allows an infinite absolutely continuous invariant measure. However, the map can be arranged in a way that there is an invariant component that supports a finite absolutely continuous invariant measure μ. Near the fixed point, the region of the component is of the form

$$\{z = (x, y) : |y| < x^2\}.$$

Since $|z_n| = O(1/\sqrt{n})$ and for $z = (x, y)$, $|y| \leq x^2$, we have $\nu(\tau > k) = O\Big(\frac{1}{k^{3/2}}\Big)$, and $\sum_{k=n+1}^{\infty} \nu(\tau > k) = O\Big(\frac{1}{n^{1/2}}\Big)$.

On the other hand, a similar computation gives $\det DT(z) = 1 + 5x^2 + 7y^2 + O(|z|^4)$. Since $|y| \leq x^2$, $|z| = |x| + O(|z|^2)$. Hence $\det DT(z) = 1 + 5|z|^2 + O(|z|^4)$, and therefore $|\det DT_1^{-n}(z)| = O(1/n^{5/2})$.

We have $||R_n|| \sim \frac{1}{n^{5/3}}$ and a decay rates of order $O(1/\sqrt{n})$.