Hyperbolicity of Renormalization for C^r Unimodal Maps

A guided tour

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History

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- Davie (1996): Using hard analysis, extended the hyperbolicity picture (local stable and unstable manifolds) from Lanford's Banach space to the space of $C^{2+\epsilon}$ maps.

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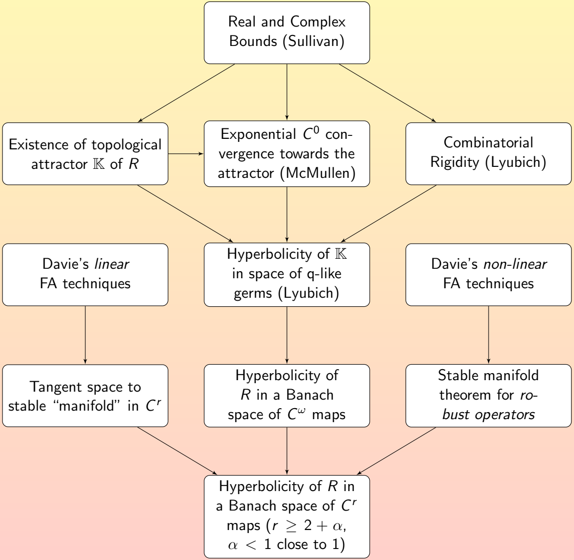
Our Main Results

- dF, de Melo, Pinto (2006): Established Lanford's conjecture in the space of C^r quadratic unimodal maps. Here r is any real number $\geq 2 + \alpha$, where $\alpha < 1$ is the largest of the Hausdorff dimensions of the post-critical sets of maps in the attractor. The proof combines Lyubich's theorem with Davie's *tour de force*.
- The authors also went beyond the conjecture, proving that the local stable manifolds form a C^0 lamination whose holonomy is $C^{1+\beta}$ for some $\beta > 0$. In particular, every smooth curve which is transversal to such lamination intersects it at a set of constant Hausdorff dimension less than one.

Structure of Proof of Hyperbolicity Conjecture



Spaces of Unimodal Maps

Normalization: $I = [-1, 1]$, $f : I \to I$ unimodal, even; $f'(0) = 0$, $f(0) = 1$.

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1. Underlying Banach spaces:
   - $\mathbb{A}^r = \{v \in C^r(I) : v \text{ is even and } v(0) = 0\}$, with C^r norm.
   - $\mathbb{B}^r = \left\{v = \varphi \circ q \in C^r(I) : \varphi \in C^r([0,1]), \varphi(0) = 0, q(x) = x^2\right\}$, the norm of v being the C^r norm of φ.

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2. Corresponding spaces of unimodal maps:
   - $\mathbb{U}^r \subset 1 + \mathbb{A}^r \subset C^r(I)$ consists of all $f : I \to I$ of the form $f(x) = 1 + v(x)$, $v \in \mathbb{A}^r$ with $v''(0) < 0$ $(r \geq 2)$.
   - $\mathbb{V}^r \subset 1 + \mathbb{B}^r$ consists of those $f = \phi \circ q$ such that $\phi([0,1]) \subseteq (-1, 1]$, $\phi(0) = 1$ and $\phi'(x) < 0$ for all $0 \leq x \leq 1$.

Remark: For each $s \leq r$, the inclusion $\mathbb{B}^r \hookrightarrow \mathbb{A}^s$ is (linear and) continuous.

Renormalizing unimodal maps

- $f \in \mathbb{U}^r$ is said to be *renormalizable* if there exist $p = p(f) > 1$ and $\lambda = \lambda(f) = f^p(0)$ such that $f^p|[-|\lambda|, |\lambda|]$ is unimodal and maps $[-|\lambda|, |\lambda|]$ into itself.

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- The set of all unimodal permutations is denoted $\mathbf{P}$.

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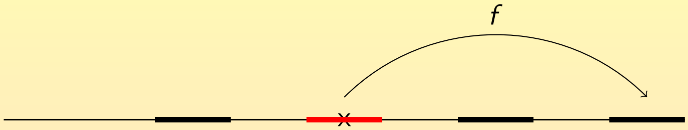

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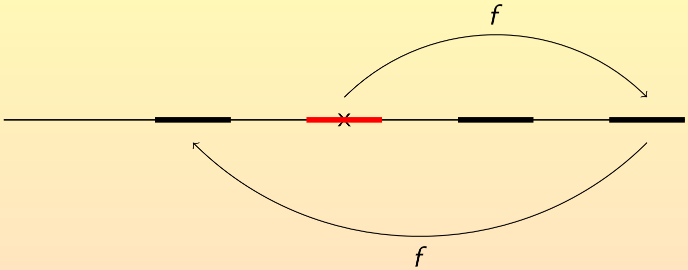

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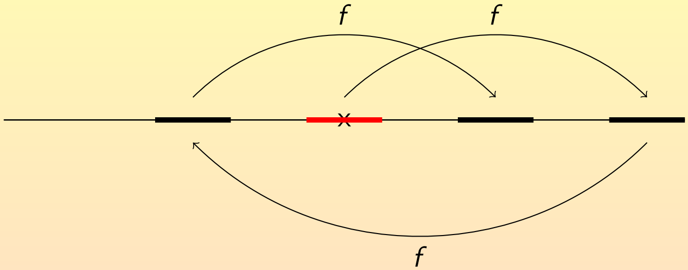

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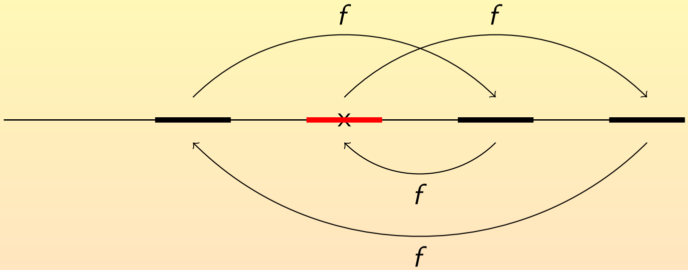

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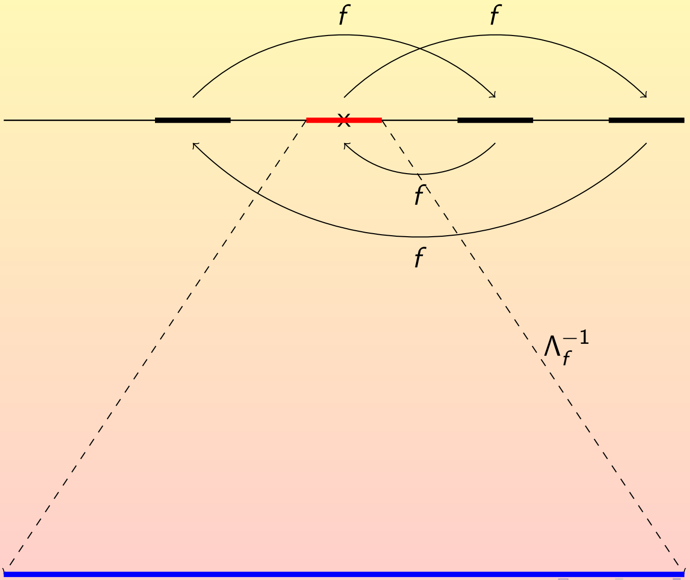

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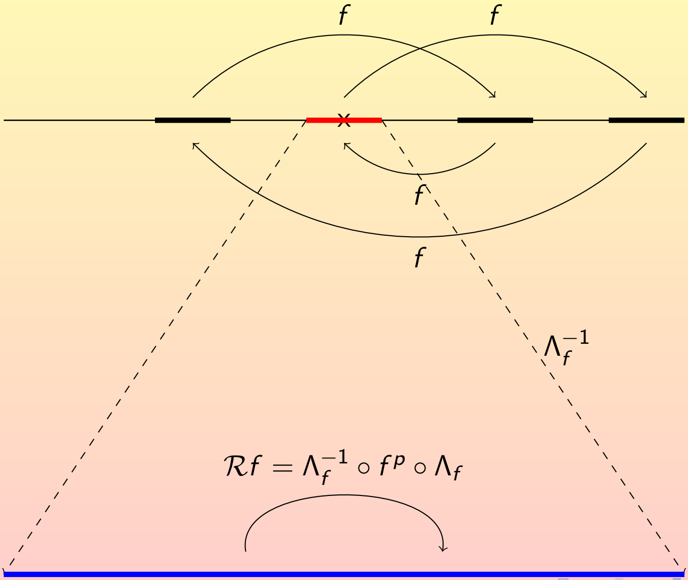

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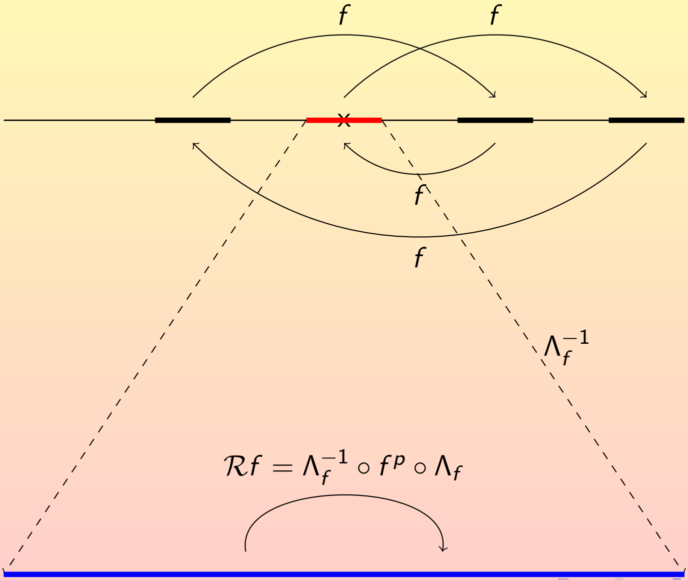

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- The set of $f \in \mathbb{U}^r$ that are renormalizable with the same unimodal permutation $\theta \in \mathbf{P}$ is a connected subset of $\mathbb{U}^r$ denoted $\mathbb{U}^r_\theta$.

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- For $\Theta \subseteq \mathbf{P}$ finite and $\theta_0, \theta_1, \ldots, \theta_n, \ldots \in \Theta$, write

$$\mathbb{U}^r_{\theta_0,\theta_1,\cdots,\theta_n,\cdots} \ = \ \mathbb{U}^r_{\theta_0} \cap R^{-1}\mathbb{U}^r_{\theta_1} \cap \cdots \cap R^{-n}\mathbb{U}^r_{\theta_n} \cap \cdots \ ,$$

and define

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- The maps in $\mathcal{D}^r_\Theta$ are *infinitely renormalizable* maps with (bounded) combinatorics belonging to Θ.

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2. $\mathbb{A}_V \subset \mathcal{H}_0(V)$: closed linear subspace of functions of the form $\varphi = \phi \circ q$, where $q(z) = z^2$ and $\phi : q(V) \to \mathbb{C}$ holomorphic with $\phi(0) = 0$.
3. $\mathbb{U}_V$: the set of functions of the form $f = 1 + \varphi$, where $\varphi = \phi \circ q \in \mathbb{A}_V$ and ϕ is *univalent* on some neighborhood of $[-1, 1]$ contained in V, such that the restriction of f to $[-1, 1]$ is unimodal.

Note that $\mathbb{U}_V \subset 1 + \mathbb{A}_V \simeq \mathbb{A}_V$ is open, so $\mathbb{U}_V$ is a Banach manifold. We denote by Ω_a the set of points in the complex plane whose distance from the interval $[-1, 1]$ is smaller than a.

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Theorem (Sullivan's Complex Bounds)

Let $\Theta \subseteq \mathbf{P}$ be a non-empty finite set. Then there exist $a > 0$, a compact subset $\mathbb{K} = \mathbb{K}_\Theta \subseteq \mathbb{A}_{\Omega_a} \cap \mathcal{D}_\Theta^\omega$ and $\mu > 0$ with the following properties.

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(iii) *For all $g \in \mathcal{D}^r_\Theta \cap \mathbb{V}^r$, with $r \geq 2$, there exists $f \in \mathbb{K}$ with the property that $||R^n(g) - R^n(f)||_{C^0(I)} \to 0$ as $n \to \infty$.*

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Complex Bounds (cont.)

Remark:

Item (ii) above means that there exists a homeomorphism $H : \mathbb{K} \to \Theta^{\mathbb{Z}}$ such that the diagram

$$\begin{array}{ccc} \mathbb{K} & \xrightarrow{R} & \mathbb{K} \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H} \\ \Theta^{\mathbb{Z}} & \xrightarrow[\sigma]{} & \Theta^{\mathbb{Z}} \end{array}$$

commutes.

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If f and g are infinitely renormalizable quadratic-like maps with the same bounded combinatorial type in $\Theta \subset P$, and with conformal moduli greater than or equal to μ, we have

$$\|R^n f - R^n g\|_{C^0(I)} \leq C\lambda^n$$

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Lyubich's Theorem

Lyubich endowed the space $\mathcal{QG}$ of quadratic-like germs (modulo affine equivalence) with a complex manifold structure modeled in a topological vector space (arising as the direct limit of a certain net of Banach spaces of analytic functions). The attractor $\mathbb{K}$ is naturally embedded in $\mathcal{QG}$ as a compact set.

Theorem (Lyubich)

In the space $\mathcal{QG}$, the renormalization operator is complex-analytic, and its limit set $\mathbb{K}$ is a hyperbolic basic set.

This deep theorem is the crowning jewel of this theory.

Hyperbolicity in a Banach space of real-analytic maps

Theorem (Hyperbolicity in a real Banach space)

There exist $a > 0$, an open set $\mathbb{O} \subset \mathbb{A} = \mathbb{A}_{\Omega_a}$ containing $\mathbb{K} = \mathbb{K}_\Theta$ and a positive integer N with the following property. There exists a real analytic operator $T : \mathbb{O} \to \mathbb{A}$ having $\mathbb{K}$ as a hyperbolic basic set with co-dimension one stable manifolds at each point, such that $T(f)|[-1,1] = R^N(f|[-1,1])$, for all $f \in \mathbb{O}$, is the N-th iterate of the renormalization operator.

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Proof Ingredients: Lyubich's theorem + the complex bounds + skew-product renormalization.

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Hyperbolic Picture

Definition (Hyperbolic Picture)

We say that the hyperbolic picture holds true for T (and $\mathbb{K}$) if there exists $0 < \theta < 1$ such that:

(i) *For all $x \in \mathbb{K}$, $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$ are Banach submanifolds of $\mathcal{B}$.*

(ii) *If $y \in W^s(x)$ then*

$$\|T^n(x) - T^n(y)\| \leq C\theta^n \|x - y\| \ .$$

Moreover, $T(W^u_\epsilon(x)) \supseteq W^u_\epsilon(T(x))$, the restriction of T to $W^u_\epsilon(x)$ is one-to-one and for all $y \in W^u_\epsilon(x)$ we have

$$\left\|T^{-n}(x) - T^{-n}(y)\right\| \leq C\theta^n \|x - y\| \ .$$

Hyperbolic Picture (cont.)

(iii) *If* $y \in \mathcal{B}(x, \epsilon)$ *is such that* $T^i(y) \in \mathcal{B}(T^i(x), \epsilon)$ *for* $i \leq n$ *then*

$$\mathrm{dist}\,(T^n(y), W^u_\epsilon(T^n(x))) \leq C\theta^n, \text{ as well as } \mathrm{dist}\,(y, W^s_\epsilon(x)) \leq C\theta^n \ .$$

(iv) *The family of local stable manifolds (and also the family of local unstable manifolds) form a* C^0 *lamination: the tangent spaces to the leaves vary continuously.*

Hyperbolic Picture in C^r

Theorem (Hyperbolic Picture in $\mathbb{U}^r$)

If $r \geq 2 + \alpha$, where $\alpha > 0$ is close to one, then the hyperbolic picture holds true for the renormalization operator acting on $\mathbb{U}^r$. Furthermore,

(i) *the local unstable manifolds are real analytic curves;*

(ii) *the local stable manifolds are of class C^1, and together they form a continuous lamination whose holonomy is $C^{1+\beta}$ for some $\beta > 0$;*

The proof is long, but not painful.

Key Steps

- Starting from the hyperbolicity of $T = R^N$ in $\mathbb{A}$ (our Banach space of C^ω maps), we first extend the hyperbolic splitting (stable and unstable subspaces) at each point of the attractor from $\mathbb{A}$ to $\mathbb{A}^r$. This *linear* part depends on the notion of *compatibility*.
- Second, we prove that the stable sets of points of $\mathbb{K}$ in $\mathbb{A}^r$ are graphs of C^1 functions defined over the corresponding extended stable subspaces obtained in the first step. This *non-linear* part depends on the notion of *robustness*.

Compatibility

The notion of compatibility involves the following data.

- Banach spaces $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}$ (typically, each inclusion compact with a dense image).
- A C^1 operator $T : \mathcal{O} \to \mathcal{A}$ ($\mathcal{O} \subset \mathcal{A}$ open) with a hyperbolic basic set $\mathbb{K} \subset \mathcal{O}$.

For each $x \in \mathbb{K}$, we want to extend $DT(x)$ and its hyperbolic splitting from $\mathcal{A}$ to $\mathcal{B}$.
The basic principle at work here is: *boundedness in $\mathcal{C}$ yields pre-compactness in $\mathcal{B}$*.

Compatibility (cont.)

Definition

Let $\theta < \rho < \lambda$ where θ, λ are the contraction and expansion exponents of $T|\mathbb{K}$. The pair $(\mathcal{B}, \mathcal{C})$ is ρ-compatible with $(T, \mathbb{K})$ if:

1. *The inclusions $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ are compact operators.*
2. *There exists $M > 0$ such that each linear operator $L_x = DT(x)$ extends to a linear operator $\hat{L}_x : \mathcal{C} \to \mathcal{C}$ with*

$$\left\|\hat{L}_x\right\|_{\mathcal{C}} < M \ , \ \hat{L}_x(\mathcal{B}) \subset \mathcal{B} \ , \ \textit{and} \ \left\|\hat{L}_x(v)\right\|_{\mathcal{B}} < M\|v\|_{\mathcal{B}} \ .$$

3. *The map $\tilde{L} : \mathbb{K} \to \mathcal{L}(\mathcal{B}, \mathcal{C})$ given by $\tilde{L}_x = \hat{L}_x|\mathcal{B}$ is continuous.*
4. *There exists $\Delta > 1$ such that $\mathcal{B}(\Delta) \cap \mathcal{A}$ is $\mathcal{C}$-dense in $\mathcal{B}(1)$.*
5. *There exist $K > 1$ and a positive integer m such that*

$$\left\|\hat{L}_x^{(m)}(v)\right\|_{\mathcal{B}} \le \max\left\{\frac{\rho^m}{2(1+\Delta)}\|v\|_{\mathcal{B}}, K\|v\|_{\mathcal{C}}\right\} \ .$$

Extending the Splitting

Theorem

If $(\mathcal{B},\mathcal{C})$ is ρ-compatible with $(T,\mathbb{K})$, then there exists a continuous splitting $\mathcal{B} = \hat{E}^u_x \oplus \hat{E}^s_x$ such that

- *$\hat{E}^u_x$ is the inclusion of $E^u_x \subset \mathcal{A}$ in $\mathcal{B}$.*
- *The splitting is invariant under $\hat{L}_x$.*
- *We have $\|\hat{L}^n_x(v)\|_{\mathcal{B}} \geq C\lambda^n \|v\|_{\mathcal{B}}$, for all $v \in \hat{E}^u_x$.*
- *We have $\|\hat{L}^n_x(v)\|_{\mathcal{B}} \leq C\hat{\theta}^n \|v\|_{\mathcal{B}}$, for all $v \in \hat{E}^s_x$ and some $0 < \hat{\theta} < \rho$.*

Remark: In particular, if $\rho \leq 1$, then $\hat{\theta} < 1$, and so therefore the hyperbolic splitting in $\mathcal{A}$ remains hyperbolic in $\mathcal{B}$.

Extending the Splitting of Renormalization

Theorem

Let T and $\mathbb{K}$ be as above, and let λ be the expansion constant of $T|\mathbb{K}$.

- *For all $\alpha > 0$ the pair of spaces $(\mathbb{A}^{2+\alpha}, \mathbb{A}^0)$ is 1-compatible with $(T, \mathbb{K})$.*
- *For all $1 < \rho < \lambda$ there exists $\alpha > 0$ sufficiently small such that $(\mathbb{A}^{2-\alpha}, \mathbb{A}^0)$ is ρ-compatible with $(T, \mathbb{K})$.*

Robustness (Informal Def.)

An operator T is *robust* if:

- It acts on four Banach spaces simultaneously: $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C} \subset \mathcal{D}$.
- In $\mathcal{A}$, T has a hyperbolic basic set $\mathbb{K}$ (with expansion contant λ).
- $(\mathcal{B}, \mathcal{D})$ is 1-compatible with $(T, \mathbb{K})$
- $(\mathcal{C}, \mathcal{D})$ is ρ-compatible with $(T, \mathbb{K})$ for some $\rho < \lambda$.
- As a map from $\mathcal{B}$ to $\mathcal{C}$, T is C^1.
- T satisfies a uniform Gateaux condition in $\mathcal{C}$ for directions in $\mathcal{B}$.
- The extension of DT to $\mathcal{B}$ is a *reasonably good approximation* of T near $\mathbb{K}$.

Stable Manifold Theorem for Robust Operators

Theorem

Let $T : \mathcal{O}_{\mathcal{A}} \to \mathcal{A}$ *be a* C^k *with* $k \geq 2$ *(or real analytic) hyperbolic operator over* $\mathbb{K} \subset \mathcal{O}_{\mathcal{A}}$*, and robust with respect to* $(\mathcal{B}, \mathcal{C}, \mathcal{D})$*. Then the hyperbolic picture holds true* T *acting on* $\mathcal{B}$*. The local unstable manifolds are* C^k *with* $k \geq 2$ *(or real analytic) curves, and the local stable manifolds are of class* C^1 *and form a* C^0 *lamination.*

The proof is long *and* painful. Uncountably many norm estimates.

Renormalization is Robust

Theorem

Let $T = R^N : \mathbb{O} \to \mathbb{A}$ be the renormalization operator in the Banach space of C^ω maps constructed before. If $s > s_0$ with $s_0 < 2$ sufficiently close to 2 and $r > s + 1$ not an integer, then T is a robust operator with respect to $(\mathbb{A}^r, \mathbb{A}^s, \mathbb{A}^0)$.

Remark: Here, we can take $s_0 = 1 + D$, where $D = \sup_{f \in \mathbb{K}} HD(\mathcal{I}_f)$, where $\mathcal{I}_f$ is the post-critical set of f.

Key Geometric Estimates

The proof that the renormalization operator T satisfies both *compatibility* and *robustness* depends on two key estimates concerning the geometry of the post-critical set of maps in the attractor. These come into play when we analyze the (formal) derivative of a power of T at $f \in \mathbb{K}$, namely:

$$DT^k(f)v = \frac{1}{\lambda_k} \sum_{j=0}^{p_k-1} Df^j(f^{p_k-j}(\lambda_k x))v(f^{p_k-j-1}(\lambda_k x))$$

$$+\frac{1}{\lambda_f}[x(T^k f)'(x) - T^k f(x)] \sum_{j=0}^{p_k-1} Df^j(f^{p_k-j}(0))v(f^{p_k-j-1}(0)) \ ,$$

Key Geometric Estimates (cont.)

From bounded geometry (the real bounds), the terms appearing in the expression of $DT^k(f)$ can be estimated using the lengths of renormalization intervals $\Delta_{j,k}$ at level k. Thus:

$$|Df^j(f^{p_k-j}(\lambda_k x))| \asymp \frac{|\Delta_{0,k}|}{|\Delta_{p_k-j,k}|} \tag{3}$$

and also

$$|Df^{p_k-j-1}(\lambda_k x)| \leq C|\Delta_{0,k}|\frac{|\Delta_{p_k-j-1,k}|}{|\Delta_{1,k}|} \tag{4}$$

for all $0 \leq j \leq p_k - 1$.

Key Geometric Estimates (cont.)

Given those facts, to verify compatibility and robustness one needs the following (where $D = \sup_{f\in\mathbb{K}} HD(\mathcal{I}_f)$).

Theorem

- *For each $\alpha > 0$ there exist constants C_0 and $0 < \mu < 1$ such that*

$$\sum_{i=0}^{p_k-1} \frac{|\Delta_{i,k}|^{2+\alpha}}{|\Delta_{i+1,k}|} \leq C_0\mu^k \ . \tag{5}$$

- *For each $\mu > 1$ close to one, there exist $0 < \alpha < 1 - D$ close to zero and $C_1 > 0$ such that for all $f \in \mathbb{K}$ we have*

$$\sum_{i=0}^{p_k-1} \frac{|\Delta_{i,k}|^{2-\alpha}}{|\Delta_{i+1,k}|} \leq C_1\mu^k \ .$$