Renormalization of Hamiltonians and vector fields

(Notes for a mini-course, Warwick U, Dec 2010)

(H. Koch, UT Austin)

1. **Invariant tori and equivalence**
2. **Why renormalization**
3. **Special cases and results**
4. **Scaling**
5. **Nonresonant modes**
6. **Reduction to normal form**
7. **Combining the steps**
8. **Diophantine frequencies**
9. **Invariant tori**
10. **Some open problems**

---

1. Invariant tori and equivalence

Introducing the flow, invariant tori, and equivalence of Hamiltonians.

1.1. Invariant tori

Consider the vector field on $\mathcal{M} = \mathbb{T}^d \times \mathbb{R}^d$ generated by a differentiable Hamiltonian $H : \mathcal{M} \to \mathbb{R}$,

$$\begin{aligned} \dot{\boldsymbol{q}} &= \nabla_{\boldsymbol{p}} \boldsymbol{H}(\boldsymbol{q}, \boldsymbol{p}) \,, \\ \dot{\boldsymbol{p}} &= -\nabla_{\boldsymbol{q}} \boldsymbol{H}(\boldsymbol{q}, \boldsymbol{p}) \,, \end{aligned}$$

and the corresponding flow Φ,

$$\bigl(q(t), p(t)\bigr) = \Phi^t(q_0, p_0) \,.$$

The time-t map Φ^t is a symplectic map on $\mathcal{M}$, that is, it preserves the 2-form

$$\sum_{j=1}^{d} dq_j \wedge dp_j \,.$$

Integrable example: For $\omega \in \mathbb{R}^d$ the Hamiltonian

$$K^{(M)}(q,p) = \omega \cdot p + \tfrac{1}{2}(Mp)\cdot p\,, \qquad M = M^*\,.$$

The corresponding vector field and flow are

$$\begin{aligned}\dot q &= \omega + Mp\,,\\ \dot p &= 0\,,\\ \Psi^t(q_0,p_0) &= \bigl(q_0 + t[\omega + Mp_0], p_0\bigr)\,.\end{aligned}$$

Notice that $\mathcal{M}$ is foliated by invariant tori $\mathbb{T}^d \times \{p_0\}$ with frequency vectors $\omega + Mp_0$. From now on restrict Ψ^t to $p_0 = 0$.

By an invariant torus for H, with rotation vector ω, we mean a locally one-to-one map $\Gamma : \mathbb{T}^d \times \{0\} \to \mathcal{M}$ that satisfies

$$\boldsymbol{\Phi^t \circ \Gamma = \Gamma \circ \Psi^t}\,.$$

1.2. Persistence of smooth invariant tori

Let $\beta > 0$. A nonzero vector $\omega \in \mathbb{R}^d$ is of class Diophantine(β) if there exists $C > 0$ such that

$$|\omega \cdot \nu| \geq C\|\nu\|^{1-d-\beta}\,, \qquad \nu \in \mathbb{Z}^d \setminus \{0\}\,.$$

KAM Theorem. *If ω is Diophantine, M nonsingular, H analytic and near $K^{(M)}$, then H has an analytic invariant torus with frequency vector ω.*

The proof yields symplectic transformations $U_1, U_2, \ldots$ such that

$$H \circ U_1 \circ U_2 \circ \ldots \circ U_n \;\to\; K^{(0)},$$

on domains shrinking down to $\mathbb{T}^d \times \{0\}$. The invariant torus is

$$\Gamma = U_1 \circ U_2 \circ \ldots$$

Renormalization does something similar, except that the domain does not shrink. So one gets a dynamical system on a space of Hamiltonians.
(But this is not the only goal of renormalization.)

1.3. Symplectic changes of coordinates

Consider symplectic diffeos $\Lambda : \mathcal{M} \to \mathcal{M}$ that can be decomposed

$$\Lambda = \mathcal{T} \circ \mathcal{U}\,, \qquad \mathcal{T}(q,p) = \Big(Tq, (T^*)^{-1}p\Big)\,,$$

where $\pm T$ is an integer matrix in $\mathrm{SL}(d,\mathbb{Z})$, and $\mathcal{U}$ is homotopic to the identity.

Under such a change of coordinates,

$$\widetilde{H} = H \circ \Lambda\,, \qquad \widetilde{\Phi}^t = \Lambda^{-1} \circ \Phi^t \circ \Lambda\,,$$

and

$$\widetilde{\Gamma} = \Lambda^{-1} \circ \Gamma \circ \mathcal{T}\,, \qquad \widetilde{\omega} = T^{-1}\omega\,. \tag{$*$}$$

Thus, as far as frequencies are concerned,

$\mathcal{T}$ is "relevant" (changes frequencies),
$\mathcal{U}$ is "irrelevant" (does not change frequencies).

<u>Formal proof</u> that $\widetilde{H}$ has an invariant torus as described by $(*)$, assuming that H has an invariant ω-torus Γ. Define $\widetilde{\omega}$ and $\widetilde{\Gamma}$ by equation $(*)$. For the linear flows we have

$$\begin{aligned}\big(\mathcal{T} \circ \widetilde{\Psi}^t\big)(q,0) &= \mathcal{T}\big(q + t\widetilde{\omega}, 0\big) = \big(T(q + t\widetilde{\omega}), 0\big)\\ &= (Tq + t\omega, 0) = \Psi^t(Tq, 0)\\ &= \big(\Psi^t \circ \mathcal{T}\big)(q,0)\,.\end{aligned}$$

Thus,

$$\begin{aligned}\widetilde{\Gamma} \circ \widetilde{\Psi}^t &= \Lambda^{-1} \circ \Gamma \circ \mathcal{T} \circ \widetilde{\Psi}^t = \Lambda^{-1} \circ \Gamma \circ \Psi^t \circ \mathcal{T}\\ &= \Lambda^{-1} \circ \Phi^t \circ \Gamma \circ \mathcal{T} = \widetilde{\Phi}^t \circ \Lambda^{-1} \circ \Gamma \circ \mathcal{T}\\ &= \widetilde{\Phi}^t \circ \widetilde{\Gamma}\,,\end{aligned}$$

which shows that $\widetilde{\Gamma}$ is an invariant $\widetilde{\omega}$-torus for $\widetilde{H}$.

1.4. Equivalence

Consider the group $\mathcal{G}$ of transformations $H_1 \mapsto H_4$ generated by

Scaling of energy/time: (E will be ignored later)

$$H_2 = \eta^{-1} H_1 - E\,.$$

Scaling of momenta:

$$H_3 = \mu^{-1} H_2 \circ \mathcal{S}_\mu\,, \qquad \mathcal{S}_\mu(q,p) = (q,\mu p)\,.$$

Symplectic changes of coordinates:

$$H_4 = H_3 \circ \mathcal{U}\,, \qquad \mathcal{U} \text{ homotopic to I}\,.$$

This defines an equivalence relation

$$\boldsymbol{H_4 \sim H_1 \pmod{\mathcal{G}}}\,.$$

Notice: equivalent Hamiltonians have invariant tori with the same frequency ratios ω_j/ω_d .

Interesting question: Can $H \circ \mathcal{T} \sim H$ hold for some nontrivial H and T?

---

2. Why renormalization

We describe observations that motivated renormalization, and then give a formal definition of the RG transformation.

2.1. Golden mean tori

Consider $d = 2$ and periodic orbits with rotation numbers approximating the inverse golden mean

$$\frac{a_n}{b_n} = \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8} \quad \to \quad \vartheta^{-1} = \frac{\sqrt{5}-1}{2} = 0.618033\ldots$$

The continued fraction approximants $\frac{a_n}{b_n}$ for ϑ^{-1} can be obtained via

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \end{bmatrix}, \ldots$$

We are interested in invariant tori with rotation vectors parallel to ω,

$$\omega = \begin{bmatrix} \vartheta^{-1} \\ 1 \end{bmatrix}, \qquad T\omega = \vartheta\omega\,, \qquad T = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

As a concrete example, consider the Hamiltonians

$$H^{(\beta)}(q,p) = \omega \cdot p + \tfrac{1}{2}p_1^2 + \beta\big[\cos(q_1) + \cos(q_1 - q_2)\big]\,,$$

and restrict to a "surface" of fixed energy, say $H = 0$. This is essentially the Hamiltonian that was used in [D.F. Escande '85].

For $\beta = 0$, we have an invariant ω-torus at $p = 0$, and periodic $\frac{a_n}{b_n}$-orbits at

$$p_1 = \frac{a_n}{b_n} - \vartheta^{-1} \approx (-\vartheta)^{-n}\,, \qquad p_2 = -\vartheta^{-1}p_1 - \tfrac{1}{2}p_1^2\,.$$

These orbits accumulate at a rate ϑ^{-n}, in the direction $\Omega = \left[\begin{smallmatrix} 1 \\ \vartheta^{-1} \end{smallmatrix}\right]$.
By KAM, the ω-torus persists for small $\beta > 0$.
So do the $\frac{a_n}{b_n}$-orbits, and the asymptotic accumulation ratio remains ϑ^{-1}.

2.2. Breakup

Observations for Hamiltonian flows [D.F. Escande and F. Doveil '81, ...] and area-preserving maps [L.P. Kadanoff '81, R.S. MacKay '82, ...] are:

Increasing β past a critical β_∞, the golden invariant torus <u>breaks up</u>.
After that, the $\frac{a_n}{b_n}$-orbits bifurcate from elliptic to hyperbolic.
These bifurcations occur at values $\beta_n \downarrow \beta_\infty$, and

$$\frac{\beta_{n+1} - \beta_n}{\beta_n - \beta_{n-1}} \to \delta_2^{-1}\,.$$

At $\beta = \beta_\infty$, the $\frac{a_n}{b_n}$-orbits accumulate at the ω-torus in an asymptotically geometric fashion, characterized by scaling constants $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

The scaling values are observed to be <u>**universal**</u>,
that is, independent of the family $\beta \mapsto H^{(\beta)}$.

$$\begin{aligned}
\delta_2 &= 1.6279502\ldots \\
\delta_1 &= -1/(\vartheta\mu_*) \\
\mu_* &= 0.230460196\ldots \\
\lambda_1 &= \vartheta \\
\lambda_3 &= -0.326063\ldots \\
\lambda_2 &= \mu_*/\lambda_3 \\
\lambda_4 &= \mu_*/\lambda_1
\end{aligned}$$

Expected explanation in a space $\mathcal{A}$ of analytic Hamiltonians:

The RG transformation $\mathcal{R}$ should be hyperbolic and satisfy

$$\mathcal{R}(\Sigma_{n+1}) \subset \Sigma_n\,, \qquad \mathcal{R}(H_*) = H_*\,.$$

2.3. Renormalization: basic ideas

The simplest map $\mathcal{R}$ with the property $\mathcal{R}(\Sigma_{n+1}) \subset \Sigma_n$ is

$$H \mapsto H \circ \mathcal{T}\,, \qquad \mathcal{T}(q,p) = \Big(Tq, (T^*)^{-1}p\Big)\,.$$

Namely, if γ is an orbit for $H \circ \mathcal{T}$ with rotation vector w, then $\mathcal{T}\gamma$ is an orbit for H, with rotation vector Tw.

So, as proposed first by [D.F. Escande and F. Doveil '81], we want

$$\mathcal{R}(H) \sim H \circ \mathcal{T} \quad (\text{mod } \mathcal{G})\,.$$

Re-normalization: Define $\mathcal{R}(H)$ to be the unique $\widetilde{H} \in \mathcal{A}^+$ that lies on the orbit of $\mathcal{G}$ through $H \circ \mathcal{T}$. Here, $\mathcal{A}^+$ is a subspace of $\mathcal{A}$ that is transversal to the orbits of $\mathcal{G}$. The Hamiltonians in $\mathcal{A}^+$ are said to be in "normal form".

More specifically,

$$\mathcal{R}(H) = \tfrac{1}{\eta\mu} H \circ \mathcal{T} \circ \mathcal{S}_\mu \circ \mathcal{U}\,,$$

where $\mathcal{S}_\mu(q,p) = (q, \mu p)$, and where $\mathcal{U}$ is a symplectic change of variables, homotopic to the identity. The quantities η, μ, and $\mathcal{U}$ depend on H.

For a local analysis, $\mathcal{U} \approx U$ for some fixed U, so we can write

$$\mathcal{R}(H_0) = H \circ \mathcal{U}_H\,, \qquad H = \frac{1}{\eta\mu} H_0 \circ \mathcal{T} \circ \mathcal{S}_\mu \circ U\,,$$

with $\mathcal{U}_H$ close to the identity.

Main issue: Find an appropriate normal form subspace $\mathcal{A}^+$ and control the map $H \mapsto \mathcal{U}_H$.

Here, and in what follows, T can be any $d \times d$ integer matrix with determinant ± 1, whose eigenvalues satisfy

$$\vartheta_1 > 1 > |\vartheta_2| \geq \ldots \geq |\vartheta_d| \,.$$

The largest eigenvalue $\vartheta = \vartheta_1$ is assumed to be simple. We will also need the corresponding eigenvectors,

$$T\Omega_j = \vartheta_j \Omega_j \,, \qquad j = 1, 2, \ldots, d \,.$$

---

3. Special cases and results

We start by renormalizing integrable Hamiltonians; then describe results for near-integrable and near-critical Hamiltonians. The connection with the commuting maps approach is described as well.

3.1. Integrable Hamiltonians

Start with the simple Hamiltonian

$$K^{(m)}(q,p) = \omega \cdot p + \frac{m}{2}(\Omega_d \cdot p)^2 \,, \qquad \omega = \Omega_1 \,.$$

If m is chosen nonzero, assume that ϑ_d is real and simple. Notice that $K^{(m)}$ is degenerate, but for $d = 2$ and $m \neq 0$ it is isoenergetically nondegenerate (meaning that $\ldots$).

Consider <u>integrable</u> Hamiltonians

$$H(q,p) = K^{(m)}(q,p) + h(z_1, z_2, \ldots, z_d) \,, \qquad z_j = \Omega_j \cdot p \,.$$

For such Hamiltonians we can choose $\mathcal{U} = \mathrm{I}$ in the definition of $\mathcal{R}$.
Then $\widetilde{H} = \mathcal{R}(H)$ is of the form

$$\widetilde{H}(q,p) = \frac{1}{\eta\vartheta_1}(\omega \cdot p) + \frac{\mu}{\eta\vartheta_d^2}\,\frac{m}{2}(\Omega_d \cdot p)^2 + \frac{1}{\eta\mu} h\big(\mu\vartheta_1^{-1} z_1, \ldots, \mu\vartheta_d^{-1} z_d\big) \,.$$

In particular, $K^{(m)}$ is a fixed point of $\mathcal{R}$ with the choice

$$\eta = \vartheta_1^{-1}, \qquad \mu = \vartheta_1^{-1}\vartheta_d^2 .$$

Consider these values fixed for now, in order to simplify the discussion.
For $h(z) = z_j^k$ we have

$$D\mathcal{R}\big(K^{(m)}\big)h = \frac{1}{\eta\mu}\big(\mu\vartheta_j^{-1}\big)^k h = \frac{\vartheta_1}{\vartheta_j}\big(\mu\vartheta_j^{-1}\big)^{k-1} h = \frac{\vartheta_1}{\vartheta_j}\left(\frac{\vartheta_d^2}{\vartheta_1\vartheta_j}\right)^{k-1} h .$$

<u>expanding</u> directions: For each $j > 1$, there is an eigenvector $h = \Omega_j \cdot p$ with eigenvalue ϑ_1/ϑ_j . They are in some sense trivial: If the Hamiltonian has a nondegenerate quadratic part, then these eigenvalues could be eliminated via p-translations. But it is easier to deal with this "later".

<u>neutral</u> directions: $h = \omega \cdot p$ yields an eigenvalue 1. This eigenvalue can be eliminated by choosing η appropriately. $h = (\Omega_d \cdot p)^2$ also yields an eigenvalue 1. This eigenvalue can be eliminated by choosing μ appropriately. So the neural directions are trivial and will mostly be ignored.

<u>contracting</u> directions: all others.

Remarks.

- The RG analysis will mostly be done with $m = 0$. Then there is no restriction on μ. We can take μ as small (but positive) as is convenient. In particular, $h = (\Omega_d \cdot p)^2$ then contracts under $D\mathcal{R}\big(K^{(0)}\big)$.
- No new spectrum will appear when we extend $\mathcal{R}$ to near-integrable Hamiltonians.

3.2. Near-integrable Hamiltonians

The goal here it to prove KAM-type theorems.
For frequency vectors $\omega \in \mathbb{R}^d$ that admit a <u>periodic</u> continued fractions expansion

$$\omega = \lim_{m\to\infty} c_m T^m w , \qquad T \in \mathrm{SL}(d, \mathbb{Z}) , \quad w \in \mathbb{Q}^d .$$

(1) Define $\mathcal{R}$ in some neighborhood of a trivial fixed point K.
The infinitely renormalizable Hamiltonians near K are the ones lying on the local stable manifold of $\mathcal{R}$ at K.

(2) Prove that every infinitely renormalizable Hamiltonian has an analytic ω-torus.
Show that "sufficiently nondegenerate" Hamiltonians in the domain of $\mathcal{R}$ can be p-translated to make them infinitely renormalizable.

Generalizations include: Hamiltonians with shearless tori, Diophantine frequency vectors (see below), Brjuno frequency vectors, Hamiltonians with shearless tori, non-Hamiltonian vector fields, skew flows, lower dimensional tori, ...

Work by: H.K., J. Lopes Dias, J. Abad, K. Khanin, J. Marklof, D. Gaidashev, S. Kocić, ...

Concerning Diophantine frequency vectors:
The continued fractions expansion is of the form

$$\omega = \lim_{m\to\infty} c_m T_1 T_2 \cdots T_m w\,, \qquad T_n \in \mathrm{SL}(d,\mathbb{Z})\,, \quad w \in \mathbb{Q}^d\,.$$

For the standard continued fractions expansion in $d=2$, applying T_n to $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ with $0<a<b$ corresponds to prepending a digit k_n to $r=a/b$,

$$T_n = \begin{bmatrix} 0 & 1 \\ 1 & k_n \end{bmatrix}\,, \qquad r = [r_1, r_2, \ldots] \mapsto [k_n, r_1, r_2, \ldots] = \frac{1}{k_n + r}\,.$$

For practical purposes, a Hamiltonian H_0 is infinitely renormalizable if

$$\|H_n - K_n\| \to 0\,, \qquad H_n = \mathcal{R}_n(H_{n-1})\,,$$

where $\{K_n\}$ is a suitable sequence of trivial Hamiltonians.

More details will be given later.

3.3. Connection with commuting maps

Consider Hamiltonians

$$H(q,p) = \omega \cdot p + h(q, z_2, \ldots, z_d)\,, \qquad z_j = \Omega_j \cdot p\,.$$

Let ω' be the expanding eigenvector of T^*, normalized such that $\omega \cdot \omega' = 1$. Then $\omega' \cdot \Omega_j = 0$ for all $j > 1$. Thus,

$$\frac{d}{dt}\omega' \cdot q = \omega' \cdot \nabla_p H = \omega' \cdot \omega + \omega' \cdot \nabla_p h = 1\,.$$

So $\omega' \cdot q$ is "time". This implies e.g. that the maps

$$F_j(q,p) = \Phi^{2\pi\omega'_j}(q - 2\pi\delta_j, p)\,, \qquad j = 1, 2, \ldots, n,$$

leave the section

$$X = \bigl\{(q,p) \in H^{-1}(0) : \omega' \cdot q = 0\bigr\}$$

invariant. (Here we consider the lift to $\mathbb{R}^d \times \mathbb{R}^d$.) Furthermore, they commute with each other, and their restriction to X is symplectic. The same holds for the maps $\widetilde{F}_j$ determined by the renormalized Hamiltonian

$$\widetilde{H} = \frac{\vartheta_1}{\mu} H \circ \Lambda\,, \qquad \Lambda = \mathcal{T} \circ \mathcal{S}_\mu \circ \mathcal{U}\,.$$

And a straightforward computationⓇ shows that

$$\boldsymbol{\widetilde{F}_j = \Lambda^{-1} \circ F_1^{T_{1,j}} \circ F_2^{T_{2,j}} \circ \ldots \circ F_d^{T_{d,j}} \circ \Lambda}\,.$$

In the golden mean case,

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \begin{bmatrix} \widetilde{F}_1 \\ \widetilde{F}_2 \end{bmatrix} = \begin{bmatrix} \Lambda^{-1} \circ F_2 \circ \Lambda \\ \Lambda^{-1} \circ F_1 \circ F_2 \circ \Lambda \end{bmatrix}.$$

The map $(F_1, F_2) \mapsto (\widetilde{F}_1, \widetilde{F}_2)$ is MacKay's RG transformation for (commuting) pairs of area-preserving maps.

3.4. Near-critical Hamiltonians

Restrict to the golden mean case

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \omega = \begin{bmatrix} \vartheta^{-1} \\ 1 \end{bmatrix}, \qquad \Omega = \begin{bmatrix} 1 \\ -\vartheta^{-1} \end{bmatrix}.$$

Early computations with highly approximate RG transformations:
A. Mehr, D.F. Escande, C. Chandre, M. Govin, H.R. Jauslin, ...

For simplicity, consider only Hamiltonians of the form

$$H(q,p) = \omega \cdot p + \sum_{\nu,k} h_{\nu,k} \cos(\nu \cdot q)(\Omega \cdot p)^k .$$

To find the critical fixed point for $\mathcal{R}$, we decompose

$$\mathcal{R} = \mathcal{N} \circ \mathcal{L} \circ \mathcal{K} ,$$

where $\mathcal{K}$ is trivial, $\mathcal{L}$ linear, and $\mathcal{N}$ close to the identity:

$$\begin{aligned} (\mathcal{K}H)(q,p) &= c_H H(q, p/c_H) , \qquad\qquad c_H = 2h_{0,2} , \\ \mathcal{L}H &= \vartheta \mu_0^{-1} H \circ \mathcal{T} \circ \mathcal{S}_{\mu_0} \circ U_0 , \\ \mathcal{N}(H) &= H \circ \mathcal{U}_H . \end{aligned}$$

The construction of $\mathcal{U}_H$ involves a Newton-type iteration,

$$\mathcal{U}_H = U_1 \circ U_2 \circ U_3 \circ \ldots$$

This $\mathcal{R}$ was investigated numerically in [J. Abad, H.K., P. Wittwer '98].

Theorem. [H.K. '04] *Existence of a nontrivial fixed point H_* for $\mathcal{R}$. Bound from p.6 on μ_* .*

Theorem. [H.K. '07] *If H is near H_* and $\mathcal{R}^n(H) \to H_*$ exponentially, then H has a non-smooth golden invariant torus; that is, Γ is not differentiable. Bounds from p.6 on $\lambda_1, \ldots, \lambda_4$.*

And for completeness ...

Theorem [G. Arioli, H.K. '09]. *MacKay's RG transformation for pairs of area-preserving maps has a fixed point* (F_*, G_*). *The maps* F_* *and* G_* *are analytic, area-preserving, reversible, and they commute. The associated scaling is*

$$\Lambda_* = \begin{bmatrix} -0.7067956691\ldots & 0 \\ 0 & -0.3260633966\ldots \end{bmatrix} .$$

The proofs of these theorems are computer-assisted.

Other cases:

- One expects similar results with $T = \left[\begin{smallmatrix} 0 & 1 \\ 1 & k \end{smallmatrix}\right]$.
- Possibly also for $d \times d$ matrices with a real contracting eigenvalue.
- For the spiral mean, a 3×3 matrix with non-real contracting eigenvalue-pair, a very rough RG analysis by [C. Chandre, H.R. Jauslin, G. Benfatto '99] suggests the existence of a strange non-chaotic RG attractor.
- Critical phenomena were observed for shearless golden tori [D. Del-Castillo-Negrete, J.M. Greene, P.J. Morrison, A. Apte, A. Wurm, ...] A numerical RG analysis [D. Gaidashev, H.K. '04] yields a RG period 12.

---

4. Scaling

The goal here is to control the scaling

$$H \mapsto \frac{1}{\eta\mu} H \circ \mathcal{T} \circ \mathcal{S}_\mu \,.$$

for "resonant" Hamiltonians. To simplify notation, let

$$\mathcal{T}_\mu = \mathcal{T} \circ \mathcal{S}_\mu \,, \qquad \mathcal{T}_\mu(q,p) = \bigl(Tq, \mu(T^*)^{-1}p\bigr) \,.$$

4.1. Analytic Hamiltonians

We need to make a technical choice between

(A) Renormalization with fixed T. Then it makes sense e.g. to use eigenvectors. Or an RG that preserves the size of p-quadratic term. Then we cannot take $|\mu| > 0$ small.

(B) Other cases, including RG sequences with varying matrices T_n. We can assume that $\|T\|$ is large; if not, replace T by some power of T.

We choose here (B), and $|\mu| > 0$ small.

We consider Hamiltonians that are analytic on

$$D_\rho = \left\{(q,p) \in \mathbb{C}^d \times \mathbb{C}^d : \quad |\operatorname{Im} q_j| < \rho\,, \quad |p_j| < \rho\,, \quad \forall j\right\}.$$

Problem: $\mathcal{T}_\mu$ does not map D_ρ into itself: Some q-direction gets expanded by T. So if H is analytic on D_ρ then $H \circ \mathcal{T}_\mu$ need not be.

Given any $\rho > 0$, denote by $\mathcal{A}_\rho$ the space of functions

$$H(q,p) = \sum_{(\nu,\alpha)\in I} H_{\nu,\alpha} e^{i\nu\cdot q} p^\alpha\,, \qquad p^\alpha = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_d^{\alpha_d}\,,$$

where $I = \mathbb{Z}^d \times \mathbb{N}^d$, equipped with the norm

$$\|H\|_\rho = \sum_{(\nu,\alpha)\in I} |H_{\nu,\alpha}| e^{\rho|\nu|} \rho^{|\alpha|}\,, \qquad |\alpha| = \sum_j |\alpha_j|\,.$$

Notice: If $r > \rho$ then the inclusion map from $\mathcal{A}_r$ into $\mathcal{A}_\rho$ is compact.

Define for $J \subset I$ the projection

$$(\mathbb{P}(J)H)(q,p) = \sum_{(\nu,\alpha)\in J} H_{\nu,\alpha} e^{i\nu\cdot q} p^\alpha\,.$$

Special cases: the torus-average $\mathbb{E}_0$ and

$$(\mathbb{E}_k H)(q,p) = \sum_{|\alpha|\geq k} H_{0,\alpha} p^\alpha\,.$$

Convention. Throughout the analysis,
restrict $c < \rho \le r < c^{-1}$ for some fixed positive $c < 1$.
Furthermore, the dimension d is considered fixed. With this in mind,
C denotes a universal constant that may vary from one place to another.

4.2. Resonant modes

For a single "mode" $H(q,p) = e^{i\nu\cdot q}p^\alpha$, we have

$$H \circ \mathcal{T}_\mu = e^{i(T^*\nu)\cdot q}\bigl(\mu(T^*)^{-1}p\bigr)^\alpha,$$

and

$$\|H \circ \mathcal{T}_\mu\|_r \le e^A \|H\|_\rho, \quad A = r|T^*\nu| - \rho|\nu| + |\alpha| \ln\bigl(N_T|\mu|\tfrac{r}{\rho}\bigr).$$

So the mode is "nice" if $|\mu|$ is sufficiently small, and
either $|T^*\nu| \ll |\nu|$ or $|T^*\nu| \ll |\alpha|$. We call these modes "resonant" …

Definition. *Given a positive $\tau \ll 1$ the resonant modes are those indexed by*

$$I^+ = \bigl\{(\nu,\alpha) \in I : \quad |T^*\nu| \le \tau|\nu| \quad \text{or} \quad |T^*\nu| \le C\tau|\alpha| \bigr\}.$$

The resonant projection is defined as $\mathbb{I}^+ = \mathbb{P}(I^+)$.

Consider $r \ge \rho$. Fix $\gamma > 0$. Assume that (ν,α) belongs to I^+.

Case 1: $\nu = 0$.
Here e^A is bounded by $\bigl(CN_T|\mu|\bigr)^{|\alpha|}$.

Case 2: $0 < |T^*\nu| \le \tau|\nu|$.
Here $|\nu| \ge \tau^{-1}$, and we gain a small factor $e^{-C\tau^{-1}}$.

Case 3: $0 < \tau|\nu| < |T^*\nu| \le C\tau|\alpha|$.
Here $|\alpha| > C^{-1}$. For small $|\mu|$ we gain a factor $\bigl(CN_T|\mu|\bigr)^\gamma$.

So under a smallness condition on τ and $|\mu|$ like

$$e^{-C\tau^{-1}} < \bigl(CN_T|\mu|\bigr)^\gamma < 1,$$

we obtain the

Proposition. *$H \mapsto H \circ \mathcal{T}_\mu$ is compact from $\mathbb{I}^+\mathcal{A}_\rho$ to $\mathcal{A}_r$. For $H \in \mathbb{I}^+\mathcal{A}_\rho$,*

$$\|\mathbb{E}_k H \circ \mathcal{T}_\mu\|_r \le C\big(CN_T|\mu|\big)^k \|\mathbb{E}_k H\|_\rho\,,$$
$$\big\|[(\mathbb{I}-\mathbb{E}_0)H] \circ \mathcal{T}_\mu\big\|_r \le C\big(CN_T|\mu|\big)^\gamma \big\|(\mathbb{I}-\mathbb{E}_0)H\big\|_\rho\,.$$

Recall that
$$\mathcal{R}(H) = \frac{1}{\eta\mu} H \circ \mathcal{T}_\mu \circ \mathcal{U}\,.$$

The above proposition shows that the first RG step $H \mapsto \frac{1}{\eta\mu} H \circ \mathcal{T}_\mu$ is <u>analyticity-improving</u> on the subspace of resonant Hamiltonians. Furthermore, the second bound implies that the q-dependent parts of H <u>contract strongly</u> under renormalization. This fact will be used e.g. in the construction of invariant tori.

The idea is to use "resonant" as our "normal form".

---

5. Nonresonant modes

Converting a Hamiltonian to "normal form" means eliminating nonresonant modes. The following Basic Fact is crucial for this procedure.

Definition. The <u>nonresonant modes</u> are those indexed by

$$I^- = \Big\{(\nu,\alpha) \in I : \quad |T^*\nu| > \tau|\nu| \quad \textbf{and} \quad |T^*\nu| > C\tau|\alpha| \ \Big\}\,.$$

The <u>nonresonant projection</u> is $\mathbb{I}^- = \mathbb{P}(I^-)$.

Notice that $I^- \cup I^+ = I$ and $\mathbb{I}^- + \mathbb{I}^+ = \mathbb{I}$.
The constant $0 < \tau < 1$ is mainly determined by the following.

Assume that there exists some nonzero $\omega \in \mathbb{R}^d$ such that $\sqrt{d}\,T^*$ contracts distances in $\omega^\perp$ by a factor at least $\frac{1}{2}\tau$. Choose $\sigma > 0$ such that

$$\sqrt{d}\,\|T^*\| \le \tfrac{1}{2}\sigma^{-1}\tau\,.$$

Basic Fact. *On $\mathbb{I}^{-}\mathcal{A}_\rho$, all derivatives are bounded by $\omega \cdot \nabla_q$, and ...*

Proof, ignoring factors like C and $\sqrt{d}$. Let $(\nu, \alpha) \in I^-$. Consider the decomposition $\nu = \nu_\parallel + \nu_\perp$ into a vector $\nu_\parallel$ parallel to ω and a vector $\nu_\perp$ perpendicular to ω. Using the contraction property of T^*, and $\|T^*\nu\| > \tau\|\nu\|$, we get

$$\|T^*\nu_\perp\| \leq \frac{\tau}{2}\|\nu_\perp\| \leq \frac{\tau}{2}\|\nu\| < \frac{1}{2}\|T^*\nu\|.$$

Thus,

$$\tfrac{1}{2}\|T^*\nu\| \leq \|T^*\nu_\parallel\| \leq \|T^*\|\|\nu_\parallel\| \leq \tfrac{1}{2}\sigma^{-1}\tau\|\nu_\parallel\| = \tfrac{1}{2}\sigma^{-1}\tau\|\omega \cdot \nu\|.$$

Combined with the definition of I^-, this yields

$$|\omega \cdot \nu| > C\sigma|\nu|, \qquad |\omega \cdot \nu| > C\sigma|\alpha|,$$

Thus, on $\mathbb{I}^{-}\mathcal{A}_\rho$ we have

$$\left\|\frac{\nabla_q}{\omega \cdot \nabla_q}\right\| \leq \frac{|\nu|}{|\omega \cdot \nu|} \leq \frac{1}{C\sigma}, \qquad \rho\left\|\frac{\nabla_p}{\omega \cdot \nabla_q}\right\| \leq \frac{|\alpha|}{|\omega \cdot \nu|} \leq \frac{1}{C\sigma}.$$

QED

---

6. Reduction to normal form

We consider the problem of reducing a near-resonant Hamiltonian to normal (resonant) form via a small change of coordinates. This is done more generally for vector fields.

The vector field defined by a Hamiltonian H is

$$X_H = \mathbb{J}\nabla H = \begin{bmatrix} \nabla_q H \\ -\nabla_p H \end{bmatrix}.$$

6.1. Vector fields

Denote by $\mathcal{A}_\rho$ the space of analytic vector fields on D_ρ,

$$X(q,p) = \sum_{\nu,\alpha} X_{\nu,\alpha} e^{i\nu\cdot q} z^\alpha\,, \qquad \|X\|_\rho = \sum_{\nu,\alpha} \|X_{\nu,\alpha}\| e^{\rho|\nu|}\rho^{|\alpha|}\,,$$

where $\|.\|$ denotes the ℓ^∞ norm on $\mathbb{C}^d \times \mathbb{C}^d$.
Some basic properties are

Proposition. *Let $Y \in \mathcal{A}_r$ and $Z \in \mathcal{A}_\rho$ with $0 \le \rho \le r$. Then*
(a) $\|(DY)Z\|_\rho \le (r-\rho)^{-1}\|Y\|_r\|Z\|_\rho$ *if* $\rho < r$.
(b) $\|Y \circ [\mathrm{I} + Z]\|_\rho \le \|Y\|_r$ *if* $\rho + \|Z\|_\rho \le r$.

The flow $t \mapsto \Phi^t_Y$ associated with a vector field is obtained by solving

$$\frac{d}{dt}\Phi^t_Y = Y \circ \Phi^t_Y\,, \qquad \Phi^0_Y = \mathrm{I}\,.$$

Converting this to an integral equation for $Z(t) = \Phi^t_Y - \mathrm{I}$,
and using property (b) above, one obtains

Proposition. *Let $\rho, r, \theta > 0$ with $\rho + \theta\|Y\|_r < r$. Then*

$$\|\Phi^t_Y - \mathrm{I}\|_\rho \le \|tY\|_r\,, \qquad |t| \le \theta\,.$$

If X is a vector field, defined on the range of a diffeomorphism Φ, then the pullback of X under Φ is given by

$$\Phi^* X = (D\Phi)^{-1}(X \circ \Phi)\,.$$

For the time-t map Φ^t_Y associated with a vector field Y,

$$(\Phi^t_Y)^* X = e^{t\widehat{Y}} X = \sum_n \frac{1}{n!}\big(t\widehat{Y}\big)^n X\,,$$

where

$$\widehat{Y} X = [Y, X] = (DX)Y - (DY)X\,.$$

Denote by $\mathcal{A}'_r$ the space of all $Y \in \mathcal{A}_r$ such that DY is a bounded linear operator on $\mathcal{A}_r$, equipped with the norm

$$\|Y\|'_r = \|Y\|_r + \|DY\|_r \,.$$

Proposition. *Let $0 < \delta < r$ and $\varepsilon < \frac{1}{6}$ and $\|Y\|'_r \le \delta\varepsilon$. Then*

$$\begin{aligned} \big\|(\Phi^1_Y)^* X - X\big\|_{r-\delta} &\le C\varepsilon \|X\|_r \,, \\ \big\|(\Phi^1_Y)^* X - X - [Y, X]\big\|_{r-\delta} &\le C\varepsilon^2 \|X\|_r \,. \end{aligned}$$

Proof sketch. To estimate $\frac{1}{n!}(\widehat{Y})^n X$, we use up a domain δ/n for each operator $\widehat{Y}$. This gives a factor $(\delta + n)\varepsilon$ per operator, and thus a factor

$$\frac{1}{n!}(\delta + n)^n \varepsilon^n \le \frac{1}{2} e^\delta (e\varepsilon)^n \,.$$

QED

6.2. Eliminating nonresonant modes

The RG transformation for vector fields is

$$\mathcal{R}(X_0) = \mathcal{U}_X^* X \,, \qquad X = \eta^{-1} \mathcal{T}_\mu^* X_0 \,.$$

As for Hamiltonians, if $X_0 \in \mathcal{A}_\rho$ is resonant, then $X \in \mathcal{A}_r$ for some $r > \rho$.

Consider now the step $X \mapsto \mathcal{U}_X^* X$. The goal is to eliminate all nonresonant modes, so the renormalized vector field $\mathcal{R}(X)$ is again in normal form (meaning resonant).

Let $\mathbb{I}^- = \mathbb{I} - \mathbb{I}^+$. Given a vector field X with a small nonresonant part $\mathbb{I}^- X$, we will construct a change of variables

$$\mathcal{U}_X = \Phi_Y \circ \Phi_{Y'} \circ \Phi_{Y''} \circ \dots \,,$$

such that

$$\mathbb{I}^{-}\mathcal{U}_X^* X = 0\,.$$

For the first step, find a <u>small</u> nonresonant Y that solves

$$\mathbb{I}^{-}\big(\,X + [Y,X]\,\big) = 0\,. \tag{$*$}$$

Changing variables with Φ_Y^1 reduces the nonresonant part of X to

$$\mathbb{I}^{-}(\Phi_Y^1)^* X = \mathbb{I}^{-}\big(\,X + [Y,X] + \text{small}^2\,\big) = \text{small}^2\,.$$

Then iterate …

The equation $(*)$ and its formal solution can be written as

$$\mathbb{I}^{-}\widehat{X}Y = \mathbb{I}^{-}X\,, \qquad Y = \big(\mathbb{I}^{-}\widehat{X}\big)^{-1}\mathbb{I}^{-}X\,.$$

Proposition. *If $X \in \mathcal{A}_r'$ is sufficiently close to $K = \left[\begin{smallmatrix}\omega\\ 0\end{smallmatrix}\right]$, then $\mathbb{I}^{-}\widehat{X} :\, \mathbb{I}^{-}\mathcal{A}_r' \to \mathbb{I}^{-}\mathcal{A}_r$ has a bounded inverse, and …*

Proof. Let $Y, Z \in \mathcal{A}_r'$. Then

$$\big\|\widehat{Z}Y\big\|_r \le \|Z\|_r'\|Y\|_r'\,.$$

Assume Y is nonresonant. Notice that $\widehat{K} = \omega\cdot\nabla_q$. According to “Basic Fact” we have

$$\big\|Y\|_r' \le C\sigma^{-1}\big\|\mathbb{I}^{-}\widehat{K}Y\big\|_r\,.$$

Thus, if $X = K + Z$ with $\|Z\|_r'$ sufficiently small,

$$\big\|\mathbb{I}^{-}\widehat{X}Y\big\|_r \ge \big\|\mathbb{I}^{-}\widehat{K}Y\big\|_r - \big\|\mathbb{I}^{-}\widehat{Z}Y\big\|_r \ge \big(\sigma C^{-1} - \|Z\|_r'\big)\|Y\|_r'\,.$$

QED

Combining this with our earlier bounds, we get

$$\big\|\mathbb{I}^{-}(\Phi_Y^1)^*X\big\|_{r-\delta} = \big\|\mathbb{I}^{-}\big((\Phi_Y^1)^*X - X - [Y,X]\big)\big\|_{r-\delta} \le C\varepsilon^2\|X\|_r\,,$$

with

$$\varepsilon = \frac{1}{r}\|Y\|_r' = \frac{1}{r}\big\|(\mathbb{I}^{-}\widehat{X})^{-1}\mathbb{I}^{-}X\big\|_r' \le \frac{2C}{r\sigma}\|\mathbb{I}^{-}X\|_r\,.$$

7. Combining the steps

Apply the above with $r = \rho + \delta$ and $\delta > 0$ small.
Assuming $X_0 \in \mathcal{A}_\rho$ is resonant,

$$X = \eta^{-1} \mathcal{T}_\mu^* X_0 \ \in \ \mathcal{A}'_r \,.$$

And if $\|X_0 - K\|_\rho$ is sufficiently small, then

$$\mathcal{R}(X_0) = \mathcal{U}_X^* X \ \in \mathcal{A}_\rho \,.$$

Given that the iterative procedures converge absolutely, we have

Theorem. *Given $\rho > 0$ there exists an open neighborhood $\mathcal{B}$ of $K = \left[\begin{smallmatrix}\omega\\0\end{smallmatrix}\right]$, such that $\mathcal{R}$ is well defined, analytic, and compact, as a map from $\mathcal{B}$ to $\mathcal{A}_\rho$.*

In addition, all eigenvectors of $D\mathcal{R}(K)$ belong to the subspace $\mathbb{E}_0\mathcal{A}_\rho$ of vector fields $X_0(q, p)$ that only depend on p. These "integrable" vector fields are resonant, so $\mathcal{U}_X$ is the identity.

Remarks.

- For matrices T with large norm, τ is of the order $\|T\|^{-c}$ with $0 < c < 1$, and σ is of the order $\|T\|^{-1-c}$.
- μ can be taken to be of the order $\sigma^{1+\varepsilon}$
- The domain $\mathcal{B}$ contains a ball whose radius is of the order σ^2.

---

8. Diophantine frequencies

We sketch a multi-dimensional continued fractions expansion

$$\omega_n = \eta_n^{-1} T_n^{-1} \omega_{n-1} \,, \qquad n = 1, 2, \ldots$$

for Diophantine frequency vectors $\omega_0 \in \mathbb{R}^n$, and then describe the corresponding renormalization procedure.

8.1. Continued fractions expansion

The following multidimensional continued fractions expansion has been developed by [K. Khanin, J. Lopes Dias, J. Marklof '05]. It is based on work by [J.C. Lagarias '94] and [D.Y. Kleinbock, G.A. Margulis '98].

Consider the group $\mathrm{SL}(d,\mathbb{R})$ and the 1-parameter subgroup

$$E^t = \mathrm{diag}\Bigl(e^{-t},\ldots,e^{-t},e^{(d-1)t}\Bigr).$$

To a given Diophantine vector $\omega_0 = \left[\begin{smallmatrix} w \\ 1 \end{smallmatrix}\right]$ in $\mathbb{R}^{d-1}\times\mathbb{R}$ associate

$$W_0 = \begin{bmatrix} \mathrm{I} & w \\ 0 & 1 \end{bmatrix} \;\in\; \mathrm{SL}(d,\mathbb{R}).$$

Consider

$$W_0E^t = \begin{bmatrix} e^{-t}\mathrm{I} & e^{(d-1)t}w \\ 0 & e^{(d-1)t} \end{bmatrix}, \qquad t\ge 0.$$

Choose a fundamental domain $F \ni W_0$ for the left action of $\mathrm{SL}(d,\mathbb{Z})$ on $\mathrm{SL}(d,\mathbb{R})$, and define $P(t)\in\mathrm{SL}(d,\mathbb{Z})$ by

$$W(t)\in F\,, \qquad W(t) = P(t)W_0E^t\,.$$

Roughly speaking, $P(t)$ is the "integer part" of the inverse of W_0E^t.
Now pick appropriate "stopping times" $0=t_0<t_1<t_2<\ldots$ and set

$$P_n = P(t_n)\,, \qquad T_n = P_{n-1}P_n^{-1}\,, \qquad \omega_n = c_n^{-1}P_n\omega_0\,,$$

with c_n chosen e.g. such that $\|\omega_n\|=1$.

Let now $\omega\in\mathrm{Diophantine}(\beta)$, and define $\theta=\beta/(d+\beta)$.

Theorem [KLM]. *For $t\ge 0$,*

$$\bigl\|W(t)\bigr\| \le c_1e^{(d-1)\theta t}\,, \qquad \bigl\|W(t)^{-1}\bigr\| \le c_2e^{\theta t}\,.$$

Corollary. *Useful bounds on $P_n\,,P_n^{-1}\,,T_n\,,T_n^{-1}$.*

Recall that

$$W_n = P_n W_0 E^{t_n}\,, \qquad P_n = W_n E^{-t_n} W_0^{-1}\,.$$

Hyperbolicity follows from

$$\begin{aligned} T_n^*\xi &= \big(P_n^{-1}\big)^* P_{n-1}^*\xi \\ &= \big(W_n^{-1}\big)^* E^{t_n} W_0^* \big(W_{n-1}E^{-t_{n-1}}W_0^{-1}\big)^*\xi \\ &= \big(W_n^{-1}\big)^* E^{t_n - t_{n-1}} W_{n-1}^*\xi \\ &= e^{-(t_n-t_{n-1})}\big(W_n^{-1}\big)^* W_{n-1}^*\xi\,, \qquad \text{if} \quad \xi \perp \omega_{n-1}\,, \end{aligned}$$

by choosing $t_1 \ll t_2 \ll \ldots$ appropriately. Notice that $\theta < 1$.

Here we have used that
- ◦ W_{n-1} maps the expanding subspace of E^t into the direction ω_{n-1} .
- ◦ So W_{n-1}^* maps $\omega_{n-1}^\perp$ into the contracting subspace of E^t.

A typical cutting sequence is

$$t_n = a(1+b)^n\,,$$

with $a, b > 0$ chosen in such a way that $\|T_n^*\xi\| \ll \|\xi\|$ for $\xi \perp \omega_{n-1}$.

8.2. Infinite renormalizability

The above can be applied to renormalize Hamiltonians [K. Khanin, J. Lopes Dias, J. Marklof '05] and more general vector fields [H.K., S. Kocić '08].

For each $n > 0$, after choosing proper RG parameters τ_n , σ_n , μ_n , we define an RG transformation $\mathcal{R}_n$ in some neighborhood of

$$K_{n-1} = \begin{bmatrix} \omega_{n-1} \\ 0 \end{bmatrix}\,.$$

Using that the projections $\mathbb{E}_k$ in the contraction bound are the same for each transformation $\mathcal{R}_n$, it is possible to (prove and) apply a stable manifold theorem for the sequence of maps $\mathfrak{R} = (\mathcal{R}_1, \mathcal{R}_2, \ldots)$.

More precisely, there exists an analytic manifold $\mathcal{W}^s$ of finite codimension, passing through $K_0 = \left[\begin{smallmatrix}\omega_0\\ 0\end{smallmatrix}\right]$, such that

$$\|X_n - K_n\|_\rho \to 0\,, \qquad X_n = \mathcal{R}_n(X_{n-1})\,.$$

exponentially, whenever X_0 belongs to $\mathcal{W}^s$.

Remarks.

- The largest set of frequency vectors for which KAM-type theorems have been proved are the "Brjuno vectors". However, appropriate bounds on the multidimensional continued fractions expansion are still missing. So the above does not (yet) generalize to Brjuno frequencies.
- The matrices $P(t) \in \mathrm{SL}(d,\mathbb{Z})$ can be viewed as integer approximations to the real matrices $\left(W_0 E^t\right)^{-1}$. If one is willing to allow frequency lattices other than $\mathbb{Z}^d$, then one could renormalize with real matrices instead. No continued fractions expansion would be needed.
- Such an RG analysis has been carried out for Brjuno vectors by [H.K., S. Kocić '10].

---

9. Invariant tori

For simplicity, we restrict again to Hamiltonians. In the near-integrable case, the goal is to construct an invariant ω_0-torus for every H_0 lying on the (local) strong $\mathfrak{R}$-stable manifold $\mathcal{W}^s$ at K_0. Here,

$$K_n(q,p) = \omega_n \cdot p\,, \qquad \omega_n = \eta_n^{-1} T_n^{-1} \omega_{n-1}\,.$$

Critical tori will be discussed at the end.

9.1. The torus equation

When using fixed time-normalization factors η_n^{-1}, the $\mathfrak{R}$-stable manifold $\mathcal{W}^s$ at K_0 is of codimension d, with the d unstable directions being the linear functions $(q,p) \mapsto w \cdot p$ with $w \in \mathbb{R}^d$.

If H is a sufficiently nondegenerate Hamiltonian near $\mathcal{W}^s$, then we can perform a p-translation to move it on $\mathcal{W}^s$.

If we choose the time-normalization factors η_n^{-1} in the RG procedure appropriately (depending on the Hamiltonian), then the number of unstable directions reduces to $d-1$. In this case, any sufficiently isoenergetically nondegenerate Hamiltonian near $\mathcal{W}^s$ can be p-translated onto $\mathcal{W}^s$.

Consider now a fixed $H_0 \in \mathcal{W}^s$ and its RG iterates $H_n = \mathcal{R}_n(H_{n-1})$. Recall that H_n is of the form

$$H_n = \frac{1}{\eta_n \mu_n} H_{n-1} \circ \Lambda_n \,, \qquad \Lambda_n = \mathcal{T}_n \circ \mathcal{S}_{\mu_n} \circ \mathcal{U}_n \,.$$

The equation for an invariant ω_n-torus Γ_n for H_n is

$$\Phi_n^t \circ \Gamma_n = \Gamma_n \circ \Psi_n^t \,,$$

where Φ_n and Ψ_n are the flows for H_n and K_n , respectively.
Using the two "basic identities"

$$\Lambda_n \circ \Phi_n^t = \Phi_{n-1}^{\eta_n^{-1} t} \circ \Lambda_n \,, \qquad \mathcal{T}_n \circ \Psi_n^t = \Psi_{n-1}^{\eta_n^{-1} t} \circ \mathcal{T}_n \,.$$

one easily gets the

Proposition. *If Γ_n is an invariant ω_n-torus for H_n, taking values in in the domain of Λ_n, then $\Gamma_{n-1} = \mathcal{M}_{n-1}(\Gamma_n)$ is an invariant ω_{n-1}-torus for H_{n-1}, where*

$$\mathcal{M}_{n-1}(F) = \Lambda_n \circ F \circ \mathcal{T}_n^{-1} .$$

So from an invariant torus Γ_m for H_m one gets a sequence of invariant tori

$$\boldsymbol{\Gamma_n = \mathcal{M}_n(\Gamma_{n+1})}, \qquad n = 0, 1, \ldots, m-1 .$$

9.2. Solving the equations

The idea is to construct a sequence $\Gamma_0, \Gamma_1, \ldots$ via inverse limits

$$\Gamma_{n,m} = \bigl(\mathcal{M}_n \circ \mathcal{M}_{n+1} \circ \ldots \circ \mathcal{M}_{m-1}\bigr)(F_m) \xrightarrow{m\to\infty} \Gamma_n ,$$

that are hopefully independent of the sequence $\{F_m\}$. Then show that Γ_n is an invariant torus for H_n.

Since we want $\Gamma_n = \mathrm{I} + \gamma_n$ with $\gamma_n(q, 0)$ periodic, consider

$$\mathcal{N}_n(f) = \mathcal{M}_n(\mathrm{I} + f) - \mathrm{I} .$$

<u>The problem</u> is that the maps $\mathcal{N}_n$ are not typically contractions on a fixed space, except under special circumstances (see below).

In particular, $\mathcal{T}_n$ is expanding in one q-direction, and so the same is to be expected for $\Lambda_n = \mathcal{T}_n \circ \mathcal{S}_{\mu_n} \circ \mathcal{U}_n$. For an RG analysis of near-integrable Hamiltonians, this can be solved by choosing a sequence of spaces $\mathcal{F}_n$ such that $\mathcal{N}_n : \mathcal{F}_{n+1} \to \mathcal{F}_n$ is a contraction. (Use norms that increase with n.) This is possible due to the fast convergence of $H_n - K_n \to 0$.

Furthermore, $\mathcal{T}_n^{-1}$ is expanding in many directions, so $f \mapsto f \circ \mathcal{T}_n$ is “bad” for smooth functions f. This can be solved by considering spaces of low-regularity functions. The drawback is that the so-constructed tori are not known to be smooth. More on this later ...

Once the sequence $\{\Gamma_n\}$ has been constructed, we need to show that Γ_n is an invariant ω_n-torus for the Hamiltonian H_n. To simplify notation, assume that $T_n = T$ and $\eta_n = \eta < 1$ for all n. Then

$$\Gamma_{n,m} = \Lambda_{n+1} \circ \ldots \circ \Lambda_m \circ F_m \circ \mathcal{T}^{-m+n} .$$

By using the "basic identities" we obtain

$$\begin{aligned} \Phi_n^t \circ \Gamma_{n,m} \Psi_n^{-t} &= \Lambda_{n+1} \circ \ldots \circ \Lambda_m \circ \left[\Phi_m^{\eta^{m-n}t} \circ F_m \circ \Psi^{-\eta^{m-n}t}\right] \circ \mathcal{T}^{-m+n} \\ &= \big(\mathcal{M}_n \circ \ldots \circ \mathcal{M}_m\big)(G_m) , \end{aligned}$$

where

$$G_m = \Phi_m^{\eta^{m-n}t} \circ F_m \circ \Psi^{-\eta^{m-n}t} .$$

Taking $m \to \infty$ we see that Γ_n is an invariant torus for H_n.

9.3. Analytic tori

Consider the near-integrable case.

The above procedure yields a torus Γ_H for each H on the $\mathfrak{R}$-stable manifold $\mathcal{W}^s$ at K_0. This torus Γ_H belongs to some low-regularity space.

However, the map $H \mapsto \Gamma_H$ is analytic on $\mathcal{W}^s$.

To prove that Γ is in fact analytic, consider translations

$$\mathcal{J}_v H = H \circ J_v , \qquad J_v(q,p) = (q - v, p) .$$

First, one verifies that $\mathcal{R}$ is "covariant" for real v,

$$\mathcal{R} \circ \mathcal{J}_v = \mathcal{J}_{T^{-1}v} \circ \mathcal{R} .$$

In particular, $\mathcal{W}^s$ is invariant under $\mathcal{J}_v$. Here, one uses that K_0 is invariant under $\mathcal{J}_v$. The next step is to verify that (by construction)

$$\Gamma_{H \circ J_v} = J_v^{-1} \circ \Gamma_H \circ J_v . \qquad v \in \mathbb{R}^d .$$

This identity implies that for real v,

$$\Gamma_H(q) = E_q \Gamma_H = E_0\big(\Gamma_H \circ J_q\big) = J_q E_0 \Gamma_{H \circ J_q} , \tag{$*$}$$

where $E_q f = f(q,0)$.
Now we can use that the map $H \mapsto \Gamma_H$ is analytic on $\mathcal{W}^s$ near K_0. Then the equation $(*)$ yields an analytic continuation of Γ_H.

9.4. Critical tori

Consider the golden mean RG, where $T = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$.
The analysis in this case is restricted to Hamiltonians

$$H(q,p) = \omega \cdot p + \sum_{\nu,k} h_{\nu,k} \cos(\nu \cdot q) z^k \,, \qquad z = \Omega \cdot p \,,$$

where ω and Ω are the expanding and contracting eigenvectors, respectively, of T. The ω-torus for the critical fixed point H_* of $\mathcal{R}$ is obtained as a fixed point of $\mathcal{M}_*$,

$$\mathcal{M}_*(F) = \Lambda_* \circ F \circ \mathcal{T}^{-1} \,, \qquad \Lambda_* = \mathcal{T}_\mu \circ \mathcal{U}_* \,.$$

Since $\omega' \cdot q$ evolves linearly in time, coordinate changes like $\mathcal{U}_*$ can be taken of the special form

$$U = \mathrm{I} + u \,, \qquad u(q,p) = \bigl(q + Q_\Omega(q,z)\Omega,\ P(q,z) \bigr) \,,$$

with Q_Ω odd and P even in q. Similarly, it suffices to consider the torus-equation for functions $F = \mathrm{I} + f$ of the same form, restricted to $z = 0$. In particular, f has a zero component in the expanding direction of $\mathcal{T}_\mu$. As a result, $f \mapsto \Lambda_* \circ f$ is a contraction on a fixed space of (low-regularity) functions. In fact, the contraction is substantial.

So it is not hard to show (via computer-assisted proof) that $\mathcal{M}_*$ has a fixed point Γ_*. In addition, following the procedure sketched above, one can construct an invariant ω-torus Γ_0 for every Hamiltonian H_0 on the (local) strong stable manifold of $\mathcal{R}$ at H_*.

Furthermore, Γ_0 can be shown to be non-differentiable.
In the case of Γ_*, the basic idea is the following.

The invariant ω-torus Γ_* for H_* satisfies the equation

$$\Gamma_* \circ \mathcal{T} = \Lambda_* \circ \Gamma_* \,. \qquad (*)$$

In particular, $p_* = \Gamma_*(0)$ is a fixed point of Λ_*. Assume for contradiction that $g(t) = \Gamma_*(t\Omega, 0)$ is differentiable at $t = 0$. After excluding the case $g'(0) = 0$, the conjugacy $(*)$ implies that $D\Lambda_*(p_*)$ must have an eigenvalue $-\vartheta^{-1} = -0.618033\ldots$ with eigenvector $(\Omega, 0)$. But our bound on this eigenvalue ($\lambda_2 = -0.706795\ldots$) shows that this is not the case.

10. Some open problems

Directly related open problems.

- Hyperbolicity of $\mathcal{R}$ at H_*.
- Hyperbolicity of the commuting-maps RG in the space of "all" reversible pairs.
- Other "quadratic" rotation vectors in $\mathbb{R}^2$, starting with $T = \left[\begin{smallmatrix} 0 & 1 \\ 1 & k \end{smallmatrix}\right]$ for large N.
- Find out what "really" happens for the spiral mean $\vartheta^3 - \vartheta - 1 = 0$.
- Breakup of shearless invariant tori.

Diophantine or Brjuno rotation vectors.

KAM-type results were proved for one rotation vector at a time. These vectors ω have full measure, but ...

- Do families intersect all these stable manifolds $\mathcal{W}^s_\omega$ in a set of large measure?
- Same question for lower dimensional tori, including some PDEs.
  But restrict to arithmetically interesting cases.

Different dynamical systems.

Not mentioning systems whose nonlinearities are purely "algebraic",

like some cocycles [R. Krikorian, A. Avila, ...]
and some parabolic flows [G. Forni, ...]

- Proof without computer (so far exist only for maps on 1-dim spaces) for a nontrivial RG problem in 2 or more dimensions.
- Hamiltonian systems with nonholonomic constraints.
- RG for nonlinear flows on other surfaces, manifolds, groups?

Figures and References can be found e.g. in the notes

H. Koch, *Renormalization of vector fields*, in: Holomorphic Dynamics and Renormalization, M. Lyubich and M. Yampolsky (eds.), Fields Institute Communications, AMS(2008) 269–330.
ftp://ftp.ma.utexas.edu/pub/papers/koch/toronto7.pdf