Statistical properties of one-dimensional maps

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Goal

Existence and statistical properties of absolutely continuous invariant measures of:

- `Non-degenerate smooth interval maps`. We will assume them to be topologically exact and with all cycles hyperbolic repelling.
  - The reference measure will be the LEBESGUE measure on the interval domain.
- `A complex rational map` f, viewed as dynamical system acting on its JULIA set $J(f)$.
  - When $J(f) = \overline{\mathbb{C}}$, the reference measure will be spherical measure.
  - When $J(f) \neq \overline{\mathbb{C}}$, the reference measure will be a conformal measure of minimal exponent.

Correlations

f a real or complex one-dimensional map as before;

ν an invariant probability measure for f.

Definition

Given real functions φ, ψ and $n \geq 1$ we put

$$C_n(\varphi, \psi) := \left| \int \varphi \circ f^n \cdot \psi d\nu - \int \varphi d\nu \cdot \int \psi d\nu \right|.$$

Roughly speaking it measures the (lack of) independence of the random variables $\varphi \circ f^n$ and ψ on the probability space defined by ν.

Decay of correlations

f a real or complex one-dimensional map as before;

ν an invariant probability measure for f.

We will say ν is:

- exponentially mixing, if for every pair of Hölder continuous real functions φ, ψ the correlations $C_n(\varphi, \psi)$ at least exponentially with n;
- polynomially mixing of exponent $\gamma > 0$, if for every pair of Hölder continuous real functions φ, ψ there is a constant $C > 0$ such that for every $n \geq 1$,

$$C_n(\varphi, \psi) \leq C n^{-\gamma};$$

- super-polinomially mixing, if for every $\gamma > 0$ it is polynomially mixing of exponent γ.

Shrinking of components

- Exponential Shrinking of Components:
  There are constants $\theta \in (0, 1)$, $C > 0$ and $\delta_0 > 0$ such that for every x, $\delta \in (0, \delta_0)$, every integer $m \geq 1$ and every connected component W of $f^{-m}(B(x, \delta))$ we have,
  $$\mathrm{diam}(W) \leq C\theta^{-m}.$$
- Polynomial Shrinking of Components of exponent $\beta > 0$:
  There are $\delta_0 > 0$ and $C > 0$ such that for every x, $\delta \in (0, \delta_0)$, every integer $m \geq 1$ and every connected component W de $f^{-m}(B(x, \delta))$ we have,
  $$\mathrm{diam}(W) \leq Cm^{-\beta}.$$
- Super-polynomial Shrinking of Components:
  Polynomial Shrinking of Components for each $\beta > 0$.

Overview

`Main goal`:

Large Derivatives $\Rightarrow$ super-polynomially mixing a.c.i.p.

`First lecture`:

Large Derivatives

$\Rightarrow$ Backward Contraction

$\Rightarrow$ Super-polynomial Shrinking of Components

Goals for today

`Goals for today`:

- describe the inducing scheme;
- as an application prove

Exponential Shrinking of Components

$\Rightarrow$ existence of an exponentially mixing a.c.i.p.

Remark

The reverse implication:

Exponential Shrinking of Components

$\Leftarrow$ existence of an exponentially mixing a.c.i.p.

is easy ("KELLER's trick").

Base map

$(\Delta_0, \mathcal{B}_0, \mathfrak{m}_0)$ measurable space;

$\mathscr{P}_0$ a countable and measurable partition of Δ_0;

$T_0 : \Delta_0 \to \Delta_0$ a measurable map such that for each $\Delta' \in \mathscr{P}_0$ the map

$$T_0 : \Delta' \to \Delta_0$$

is a bijection whose inverse is measurable.

Base map

We will suppose that the partition $\mathscr{P}_0$ is generating:

each element of the partition

$$\bigvee_{n=1}^{+\infty} T_0^{-n}\mathscr{P}_0$$

reduces to a point.

Equivalently: for each pair of different points $x, y \in \Delta_0$ there is an integer $s \geq 0$ such that

$$T_0^s(x) \text{ y } T_0^s(y)$$

belong to different elements of $\mathscr{P}_0$.

The separation time $s_0(x, y)$ of x and y is by definition the least integer $s \geq 0$ with this property

Bounded distortion

We will assume that the Jacobian $\mathrm{Jac}(T_0)$ of T_0 is defined on a subset of Δ_0 of full measure and that the following bounded distortion property holds:

> *There are constants $C > 0$ and $\beta \in (0, 1)$ such that for every pair point distinct points $x, y \in \Delta_0$*
>
> $$\left| \frac{Jac(T_0)(x)}{Jac(T_0)(y)} - 1 \right| \leq C\beta^{s_0(x,y)}.$$

Return time

Suppose we are given a function

$$R : \Delta_0 \to \mathbb{N} = \{1, 2, \ldots\}$$

which is constant on each element of $\mathscr{P}_0$ and such that

$$\text{maximum common divisor}\{R(x) : x \in \Delta_0\} = 1.$$

Tower

$$\Delta_n := \{z \in \Delta_0 : n < R(z)\}.$$

$$\Delta := \bigsqcup_{n=0}^{\infty} \{n\} \times \Delta_n.$$

$$= \{(z, n) \in \Delta_0 \times (\mathbb{N} \cup \{0\}) : n < R(z)\}.$$

$\mathfrak{c}$:= counting measure on $\mathbb{N} \cup \{0\}$.

$\mathfrak{m}$:= restriction to $\Delta \subset \Delta_0 \times (\mathbb{N} \cup \{0\})$ of the product measure $\mathfrak{m}_0 \times \mathfrak{c}$.

$$\begin{array}{rccl} T : & \Delta & \to & \Delta \\ & (x, n) & \mapsto & \begin{cases} (x, n+1) & \text{si } n+1 < R(x); \\ (T_0(x), 0) & \text{si } n+1 = R(x). \end{cases} \end{array}$$

Young's theorem

Theorem (Young, 1991)

- *If* $\int R d\mathfrak{m}_0 < +\infty$*, then* T *has an a.c.i. measure* ν *which is ergodic and mixing.*
- *If*
$$(\mathfrak{m}_0(\{x \in \Delta_0 : R(x) > n\}))_{n=1}^{+\infty} \tag{1}$$
decreases exponentially, then ν *is exponentially mixing.*
- *If* (1) *decreases polynomially with an exponent* $\gamma > 1$*, then the measure* ν *is polynomially mixing of exponent* $\gamma - 1$*.*

Pull-backs

$$
\begin{aligned}
f &\quad \text{real or complex one-dimensional map;} \\
V &\quad \text{subset of the ambient space } ([0,1] \text{ or } \overline{\mathbb{C}}); \\
m \geq 1 &\quad \text{an integer.}
\end{aligned}
$$

Definition

- A pull-back of V by f^m is a connected component of $f^{-m}(V)$.
- A pull-back W of V by f^m is diffeomorphic if the restriction of f^m to W is a diffeomorphism onto its image.

Observations:

- In this definition V could be disconnected.
- If W is a diffeomorphic pull-back of V by f^m, then $f^m(W)$ is a connected component of V.

Nice sets

f real or complex one-dimensional map.

Nice set for f: an open neighborhood V of $\mathrm{Crit}'(f)$ such that every connected component of V contains precisely a critical point of f and such that for $n \geq 1$

$$f^n(\partial V) \cap V = \emptyset.$$

Markovian properties:

- *For each pull-back W of V*

$$W \cap V = \emptyset \text{ or } W \subset V;$$

- *for each pair of pull-backs W, W' of V*

$$W \cap W' = \emptyset,\ W \subset W' \text{ or } W' \subset W.$$

Nice sets

f a real or complex one-dimensional map;

V nice set for f.

For $c \in \mathrm{Crit}(f)$ we denote by V^c the connected component of V containing c, so that

$$V = \bigsqcup_{c \in \mathrm{Crit}(f)} V^c.$$

Nice couples

f a real or complex one-dimensional map.

Nice couple for f: a pair of nice sets $(\widehat{V}, V)$ such that

$$\overline{V} \subset \widehat{V}$$

and such that for every $n \geq 1$

$$f^n(\partial V) \cap \widehat{V} = \emptyset.$$

Joint Markovian property: *for each pull-back W of $\widehat{V}$*

$$W \cap V = \emptyset \text{ or } W \subset V.$$

Induced map

f a real or complex one-dimensional map;

$(\widehat{V}, V)$ nice couple for.

A good time for $x \in V$: an integer $m \geq 1$ such that

$$f^m(x) \in V$$

and such that the pull-back of $\widehat{V}$ by f^m containing x is diffeomorphic;

D: the set of points in V having a good time;

$m(x)$: the least good time of $x \in D$;

canonical induced map associated to $(\widehat{V}, V)$:

$$\begin{aligned} F : \quad D &\longrightarrow V \\ x &\longmapsto f^{m(x)}(x) \end{aligned}$$

Induced map

- The set D is a disjoint union of diffeomorphic pull-backs of V.
- In each connected component W of D the return time m is constant on W.
  We will denote this number by $m(W)$ and we will denote by $c(W)$ the critical point such that $f^{m(W)}(W) \subset V^{c(W)}$.
  Then the map

$$f^{m(W)} : W \to V^{c(W)}$$

extends to a diffeomorphism from a neighborhood of W onto $\widehat{V}^{c(W)}$.

Induced map

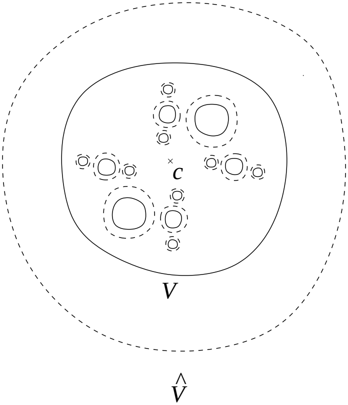

Figure: The domain of the induced map associated to $(\widehat{V}, V)$

The tower of the induced map

f a real or complex one-dimensional map.

$(\widehat{V}, V)$ a nice couple for f;

$F : D \to V$ the canonical induced map associated to $(\widehat{V}, V)$.

In the real case we denote by μ be Lebesgue measure.
In the complex case we first will assume we are given a conformal measure μ of exponent $\mathrm{HD}(J(f))$.

$J(F)$: maximal invariant set of F

$$= \{x \in D : \text{all iterates of } F \text{ are defined on } x\}.$$

Δ_0: $= J(F)$.

$\mathfrak{m}_0$: the restriction of the reference measure μ to $\Delta_0 = J(F)$.

T_0: $= F|J(F)$.

$\mathscr{P}_0$: $= \{W \cap J(F) : W \text{ connected component of } D\}$.

The tower of the induced map

Let Δ and $T : \Delta \to \Delta$ be as in Young's tower, taking

$$R := m|\Delta_0.$$

Defining

$$\begin{array}{rccc} \pi : & \Delta & \to & [0,1] \text{ or } J(f) \\ & (x,n) & \mapsto & f^n(x) \end{array}$$

we have the commutative diagram:

$$\begin{array}{ccc} \Delta & \xrightarrow{T} & \Delta \\ \pi \downarrow & & \downarrow \pi \\ [0,1] \text{ or } J(f) & \xrightarrow[f]{} & [0,1] \text{ or } J(f) \end{array}$$

and the measure $\pi_* \mathfrak{m}$ is absolutely continuous with respect to μ.

Properties of the induced map

Lemma

Let f be a topologically exact real or complex one-dimensional map.
Then for every sufficiently small nice couple $(\widehat{V}, V)$ there is $c \in \mathrm{Crit}(f)$ such that the maximum common divisor of

$$\{m(W) : W \subset V^c \text{ connected component of } D \\ \text{such that } F(W) = V^c\}$$

is equal to 1.

Properties of the induced map

f real or complex one-dimensional map;
$(\widehat{V}, V)$ nice couple for f;
$F : D \to V$ canonical induced map associated to $(\widehat{V}, V)$.

Definition

We will say that F es uniformly expanding if there are $\eta > 1$ and $C > 0$ such that for every $x \in D$ in the domain of F^n

$$|(F^n)'(x)| \geq C\eta^n.$$

Lemma

For every sufficiently small nice couple $(\widehat{V}, V)$ the canonical induced map

$$F : D \to V$$

is uniformly expanding.

Applying Young's theorem

If F is uniformly expanding, then:

- $\mathscr{P}_0$ is generating;
- F, and hence T_0, have bounded distortion;
- π is Hölder continuous.

Thus, if the nice couple $(\widehat{V}, V)$ is sufficiently small, then in order to apply Young's theorem to f we only have to:

- show that $\mu(J(F)) > 0$;
- estimate how:

$$\sum_{\substack{W \text{ connected component of } D \\ m(W) \geq n}} \mathrm{diam}(W)^{1 \text{ or } \mathrm{HD}(J(f))}$$

decreases with n.

The measure of $J(F)$

We will prove $\mu(J(F)) > 0$ in two steps:

- In the real setting we will prove

$$\mathrm{HD}(V \setminus J(F)) < 1$$

  and in the complex

$$\mathrm{HD}((V \cap J(f)) \setminus J(F)) < \mathrm{HD}(J(f)).$$

- Using results of Mauldin–Urbanski we will prove that μ is the unique conformal measure of f of minimal exponent and that the Hausdorff dimension of μ is equal to 1, or to $\mathrm{HD}(J(f))$ in the complex setting. By the previous point, $\mu(J(F)) > 0$.

Notice that we will construct a conformal measure μ of minimal exponent, so we do not need to assume it exists at the beginning.

The measure of $J(F)$

- N: the set of points in V or $V \cap J(f)$ that return at most a finite number of times to V.
- I: the set of points in $V \setminus J(F)$ or $(V \cap J(f)) \setminus J(F)$ that return infinitely many times to V.

In the real setting

$$V \setminus J(F) = N \sqcup I$$

and in the complex

$$(V \cap J(f)) \setminus J(F) = N \sqcup I.$$

The measure of $J(F)$

N: the set of points in V that return at most a finite number of times to V.

If we put,

$$K(V) := \{x : f^n(x) \notin V \text{ for every } n \geq 0\}$$

then by definition:

$$N \subset \bigcup_{n=0}^{\infty} f^{-n}(K(V)).$$

Since f is uniformly hyperbolic on $K(V)$,

$$\mathrm{HD}(K(V)) < 1 \text{ or } \mathrm{HD}(J(f))$$

and thus

$$\mathrm{HD}(N) < 1 \text{ or } \mathrm{HD}(J(f)).$$

Bad pull-backs

Definition

Given an integer $m \geq 1$ we will say that a pull-back $\widetilde{W}$ of $\widehat{V}$ by f^m is a bad pull-back of $\widehat{V}$ of order m if for every

$$m' \in \{1, \ldots, m\}$$

such that $f^{m'}(\widetilde{W}) \subset \widehat{V}$, the pull-back of $\widehat{V}$ containing $\widetilde{W}$ is not diffeomorphic.

I: the set of points in $V \setminus J(F)$ or $(V \cap J(f)) \setminus J(F)$ that return infinitely many times to V.

By definition each point of I is contained in a bad pull-back of arbitrarily large order. So for every $m_0 \geq 1$,

$$I \subset \bigcup_{\substack{\widetilde{W} \text{ bad pull-back of } \widehat{V} \\ \text{of order } m \geq m_0}} \widetilde{W}.$$

Bad pull-backs

I: the set of points in $V \setminus J(F)$ or $(V \cap J(f)) \setminus J(F)$ that return infinitely many times to V.

Since $\mathrm{HD}(\mu) = 1$ or $\mathrm{HD}(J(f))$, to prove $\mu(I) = 0$ it is enough to prove

$$\mathrm{HD}(I) < 1 \text{ or } \mathrm{HD}(J(f)).$$

Thus, to show that I, and hence $V \setminus J(F)$, has zero measure for μ it is enough to show that for some $\alpha > 0$ satisfying $\alpha < 1$ or $\mathrm{HD}(J(f))$,

$$\sum_{\widetilde{W} \text{ bad pull-back of } \widehat{V}} \mathrm{diam}(\widetilde{W})^{\alpha} < +\infty.$$

Constructing the reference measure

Suppose f is a complex map and that for some $\alpha > 0$ satisfying $\alpha < 1$ or HD$(J(f))$,

$$\sum_{\widetilde{W} \text{ bad pull-back of } \widehat{V}} \operatorname{diam}(\widetilde{W})^{\alpha} < +\infty.$$

Assume furthermore that

$$\sum_{W \text{ connected component of } D} \operatorname{diam}(W)^{\alpha} < +\infty.$$

Then the results of Mauldin–Urbanski show that there is a conformal measure $\tilde{\mu}$ of exponent 1 or HD$(J(f))$ for the induced map F.

Furthermore HD$(\mu) = 1$ or HD$(J(f))$.

Constructing the reference measure

The conformal measure $\tilde{\mu}$ for the induced map F can be spread to a conformal measure μ' for the original map f of exponent $\mathrm{HD}(J(f))$.
This measure is ergodic and satisfies $\mathrm{HD}(\mu') = \mathrm{HD}(J(f))$.

From now on we assume the above sums are finite and take $\mu := \mu'$.

Exponential case

Suppose f satisfies the Exponential Shrinking of Components condition.
Then to prove

$$\sum_{\widetilde{W} \text{ bad pull-back of } \widehat{V}} \operatorname{diam}(\widetilde{W})^{\alpha} < +\infty.$$

it is enough to show:

> *for a given $n \geq 1$ the number of bad pull-backs of V is sub-exponential in n.*

Key Lemma

Key Lemma. *Suppose f has the Exponential Shrinking of Components property. Then there is $\alpha \in (0, 1$ or $\mathrm{HD}(J(f)))$ such that*

$$\sum_{W \text{ connected component of } D} \mathrm{diam}(W)^{\alpha} < +\infty.$$

Tail estimates in the exponential case

Key Lemma $\Rightarrow$ exponential tail estimates

Let $C > 0$ and $\theta \in (0,1)$ such that $\mathrm{diam}(W) \leq C\theta^{m(W)}$ and put

$$C' := \sum_{W \text{ connected component of } D} \mathrm{diam}(W)^{\alpha} < +\infty.$$

For a given $m \geq 1$ we have

$$\sum_{\substack{W \text{ connected component of } D \\ m(W) \geq m}} \mathrm{diam}(W)^{\mathrm{HD}(J(f))}$$

$$\leq \theta^{m(\mathrm{HD}(J(f)) - \alpha)} \sum_{\substack{W \text{ connected component of } D \\ m(W) \geq m}} \mathrm{diam}(W)^{\alpha}$$

$$\leq C'\theta^{m(\mathrm{HD}(J(f)) - \alpha)}.$$