# Subtractive algorithms

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# A subtractive algorithm

The map

$$\tau\colon \mathbf{x} = (x_1, x_2, \ldots, x_n) \mapsto \texttt{sort}(x_1, x_2 - x_1, \ldots, x_n - x_1)$$

is defined on ordered $n$-tuples, all $x_i \geq 0$.

Note that $\mathbf{x}_\infty = \lim_{n\to\infty} \tau^n(\mathbf{x})$ exists. It is a fixed point of $\tau$ and therefore the first coordinate of $\mathbf{x}_\infty$ is zero.

If all coordinates of $\mathbf{x}$ are rationally independent then the second coordinate of $\mathbf{x}_\infty$ is zero as well.

# A pedestrian on a line

A pedestrian walks up and down on a line, taking steps of length $x_1, \ldots, x_n$, all rationally independent. Find the length of a minimal interval that enables an infinite walk that does not visit any point twice.

For instance, if there are only two steps $x_1, x_2$ then the length is $x_1 + x_2$ and the walk is an irrational rotation on the circle.

# An algorithm to solve this problem

Sort the steps in increasing order $x_1 < x_2 < \cdots < x_n$. Let $I = [0, y]$ be a minimal interval. Partition it into $[0, y - x_1] \cup (y - x_1, y]$


On the subinterval $[0, y - x_1]$ there is an infinite walk with steps $x_1, x_2 - x_1, \ldots, x_n - x_1$. This is the subtractive algorithm, proposed by Meester.

Source: Meester, Circle percolation, ETDS 1989.

# A discrete pedestrian

A pedestrian walks up and down on $\mathbb{Z}$, taking integral steps of length $p_1, \ldots, p_n$ such that gcd is one. Find the length of a maximal interval $I$ such that the pedestrian cannot visit all points of $I$.

For instance, if there are only two steps $p_1, p_2$ then the length is $p_1 + p_2 - 2$. Again the solution is by the subtraction operation.

Source: Tijdeman and Zamboni, Fine-Wilf words, Indag Math 2003

# Does the subtraction terminate?

Consider Meester's problem on a triple $(x_1, x_2, x_3)$. If $x_3 > x_1 + x_2$ then steps of size $x_3$ do not help. The minimal length is $x_1 + x_2$ by irrational rotation.

For general $n$-tuples, Meester's algorithm iterates

$$\tau \colon \mathbf{x} = (x_1, x_2 \ldots, x_n) \mapsto \mathtt{sort}(x_1, x_2 - x_1, \ldots, x_n - x_1)$$

until $x_1 + x_2 < x_3$.

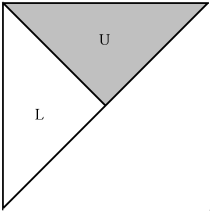
**Question**: does this algorithm terminate almost surely?
**Answer**: yes

source: for triples, Meester and Nowicki, Israel J 1989; general case: Kraaikamp and Meester, ETDS 1995

# Projective coordinates

Equivalent question: is it true that $\mathbf{x}_\infty = \lim_{n\to\infty} \tau^n(\mathbf{x})$ has third coordinate $> 0$ almost surely?

Observe that $\tau(x_1, x_2, \ldots, x_n) = \mathtt{sort}(x_1, x_2 - x_1, \ldots, x_n - x_1)$ respects projective coordinates, which reduces the degrees of freedom by one. If we normalize the third coordinate to 1, then ordered triples can be depicted by the triangle $0 < x < y < 1$:




The algorithm terminates as soon as $\tau^n(\mathbf{x}) \in L$.

# Schweiger's algorithm

In his monograph on continued fractions Schweiger generalizes $\tau$ and considers the **fully subtractive algorithm**:

$$\tau\colon (x_1, \ldots, x_a, \ldots, x_n) \mapsto \texttt{sort}(x_1, \ldots, x_a, x_{a+1} - x_a, \ldots, x_n - x_a)$$

Again, it is easy to show that $\mathbf{x}_\infty$ has first $a + 1$ coordinates equal to zero a.s. Schweiger presents two conjectures:

1. The $a + 2$ coordinate of $\mathbf{x}_\infty$ is positive a.s.
2. $\tau$ is ergodic, i.e, invariant sets are null sets or co-null sets.

1 is true and 2 is false, but conjecture 2 may be true if $n = a + 2$.

source: Fokkink-Kraaikamp-Nakada, Israel J 2011.

# Elementary properties

As always, accelerate the algorithm

$$\tau\colon (x_1, \ldots, x_a, \ldots, x_n) \mapsto \texttt{sort}(x_1, \ldots, x_a, x_{a+1} - \mathbf{k}x_a, \ldots, x_n - \mathbf{k}x_a)$$

with $\mathbf{k} = \lfloor \frac{x_{a+1}}{x_a} \rfloor$. Observe that the permutation on the coordinates is a 'rifle shuffle'.

**Lemma**

*All cylinders are full*

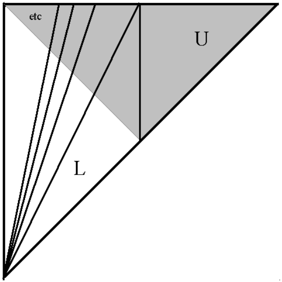
**Lemma**

*The set* $L = \{x_1 + \cdots + x_{a+1} < x_{a+2}\}$ *is invariant.*

The proof is by bounded distortion: in each iteration a positive fraction of $U$, the complement of $L$, enters $L$.

# Sketch of the partition

A sketch of the principal cylinders for the algorithm on triples:




Points that never enter $L$ are those that return infinitely often to the cylinder that is entirely contained in $U$.

# Bounded distortion

Observe that $\tau(\mathbf{x}) = \mathtt{sort}(x_1, x_2 - x_1, \ldots, x_n - x_1)$ is linear and has determinant 1.

Now we normalize the $n^{th}$ coordinate to 1, so writing $\mathbf{y} = \tau(\mathbf{x})$, in normalized coordinates the map is $T(\mathbf{x}) = \frac{1}{y_n}\mathbf{y}$, where $y_n$ is the final coordinate of $\mathbf{y}$. Therefore $DT(\mathbf{x})$ has determinant $\left(\frac{1}{y_n}\right)^n$.

To bound distortion on an $m$-cylinder $\Delta$ we have to bound $y_n$ away from zero for all $\mathbf{y} = T^m(\mathbf{x})$ in that cylinder.

# First conjecture

An principal cylinder $\Delta_{(k,\pi)}$ is given by the acceleration $k$ and the rifle shuffle $\pi$. If $\pi(a) = n$ then $1 - x_a < x_a$. So $y_n$ is bounded away from zero, since $y_n = x_a$.

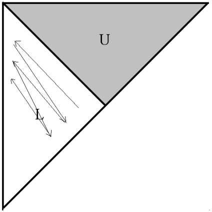
More generally, an $m$-cylinder $\Delta_{(k_1,\pi_1)(k_2,\pi_2)\cdots(k_m,\pi_m)}$ has bounded distortion if $\pi_m(a) = n$. All elements that remain in $U$ are contained in such $m$-cylinders for arbitrary large $m$.

Since cylinders are full, by bounded distortion any such $m$-cylinder loses a proportion to $L$. Therefore $L$ is an absorbing set and points that remain in $U$ have measure zero. This proves Schweiger's first conjecture

# Second conjecture

Now we know that $\mathbf{x}_\infty$ has a positive $a+2$-nd coordinate $x^\infty_{a+2}$ a.s. Define $f(\mathbf{x}) = x^\infty_{a+2}$. Then $f$ is $\tau$-invariant and non-constant if $n > a+2$ so $\tau$ is not ergodic.

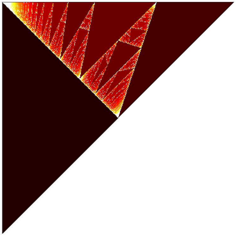
The remaining case $n = a+2$ is non-trivial.




Points in $L$ zigzag down slowly. Is there a non-trivial invariant set?

# Exotic invariant sets

The subset of triples $\mathbf{x}$ such that $\mathbf{x}_\infty = \mathbf{0}$ is a Sierpinski triangle:



Such complex-dynamic like fractals occur in general subtractive maps. Nogeira and Schweiger have found a Cantor fan, which is known from the exponential family in complex dynamics, in the Poincaré algorithm

$$(x_1, x_2, x_3) \mapsto \texttt{sort}(x_1, x_2 - x_1, x_3 - x_2)$$

source: Schweiger, On the Parry-Daniels transform, 1981; Nogeira, Poincaré algorithm, Israel J, 1995

# More general subtractive algorithms

It is natural to consider for $b \leq a$

$$\tau\colon (x_1, \ldots, x_a, \ldots, x_n) \mapsto \texttt{sort}(x_1, \ldots, x_a, x_{a+1} - x_b, \ldots, x_n - x_b)$$

Again, it is easy to show that $\mathbf{x}_\infty$ has $a + 1$ coordinates that are equal to zero. Numerical experiments suggest that almost surely $\mathbf{x}_\infty$ has $2a - b + 1$ coordinates that are equal to zero.

Unfortunately, this $\tau$ admits no Markov partition. A proof for these numerical results seems difficult.

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