Error Diffusion Algorithm

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Tomasz Nowicki

IBM T.J. Watson Research Center

Motivation

- Printing with discrete colors
- Shops with hard to switch machines
- "Fair" representative selection
- Coding
- Analog to digital (or continuous to discrete) conversion
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- Similar problems: Chairman Assignment Problem, Car Pool Problem, $\Sigma\Delta$ modulators, game theory, control theory.

Online greedy algorithm, an example

- We want to print yellow or red output dots to mimic the orange input.
- Orange is $\gamma = (0.2, 0.8)$.
- First we print pure red output $c_2 = (0, 1)$ and we get the error $\mathcal{E} = \gamma - c_2 = (0.2, -0.2)$.
- Next input is γ again and we get the modified input $x = \mathcal{E} + \gamma = (0.4, 0.6)$.
- We print red c_2 again. New error is $\mathcal{E} = x - c_2 = (0.4, -0.4)$.
- We apply the input again $x = \mathcal{E} + \gamma = (0.6, 0.4)$.
- Now we print yellow output $c_1 = (1, 0)$. $\mathcal{E} = x - c_1 = (-0.4, 0.4)$.
- We apply γ again, we get $x = \mathcal{E} + \gamma = (-0.2, 1.2)$, and we will print red.

Online greedy algorithm

- Consider a cumulative error at time t:

$$\mathcal{E}(0) = 0, \qquad \mathcal{E}(T) = \sum_{t=0}^{T-1} (\gamma(t) - c(t)), \qquad \mathcal{E}(T) = \mathcal{E}(T-1) + \gamma(T-1) - c(T-1)$$

- Find the way to to choose the outputs $c(t)$ in order to minimize the maximal error.

$$\min \sup_T ||\mathcal{E}(T)||$$

- Online: decide about the output corners without knowing the future inputs
- Greedy: decide right now how to make $||\mathcal{E}(T)||$ is minimal.

Dynamics and Voronoi regions

- Error $\mathcal{E}(t+1) = \mathcal{E}(t) + \gamma(t) - c(t)$
- Modified input $x(t) = \mathcal{E}(t) + \gamma(t)$:

$$\begin{aligned} x(t+1) &= \mathcal{E}(t+1) + \gamma(t+1) = \mathcal{E}(t) + \gamma(t) - c(t) + \gamma(t+1) \\ &= x(t) - c(t) + \gamma(t+1) \end{aligned}$$

- The corner c closest to x is called its Voronoi corner $c = \mathrm{Vor}(x)$.
  This defines a time dependent dynamical system:

$$F_\gamma(x) = x - \mathrm{Vor}(x) + \gamma$$

with the corresponding system in the error space:

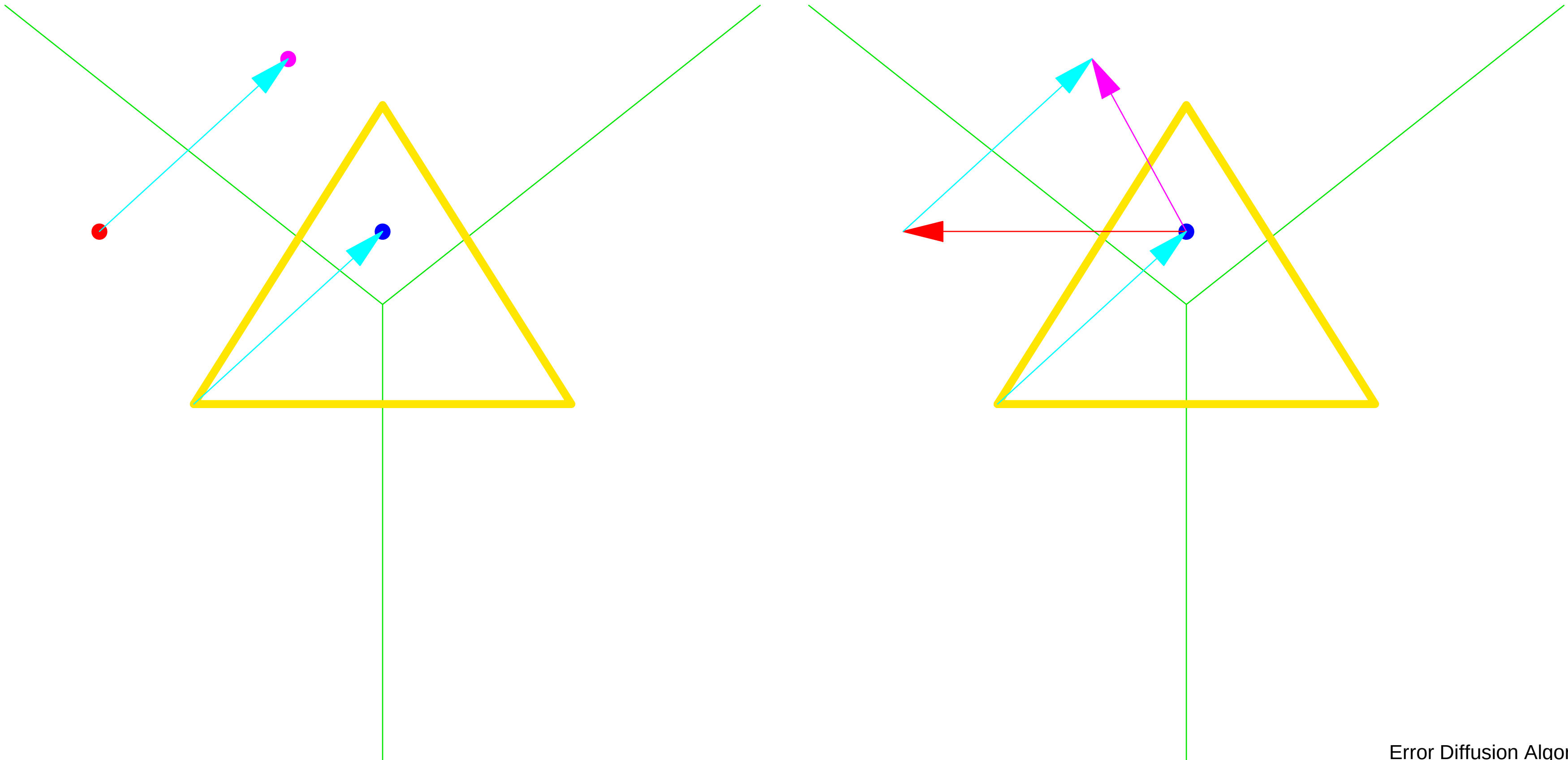
$$G_\gamma(e) = e + \gamma - \mathrm{Vor}(e + \gamma)$$

F and G

- $$F_\gamma(x) = x + t_i = x + \gamma - c_i \qquad \text{if} \qquad x \in V_i$$

- $$G_\gamma(e) = e + \gamma - c_i$$

$c_i = \mathrm{Vor}(x) = \mathrm{Vor}(e + \gamma)$



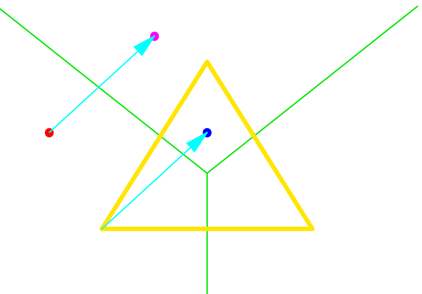
Piecewise translations

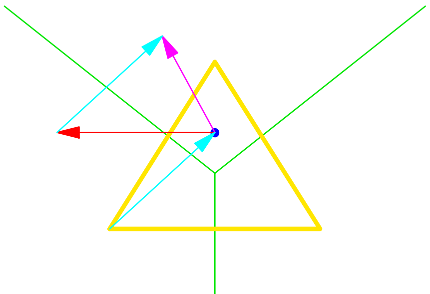
- In general:
  - $\mathbb{R}^d$ is partitioned into V_i.
  - For each V_i there is a vector t_i
  - Each $x \in \mathbb{R}^d$ we define

$$F(x) = x + t_i \qquad \text{if} \qquad x \in V_i$$

- In particular
  - Given a polytope with corners c_i: $\mathcal{P} = \mathrm{conv}(\{c_i\})$ define the (Voronoï) partition $\overline{V_i} = \{y : ||y - c_i|| \leq ||y - c_j||\}$
  - Given an "input" $\gamma \in \mathcal{P}$ define $t_i = \gamma - c_i$
  -

$$F_\gamma(x) = x + t_i = x + \gamma - c_i \qquad \text{if} \qquad x \in V_i \qquad G_\gamma(e) = e + \gamma - \mathrm{Vor}(e + \gamma)$$





Is Error Diffusion any good?

- $$\mathcal{E}(t+1) = \mathcal{E}(t) + \gamma(t) - c(t)$$

  Choose $c(t)$ to minimize the norm of $\mathcal{E}(t+1)$, i.e. closest to $\mathcal{E}(t) + \gamma(t)$ in the norm.

- Main concern:

  Is $\mathcal{E}$ bounded ?

- **Theorem**

  Adler, Kitchens, Martens, Pugh, Shub, Tresser

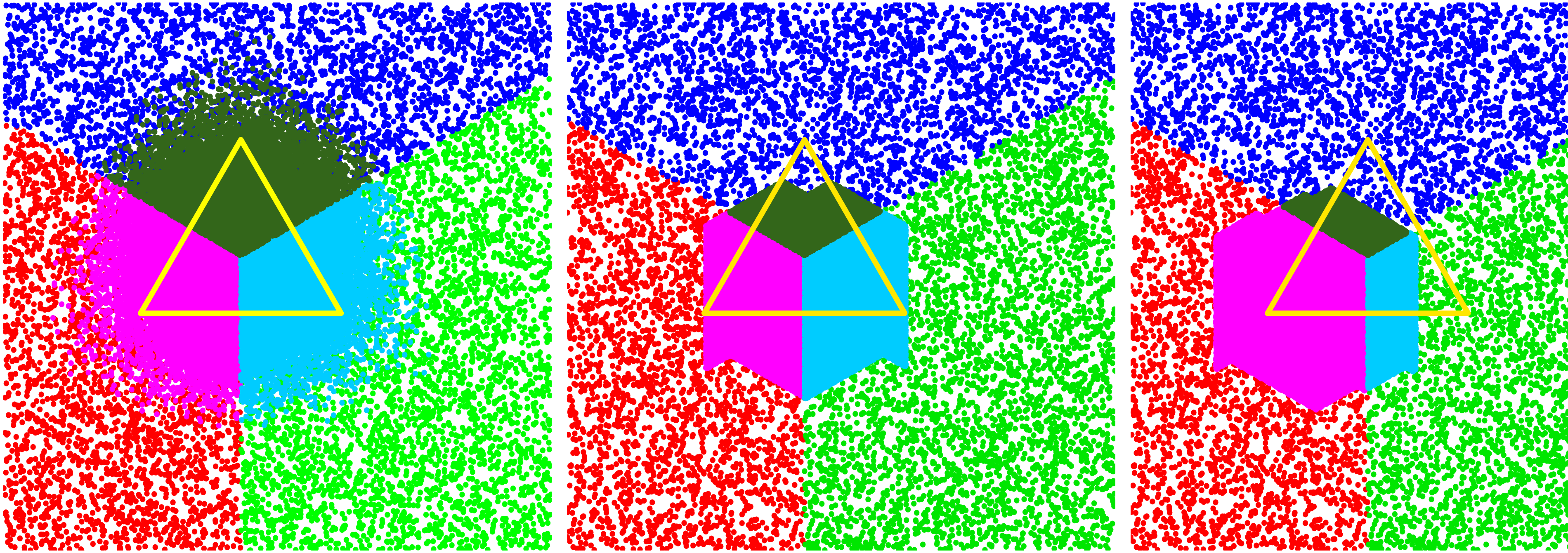
  I case of the Error Diffusion on Polytopes YES,
  for any given $\mathcal{P}$ the Error Diffusion algorithm produces the errors $\mathcal{E}$ that are (u.c.s) bounded in the Euclidean norm.

- What is the nature of the MINIMAL ABSORBING INVARIANT SET ?

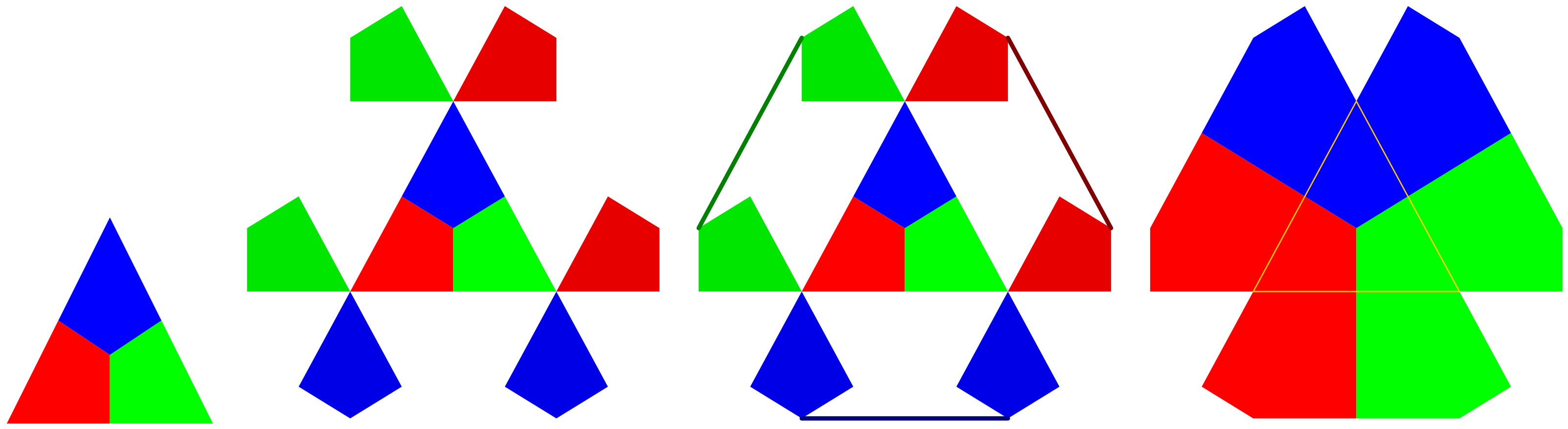
Is Error Diffusion any good?

- In the case of CAP it is the best.
- For any algorithm the maximal error in CAP can be of order log(dimension)
- $\mathcal{E} \geq H(d) = \sum_{1<k\leq d} \frac{1}{k}$
- For Error Diffusion the maximal error in CAP is at most $H(d)$.

Random and constant inputs



Dynamics of F on the simplex



Structure of the Invariant Set (any input)

- $\mathcal{Q}$ is invariant if $F_\gamma(\mathcal{Q}) \subset \mathcal{Q}$ for any γ, or $\mathcal{P}(\mathcal{Q}) = \bigcup_i ((\mathcal{P} - c_i) + (\mathcal{Q} \cap V_i)) \subset \mathcal{Q}$.
- Adler, Kitchens, Martens, Pugh, Shub, Tresser say that for any compact set of initial inputs there is a (pre-)compact invariant region.
- There exists a unique non-empty minimal invariant region.
  - Polytope itself is in this region $\mathcal{P} \subset \mathcal{Q}$
  - The corners of Voronoi regions are there $\mathcal{X} \subset \mathcal{Q}$
  - In particular the size of $\mathcal{Q}$ may depend also on the shape (angles) and not only on dimension and diameter.
- Topological correctness: $\mathcal{Q}^o = \left(\overline{\mathcal{Q}}\right)^o \qquad \overline{\mathcal{Q}} = \overline{(\mathcal{Q}^o)}$
- Convexity: Any invariant region can be convexified Voronoi-wise, $\mathrm{conv}\,\mathcal{Q} = \bigcup_i \mathrm{conv}(\mathcal{Q} \cap V_i)$ and remains invariant.
- Both $\mathcal{P}$ and $\mathcal{X}$ can be reached from any point in finite number of steps.
- Tie-breaks on medians do not matter: $\overline{\bigcup_{n \geq m} \left(\overline{\overline{\mathcal{P}}}\right)^n (A)} = \overline{\bigcup_{n \geq m} \left(\mathcal{P}^{o^4}\right)^n (A)}$.

Structure of the Invariant Set (any input)

- In each Voronoi region there is at least one extreme point of a (bounded) invariant set.
- The faces of the invariant regions contains regions invariant for the faces.
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- In dimension two there always are invariant sets which are combinatorially equivalent (with sides parallel) to the original polytope.
- There are examples when they cannot be similar to the original polytope.
- In dimension three and higher there are polytopes (simplices) with no combinatorially equivalent (no simplicial) invarian regions.

CAP revisited

- An algorithm is any function from sequences of inputs to the the sequences of outputs compatible with the problem.
- Lower bound L: for every algorithm there exists an input sequences producing at least this error.
- Upper bound U: there is an algorithm such that any input sequence produces at most such error.
- Tight bound B: for any given algorithm we take the supremum of the error over the inputs and then take the infimum over the algorithms.
- $\sup L = B = \inf U$.
- For CAP $B = H(d) = \sum_{1<k\leq d} \frac{1}{k}$, and is realized by the Error Diffusion algorithm.
- $L \geq H(d)$ for any algorithm by the example: Start with $(1, 1, 1, \ldots, 1)/d$, then when the corner is picked up for output (say the first corner) supply $(0, 1, \ldots, 1)/(d-1)$ and continue in such a nasty way.

CAP revisited

- $U \leq H(d)$ for Error Diffusion. Proof: for any given time t_0 and output c_0 renumber the outputs by the order they appear in the most recent past, assigning the time t_k of such appearance. Let $e_j(t_k)$ be the error at coordinate j just after time t_k. If $k < j$ then $t_k > t_j$ and $e_j(t_j) \leq e_j(t_l) \leq e_j(t_k) \leq e_j(t_1)$ as long as $j \geq l \geq k \geq 1$ (after last time the output was used the error can only grow).
- Let $x(t_k)$ be the modified input at time t_k. By greediness $x_j(t_k) \leq x_k(t_k)$, for all j, so also for $j < k$. For $k > j$ we have $x_j(t_k) = e_j(t_k)$
- $1 = \sum_j x_j(t_k) \leq k x_k(t_k) + \sum_{j>k} e_j(t_k)$. But then $e_k(t_k) = x_k(t_k) - 1 \geq (1 - \sum_{j>k} e_j(t_k))/k - 1$
- $-1/k \leq (\sum_{j>k} e_j(t_k))/k(k-1) + e_k(t_k)/(k-1) \leq (\sum_{j>k} e_j(t_1))/k(k-1) + e_k(t_1)/(k-1)$, $1 \leq k \leq n$. Sum up over k.
- $-H(d) \leq \sum_{k>0} \sum_{j>k} e_j(t_1)/k(k-1) + \sum_{k>0} e_k(t_1)/(k-1) = \cdots = \cdots = \sum_{j>0} e_j(t_1) = -e_0(t_1)$

Structure of the Invariant Set (simplex with constant input)

- The minimal absorbing set $\mathcal{Q}$ of F_γ with fixed $\gamma \in \mathcal{P}^0$ is a tile with respect to the lattice $L = \{\sum_{i,j} n_{ij}(c_i - c_j)\,, n_{ij} \in \mathbb{Z}\}$.
- Each union of the (Voronoï) parts of this tile is also a tile (w.r. to an explicit lattice).
- This is a
  - **Theorem** for $\mathcal{P}$ non-obtuse triangle.
  - **Theorem** for $\mathcal{P}$ an acute simplex with typical (ergodic) input.
  - **Work in progress** for general (obtuse) triangle.
  - **Work in progress** for acute simplices with general input.
  - **Conjecture** for general simplices with general input.
  - **Unknown** for general polytopes with all the corners on some lattice.

- $Q \subset \mathbb{R}^d$ is a tile with respect to the lattice $L = \mathbb{Z}(w_1, \ldots, w_d) = \{\sum_{i=1}^d n_i w_i, n_i \in \mathbb{Z}\}$, $w_i \in \mathbb{R}^d$, independent, if the map $T : Q \times L \to \mathbb{R}^d, \quad T(q, w) = q + w$ is 1-1 and onto.

Frequencies for a constant input system in a simplex

- When $\mathcal{P}$ is a simplex for any x we have $\frac{\#\{n<N:\mathrm{Vor}(F_\gamma(x))=c_i\}}{N} \to_N \gamma_i$
-

$$
\begin{aligned}
0 \quad &\leftarrow \quad \frac{F_\gamma^N(x)-x}{N} = \frac{1}{N}\sum_{n<N}(\gamma - \mathrm{Vor}(F^n(x))) \\
&= \quad \gamma - \sum_{i=0}^{d} \frac{n_i}{N} c_i = \sum_{i=0}^{d} \gamma_i c_i - \sum_{i=0}^{d} \frac{n_i}{N} c_i \\
&= \quad \sum_{i=0}^{d} (\gamma_i - \frac{n_i}{N}) c_i \\
& \quad\ \ \frac{n_i}{N} \to \gamma_i \quad n_i = \#\{n : \mathrm{Vor}(x_n) = c_i\}
\end{aligned}
$$

by the uniqueness of baricentric coordinates.

Multi-tiles (acute simplices)

- For any subset $I \subset \{1, \ldots, d\}$ define a lattice
  $L_I = L(c_i - c_0, \ldots, c_j - \gamma), \quad i \in I, \quad j \notin I$
- For any $\mathcal{Q}$ define $\mathcal{Q}_I = \mathcal{Q} \cap \bigcup_I V_i$
- **Theorem**
  - If an invariant absorbing set $\mathcal{Q}$ is a tile for the lattice $L = L_{\{1,\ldots,d\}}$ then $\mathcal{Q}_I = \mathcal{Q} \cap \bigcup_{i \in I} V_i$ is a tile for L_I with $|\mathcal{Q}_I| = |\det(L_I)| = \sum_I \gamma_i |\det(L)| = \sum_I \gamma_i |\mathcal{Q}|$.
  - If $T : \mathcal{Q} \times L$ was 1-1 then $T_I : \mathcal{Q}_I \times L_I$ is 1-1[a]
  - If $T : \mathcal{Q} \times L$ was onto then $T_I : \mathcal{Q}_I \times L_I$ is onto[b].

[a] some restrictions apply

[b] some restrictions apply (again)