Optimal Transport from Lebesgue to Poisson

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joint with Martin Huesmann

Allocation Problems

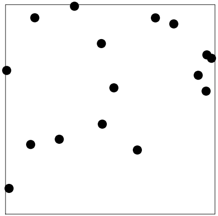
Given a finite set Y of k points together with a set $X \subset \mathbb{R}^d$ of Lebesgue measure k, we look for an 'allocation map' $T : X \to Y$ s.t.

(i) for each 'center' $y \in Y$ the associated 'cell' $T^{-1}(y)$ has unit volume:

$$\mathfrak{L}(T^{-1}(y)) = 1.$$

(ii) the transportation distance $|x - T(x)|$ is as small as possible, for instance, such that for some given $p \in (0, \infty)$

$$\int_X |x - T(x)|^p \, dx \quad \text{is minimal.}$$



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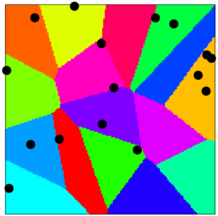
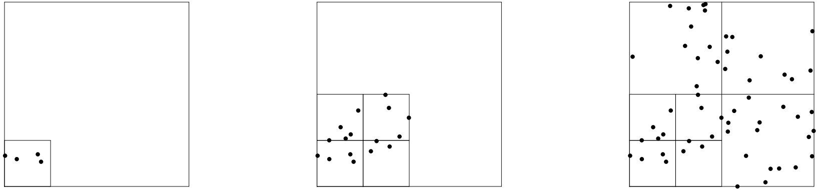


Allocation Problems

What is an appropriate basis for the respective allocation problems?



Point Processes

Poisson point process with unit intensity

$$\mu^\bullet : \Omega \to \mathcal{M}(\mathbb{R}^d), \quad \omega \mapsto \mu^\omega = \sum_{y \in Y(\omega)} \delta_y$$

- for each Borel set $A \subset \mathbb{R}^d$ of finite volume the random variable $\omega \mapsto \mu^\omega(A)$ is Poisson distributed with parameter $\mathfrak{L}(A)$
- for disjoint sets $A_1, \ldots A_k \subset \mathbb{R}^d$ the random variables $\mu^\omega(A_1), \ldots, \mu^\omega(A_k)$ are independent.

▷ Given a Borel set $A \subset \mathbb{R}^d$ with finite volume let N_A be a Poisson random variable with mean $\mathfrak{L}(A)$

▷ Throw N_A points into A, independent and uniformly distributed

▷ Patch together such A to cover $\mathbb{R}^d$.

Point Processes

A **point process** is a measurable map $\mu^\bullet : \Omega \to \mathcal{M}(\mathbb{R}^d)$, $\omega \mapsto \mu^\omega$ with values in the subset of locally finite *counting measures* on $\mathbb{R}^d$.

The point process $\mu^\bullet$ will be called **translation invariant** iff the distribution of $\mu^\bullet$ is invariant under push forwards by translations $\tau_z : x \mapsto x + z$ of $\mathbb{R}^d$, that is, iff

$$(\tau_z)_* \mu^\bullet \quad \overset{(d)}{=} \quad \mu^\bullet$$

for each $z \in \mathbb{R}^d$.

We say that $\mu^\bullet$ has **unit intensity** iff $\mathbb{E}[\mu^\bullet(A)] = \mathfrak{L}(A)$ for all Borel sets $A \subset \mathbb{R}^d$. A translation invariant point process has unit intensity if and only if its intensity

$$\beta = \mathbb{E}\left[\mu^\bullet([0,1)^d)\right]$$

is 1.

E.g. branching process with critical branching rate, started with PPP.

Couplings of Lebesgue Measure and Point Processes

Given two measures ν, μ on $\mathbb{R}^d$, we say that a measure q on $\mathbb{R}^d \times \mathbb{R}^d$ is a **coupling** of ν and μ iff the marginals satisfy

$$(\pi_1)_* q = \nu, \qquad (\pi_2)_* q = \mu.$$

That is, $q(A \times \mathbb{R}^d) = \nu(A)$, $q(\mathbb{R}^d \times A) = \mu(A)$ for all $A \subset \mathbb{R}^d$.

Note: existence of a coupling requires $\nu(\mathbb{R}^d) = \mu(\mathbb{R}^d)$.

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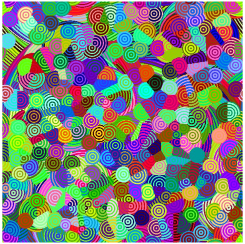
A coupling of the Lebesgue measure $\mathfrak{L} \in \mathcal{M}(\mathbb{R}^d)$ and the point process $\mu^\bullet : \ \Omega \to \mathcal{M}(\mathbb{R}^d)$ is a measurable map $q^\bullet : \ \Omega \to \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)$ s.t. for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$q^\omega \quad \text{is a coupling of } \mathfrak{L} \text{ and } \mu^\omega.$$

Stable Marriage (Hoffman/Holroyd/Peres '06):
q^ω is unstable iff $\exists (x,y),(x',y') \in \mathrm{supp}[q^\omega]$ s.t.

$$d(x,y') < d(x,y) \wedge d(x',y')$$

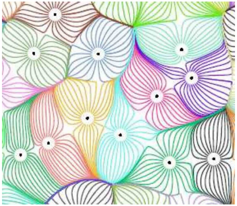


Couplings of Lebesgue Measure and Point Processes

Gravitational Allocation (Chatterjee/Peled/Peres/Romik '07, to appear in Annals of Math.):
For $d \geq 3$ consider the flow $\dot{x}(t) = F^\omega(x(t))$ in the gravitational field

$$F^\omega(x) = \sum_{z \in Z(\omega)} \frac{x-z}{|x-z|^d}.$$

Almost every particle x will finally be absorbed by one of the gravitation centers $X(z) = \{x \in \mathbb{R}^d : \ x(\infty) = z\}$.



Couplings of Lebesgue Measure and Point Processes

Fix a translation invariant point process $\mu^\bullet : \omega \mapsto \mu^\omega$ on $\mathbb{R}^d$ with unit intensity
and consider the cost function $c(x,y) = \vartheta(|x-y|)$ for some strictly increasing, continuous function $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\vartheta(0) = 0$ and $\lim_{r\to\infty} \vartheta(r) = \infty$.

Problem 1. The total cost of transportation will be infinite for each coupling since the marginals have infinite total mass.

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Problem 1. The total cost of transportation will be infinite for each coupling since the marginals have infinite total mass.

Consider the **mean cost functional** on the set Π of all couplings $q^\bullet$ of the Lebesgue measure and the point process

$$\mathfrak{C}(q^\bullet) \ := \sup_{0<\mathfrak{L}(B)<\infty} \frac{1}{\mathfrak{L}(B)} \cdot \mathbb{E}\left[\int_{\mathbb{R}^d \times B} \vartheta(|x-y|)\, dq^\bullet(x,y)\right].$$

The $\sup_B \ldots$ could be replaced by $\limsup_{B \nearrow \mathbb{R}^d} \ldots$ or by $\liminf_{B \nearrow \mathbb{R}^d} \ldots$

Existence of a Minimizer

Basic Questions.

1. Is $\inf_{q^\bullet \in \Pi} \mathfrak{C}(q^\bullet)$ finite?
2. If yes: Does there exist a minimizer? Is it unique?

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$\rightsquigarrow$ **Approximation by finite measures**

- Fix exhausting sequence of cubes $B_n \nearrow \mathbb{R}^d$
- Consider optimal coupling q_n^ω of $1_{B_n}\mathfrak{L}$ and $1_{B_n}\mu^\omega$
- Mean transportation cost for $q_n^\bullet$ should converge to $\inf_{q^\bullet \in \Pi} \mathfrak{C}(q^\bullet)$
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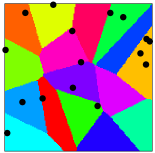

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Problem 2. In general, the total masses of the measures $1_{B_n}\mathfrak{L}$ and $1_{B_n}\mu^\omega$ will not coincide. No coupling will exist!

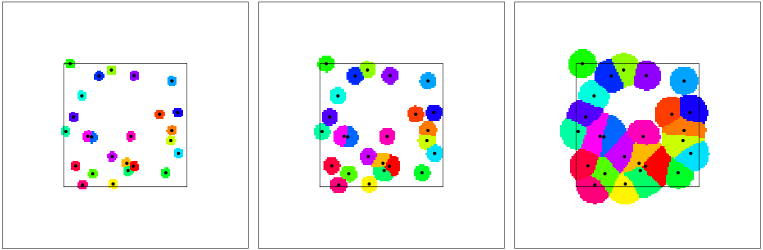
$\rightsquigarrow$ **Semicoupling**

Semicouplings

Given two measures ν, μ on $\mathbb{R}^d$ with $\nu(\mathbb{R}^d) \geq \mu(\mathbb{R}^d)$, we say that a measure q on $\mathbb{R}^d \times \mathbb{R}^d$ is a **semicoupling** of ν and μ iff the marginals satisfy

$$(\pi_1)_* q \leq \nu, \qquad (\pi_2)_* q = \mu.$$

In other words, q is a coupling of $\rho\nu$ and μ for some density $0 \leq \rho \leq 1$ on $\mathbb{R}^d$. ('Twofold minimization problem', 'free boundary value problem'.)



Cf. Figalli: 'partial coupling'

Semicouplings

Proposition 1.

For each finite set $Z \subset \mathbb{R}^d$ there exists a unique semicoupling q of $\mathfrak{L}$ and $\mu = \sum_{z \in Z} \delta_z$ which minimizes the cost functional

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \vartheta(|x-y|)\, dq(x,y).$$

Moreover, there exists a unique set $A \subset \mathbb{R}^d$ and a unique map $T : A \to \mathbb{R}^d$ s.t.

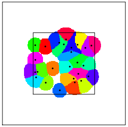
$$q = (Id, T)_*(1_A \mathfrak{L}).$$

In particular, $(\pi_1)_* q = 1_A \mathfrak{L}$.

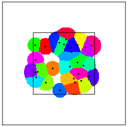
If $\vartheta(r) = r^2$ then $T = \nabla \varphi$ for some convex function $\varphi : A \to \mathbb{R}$.

Equivalently, $T : \mathbb{R}^d \to \mathbb{R}^d \cup \{\eth\}$ and $q = (Id, T)_* \mathfrak{L}$ on $\mathbb{R}^d \times \mathbb{R}^d$.

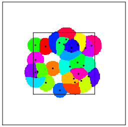
p=1



p=2



p=4



Existence of a Minimizer

- Fix exhausting sequence of cubes $B_n \nearrow \mathbb{R}^d$
- Consider optimal semicoupling q_n^ω of $\mathfrak{L}$ and $1_{B_n}\mu^\omega$
- Mean transportation cost for $q_n^\bullet$ should converge to $\inf_{q^\bullet \in \Pi} \mathfrak{C}(q^\bullet)$
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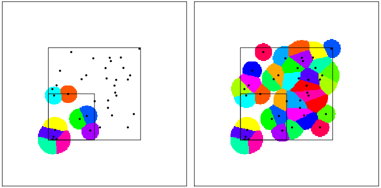

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The **asymptotic mean transportation cost** is given by

$$\mathfrak{c}_\infty = \lim_{n\to\infty} \inf_{q^\bullet \in \Pi_s} 2^{-nd} \cdot \mathbb{E}\left[\int_{\mathbb{R}^d \times [0,2^n)^d} \vartheta(|x-y|)\, dq^\bullet(x,y)\right]$$

where Π_s denotes the set of all semicouplings $q^\bullet$ of the Lebesgue measure and the point process.

Main Results

Theorem 1.

Whenever the asymptotic mean transportation cost is finite, there exists a unique translation invariant minimizer of the mean cost functional (**"optimal coupling"**).

Theorem 2. Let $\mu^\bullet$ be a Poisson point process of unit intensity and $\vartheta(r) = r^p$ for some $p \in (0, \infty)$.

The asymptotic mean transportation cost $\mathfrak{c}_\infty$ is finite if and only if

$$p < \overline{p} := \begin{cases} \infty, & \text{for } d \geq 3 \\ 1, & \text{for } d = 2 \\ \frac{1}{2}, & \text{for } d = 1. \end{cases}$$

Finiteness of Asymptotic Cost for PPP

Theorem 2.a Assume $d \geq 3$.

There exists a constant $0 < \kappa < \infty$ s.t.

$$\limsup_{r\to\infty} \frac{\log \vartheta(r)}{r^d} < \kappa \quad \Longrightarrow \quad \mathfrak{c}_\infty < \infty \quad \Longrightarrow \quad \liminf_{r\to\infty} \frac{\log \vartheta(r)}{r^d} \leq \kappa.$$

That is, $\vartheta(r) = \exp(C \cdot r^{\boldsymbol{d}})$ is borderline.

Theorem 2.b Assume $d \leq 2$.

For any concave $\hat{\vartheta} : [1, \infty) \to \mathbb{R}$ dominating ϑ

$$\int_1^\infty \frac{\hat{\vartheta}(r)}{r^{1+d/2}} dr < \infty \quad \Longrightarrow \quad \mathfrak{c}_\infty < \infty \quad \Longrightarrow \quad \liminf_{r\to\infty} \frac{\vartheta(r)}{r^{d/2}} = 0.$$

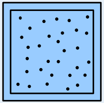
That is, $\vartheta(r) = r^{\boldsymbol{d}/2}$ is borderline.

Ajtai/Komlós/Tusnády '84, Talagrand '94, Holroyd/Peres '05, Hoffman/Holroyd/Peres '06.

Finiteness of Asymptotic Cost for PPP

CLT fluctuations

- r^d = average number of Poisson particles in box $[0, r)^d$
- $r^{d/2}$ = fluctuations of particle number
- $\epsilon \cdot r^{d-1}$ = volume of ϵ-neighborhood of box, $\epsilon = r^{1-d/2}$



Transportation cost per unit mass for $\vartheta(r) = r^p$:

$$\epsilon^p \quad \text{if } p \geq 1, \qquad\qquad r^{p-d/2} \quad \text{if } p \leq 1.$$

Large deviations

$$\mathbb{P}\left(\text{No particle in box } [0, r)^d\right) = \exp(-r^d)$$

If $\vartheta(r) \gg \exp(r^d)$ then cost of transporting Lebesgue measure from inside $[0, r)^d$ to exterior $\nearrow \infty$.

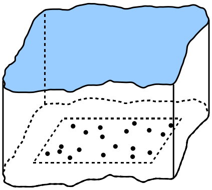
Finiteness of Asymptotic Cost for PPP

For each box B the mean transportation cost on B

$$\mathfrak{c}(B) = \inf_{q^\bullet \in \Pi} \frac{1}{\mathfrak{L}(B)} \cdot \mathbb{E}\left[\int_{\mathbb{R}^d \times B} \vartheta(|x-y|)\, dq^\bullet(x,y)\right]$$

can be estimated in terms of the modified cost $\widehat{\mathfrak{c}}(B)$:

Proposition 2. $\mathfrak{c}(B) \leq \widehat{\mathfrak{c}}(B) + \epsilon(|B|), \qquad \epsilon(|B|) \searrow 0.$



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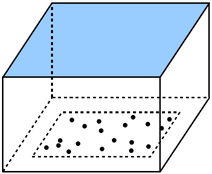

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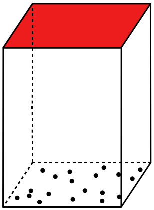

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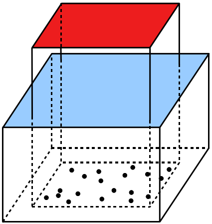
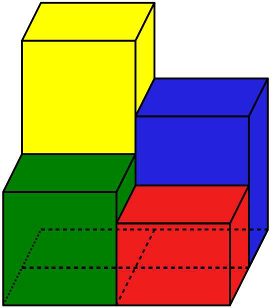


Finiteness of Asymptotic Cost for PPP

For boxes $B_n = [0, 2^n)^d$

Proposition 3. $\widehat{\mathfrak{c}}(B_{n+1}) \leq \widehat{\mathfrak{c}}(B_n) + 2^{-C\,n}.$



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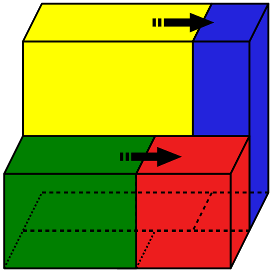

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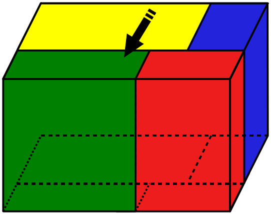
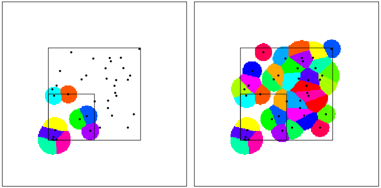


Existence of a Minimizer

Now let $\mu^\bullet$ be an arbitrary translation invariant point process of intensity $\beta \leq 1$ and with finite asymptotic cost $\mathfrak{c}_\infty$.

- Fix exhausting sequence of cubes $B_n \nearrow \mathbb{R}^d$
- Consider optimal semicoupling q_n^ω of $\mathfrak{L}$ and $1_{B_n}\mu^\omega$ ✓
- The q_n^ω should converge to optimal coupling q^ω of $\mathfrak{L}$ and μ^ω.



Problem 3.

▷ No tightness; no lower bound for the marginals of q^ω, only upper bounds $(\pi_1)_* q^\omega \leq \mathfrak{L}$, $(\pi_2)_* q^\omega \leq \mu^\omega$.

$\rightsquigarrow$ **Second Randomization**

Second Randomization

- Choose sequence $(B_n)_n$ randomly, starting at given B_0, in the n-th step adding $2^d - 1$ copies of B_{n-1} at arbitrary sides of it.
- For given $n \in \mathbb{N}$ the initial box B_0 has each possible "relative position within B_n" with equal probability.

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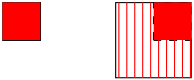

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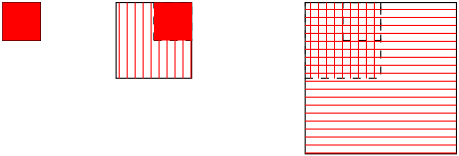

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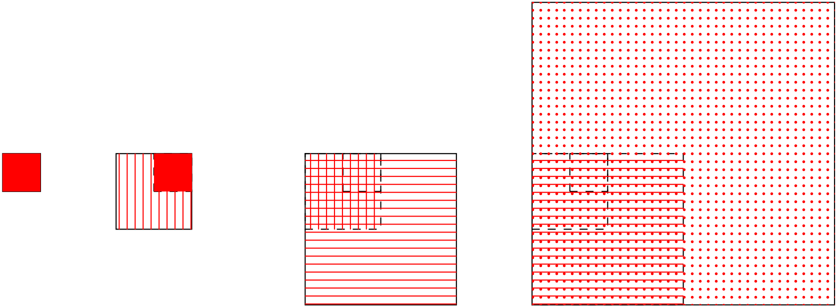

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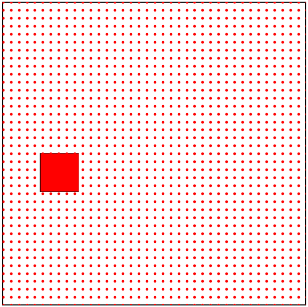

Second Randomization

Let $\Gamma = (\{0,1\}^d)^{\mathbb{N}}$ and $\nu =$ Bernoulli measure ('uniform distribution') on it. For each $z \in \mathbb{Z}^d, \gamma \in \Gamma$ and $n \in \mathbb{N}$ put

$$B_n(z,\gamma) \quad = \quad z - \sum_{k=1}^{n} 2^{k-1}\gamma_k + [0, 2^n)^d.$$

and let $q^{\omega}_{B_n(z,\gamma)}$ denote the minimizer of

$$\int_{\mathbb{R}^d \times B_n(z,\gamma)} \vartheta(|x-y|)\, dq^{\omega}(x,y)$$

which coincides with the optimal semicoupling of $\mathfrak{L}$ and $1_{B_n(z,\gamma)}\mu^{\omega}$ as constructed previously.

Annealed Limits

For z fixed put

$$q_n^\omega = \int_\Gamma q^\omega_{B_n(z,\gamma)}\, d\nu(\gamma).$$

Problem 4. Does $q_n^\omega \to q^\omega$ converge for a.e. ω?

Instead of $q_n^\omega \to q^\omega$ for $\mathbb{P}$-a.e. $\omega \in \Omega$ in the sense of convergence of measures on $\mathbb{R}^d \times \mathbb{R}^d$ we consider convergence $Q_n \to Q$ of measures on $\mathbb{R}^d \times \mathbb{R}^d \times \Omega$. Here

$$dQ_n(x, y, \omega) = dq_n^\omega(x, y)\, d\mathbb{P}(\omega).$$

(Vice versa, q_n^ω is obtained from Q_n via disintegration.)

Annealed Limits

Theorem.

Whenever the asymptotic mean transportation cost is finite, there exists a measure Q s.t.

$$Q_n \quad \longrightarrow \quad Q \qquad \text{vaguely as } n \to \infty.$$

The limit measure $Q = q^\bullet \, \mathbb{P}$

- is translation-invariant
- is a semicoupling of $\mathfrak{L}$ and $\mu^\bullet \mathbb{P}$
- is asymptotically optimal, i.e. it is a minimizer of the mean asymptotic cost.

If $\beta = 1$ then Q is indeed a coupling of $\mathfrak{L}$ and $\mu^\bullet \mathbb{P}$.

Quenched Limits

Let $q^\bullet$ be the disintegration of Q w.r.t. $\mathbb{P}$, i.e. $dq^\omega(x,y)\,d\mathbb{P}(\omega) = dQ(x,y,\omega)$.

Corollary.

For every $z \in \mathbb{Z}^d$

$$dq^\omega_{B_n(z,\gamma)}(x,y) \to dq^\omega(x,y) \qquad \text{vaguely as } n \to \infty$$

in probability w.r.t. $(\omega,\gamma) \in \Omega \times \Gamma$.

More precisely, each of the semicouplings is induced by a transport map s.t. for every $z \in \mathbb{Z}^d$

$$T^\omega_{n,z,\gamma}(x) \to T^\omega(x) \qquad \text{as } n \to \infty$$

in measure w.r.t. $(x,\omega,\gamma) \in \mathbb{R}^d \times \Omega \times \Gamma$.

Indeed, the sequence is finally stationary.

Local Optimality

Given a coupling q^ω of $\mathfrak{L}^d$ and μ^ω for fixed $\omega \in \Omega$, the following are equivalent:

- For all bounded Borel sets $A \subset \mathbb{R}^d$, the measure $1_{\mathbb{R}^d \times A} q^\omega$ is the unique optimal coupling between its marginals $q^\omega(., A)$ and $1_A \mu^\omega$.
- There exists a cyclically monotone map $T^\omega : \mathbb{R}^d \to \mathbb{R}^d$ such that
$$q^\omega = (Id, T^\omega)_* \mathfrak{L}.$$

A coupling $q^\bullet$ of Lebesgue measure and the point process is called **locally optimal** iff the previous properties are satisfied for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Optimality

A semicoupling $q^\bullet$ of Lebesgue measure and the point process is called **optimal** iff

- it is **translation invariant**: the distribution of the measure-valued random variable $q^\omega(x, y)$ is invariant under translations $(x, y) \mapsto (x + z, y + z)$ of $\mathbb{R}^d \times \mathbb{R}^d$ and
- it is **asymptotically optimal**: it minimizes the asymptotic mean transportation cost.

Examples.

- The map $T : x \mapsto \lfloor x \rfloor - 100$ defines a locally optimal + translation invariant coupling of $\mathfrak{L}$ and $\sum_{y \in \mathbb{Z}} \delta_y$.
- Any local perturbation/re-arrangement of an asymptotical optimal (semi-)coupling is again asymptotically optimal.

Optimality

A semicoupling $q^\bullet$ of Lebesgue measure and the point process is called **optimal** iff

- it is **translation invariant**: the distribution of the measure-valued random variable $q^\omega(x, y)$ is invariant under translations $(x, y) \mapsto (x + z, y + z)$ of $\mathbb{R}^d \times \mathbb{R}^d$ and
- it is **asymptotically optimal**: it minimizes the asymptotic mean transportation cost.

Theorem.

$$\text{Optimal} \quad \Longrightarrow \quad \text{locally optimal.}$$

Uniqueness

Theorem.

There exists at most one optimal semicoupling.

Proof. Assume two optimal semicouplings $q_1^\bullet$ and $q_2^\bullet$

$\Rightarrow \quad q^\bullet := \frac{1}{2}q_1^\bullet + \frac{1}{2}q_2^\bullet$ optimal semicoupling

$\Rightarrow \quad q_1^\bullet$, $q_2^\bullet$ and $q^\bullet$ locally optimal

$\Rightarrow \quad \exists$ maps T_1^ω, T_2^ω, T^ω s.t. on $\mathbb{R}^d \times \mathbb{R}^d$ for a.e. ω:

$$\begin{aligned} d\delta_{T^\omega(x)}(y)\, d\mathfrak{L}(x) &= dq^\omega(x,y) = d\left(\frac{1}{2}q_1^\omega(x,y) + \frac{1}{2}q_2^\omega(x,y)\right) \\ &= d\left(\frac{1}{2}\delta_{T_1^\omega(x)}(y) + \frac{1}{2}\delta_{T_2^\omega(x)}(y)\right) d\mathfrak{L}(x) \end{aligned}$$

$\Rightarrow \quad T_1^\omega(x) = T_2^\omega(x)$ for a.e. $x \in \mathbb{R}^d$ and thus $q_1^\omega = q_2^\omega$.

Summary

For each translation invariant point process $\mu^\bullet$ with (sub-)unit intensity consider asymptotic mean cost

$$\mathfrak{c}_\infty = \lim_{n\to\infty} \inf_{q^\bullet\in\Pi_s} 2^{-nd} \cdot \mathbb{E}\left[\int_{\mathbb{R}^d\times[0,2^n)^d} \vartheta(|x-y|)\, dq^\bullet(x,y)\right]$$

If $\mathfrak{c}_\infty < \infty$ then $\exists!$ optimal (semi-)coupling $q^\bullet$ of $\mathfrak{L}$ and $\mu^\bullet$:

(i) translation invariant

(ii) minimizing $\sup_B \frac{1}{\mathfrak{L}(B)} \cdot \mathbb{E}\left[\int_{\mathbb{R}^d\times B} \vartheta(|x-y|)\, dq^\bullet(x,y)\right]$ (which is independent of B under (i))

It is given in terms of a unique transport map $T: \mathbb{R}^d \mapsto \mathbb{R}^d(\cup\{\eth\})$.

Summary

For the Poisson point process with intensity $\beta \le 1$:

- If $d \ge 3$ or $\beta < 1$:

$$\mathfrak{c}_\infty < \infty \quad \Longleftrightarrow \quad \vartheta(r) \lessapprox \exp(C\, r^d)$$

- If $d \le 2$ and $\beta = 1$:

$$\mathfrak{c}_\infty < \infty \quad \Longleftrightarrow \quad \vartheta(r) \ll r^{d/2}.$$