A truly pathwise approach to polymer localization
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A truly pathwise approach to polymer localization

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So far, localization statements for directed polymers in random medium deal with the location of the endpoint ("favourite site" for polymer). We introduce a pathwise property, roughly: there exists a "favorite path" depending on the environment as well as the model parameters and time horizon, such that the polymer path has a significant overlap with the favorite path.

In a joint work with Mike Cranston, we establish this property in the parabolic Anderson model. We also obtain complete localization, i.e., the overlap tends to its maximal value 1 as the product (diffusivity $\times$ temperature^2) tends to 0.

Outline

The Parabolic Anderson Model

$X = (X(t), t \geq 0)$: symmetric simple random walk in $\mathbb{Z}^d$ with jump rate $\kappa > 0$ starting at 0. Law P_κ, expectation E_κ.
$W_x, x \in \mathbb{Z}^d$: i.i.d. Brownian motions. Expectation: E.

Anderson polymer model: the Gibbs measure on $\mathcal{D}_T = \mathcal{D}([0,T], \mathbb{Z}^d)$ with

$$\mu_{\kappa,\beta,T}(f) = \frac{1}{Z_{\kappa,\beta,T}} E_\kappa \left[f(X) \exp\left\{ \beta \int_0^T dW_{X(s)}(s) \right\} \right].$$

for $f : \mathcal{D}_T \to \mathbb{R}$. ($T > 0$ is time horizon.)

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for $f : \mathcal{D}_T \to \mathbb{R}$. ($T > 0$ is time horizon.) The model has two parameters:

- inverse temperature $\beta > 0$ measuring the fluctuations of the environment,
- diffusivity $\kappa \in (0, \infty)$ of the path under the *a priori* measure P_κ.

Denote

$$H_T(X) = \int_0^T dW_{X(s)}(s); \quad \text{Then, } Z_{\kappa,\beta,T} = E_\kappa \Big[\exp\{\beta H_T(X)\} \Big]$$

Lyapunov exponents

$$N(T,X) = \text{ number of jumps of } X \text{ on } [0, T].$$

Proposition (Microcanonical)

For $r \geq 0$, the following limit exists a.s. and in L^p, $p \in [1,\infty)$:

$$\Gamma(\beta, r) \quad = \quad \lim_{T\to\infty} T^{-1} \ln E_\kappa\left[\exp\{\beta H_T(X)\} | N(T,X) = [rT]\right]$$

It is deterministic, convex in β, continuous in r, independent of κ, and

$$\Gamma(\beta, r) = a\,\Gamma(a^{-1/2}\beta, a^{-1} r)\,, \qquad a > 0.$$

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Let I_κ be the Cramér transform of the Poisson distribution with parameter κ,

$$\mathrm{I}_\kappa(r) = r\ln(r/\kappa) - r + \kappa, \qquad r \geq 0.$$

Proposition (Variational formula for free energy)

$$\Psi(\kappa, \beta) \stackrel{\exists}{=} \lim_{T\to\infty} T^{-1} \ln Z_{\kappa,\beta,T} = \sup\{\Gamma(\beta, r) - \mathrm{I}_\kappa(r);\ r \geq 0\}$$

From scaling relation for Γ,

$$\Psi(\kappa, \beta) = \beta^2 \Psi(\beta^{-2}\kappa, 1) = \kappa \Psi(1, \kappa^{-1/2}\beta).$$

Define $I_{\kappa,\beta}$,

$$I_{\kappa,\beta}(r) = -\Gamma(\beta, r) + I_\kappa(r) + \Psi(\kappa, \beta),$$

a convex function in r.

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Theorem (Large deviations principle for the number of jumps)

Then

$$\lim_{T\to\infty, n/T\to r} T^{-1} \ln \mu_{\kappa,\beta,T}\big(N(T,X) = n\big) = -\mathrm{I}_{\kappa,\beta}(r), \text{ a.s..}$$

Moreover, for a.e. realization of the environment, and all subsets $B \subset \mathbb{R}_+$,

$$\begin{aligned} -\inf_{B^o} \mathrm{I}_{\kappa,\beta}(r) &\le \liminf_{T\to\infty} T^{-1} \ln \mu_{\kappa,\beta,T}\big(N(T,X)/T \in B\big) \\ &\le \limsup_{T\to\infty} T^{-1} \ln \mu_{\kappa,\beta,T}\big(N(T,X)/T \in B\big) \le -\inf_{\bar{B}} \mathrm{I}_{\kappa,\beta}(r). \end{aligned}$$

Remarks

Literature: R.Carmona-Molchanov ’94 (intermittency)
Biskup, P.Carmona, Cranston, Gärtner, van der Hofstad, den Hollander, Hu, König, Koralov, Mörters, Mountford, Rovira, Shiga, Tindel, Viens, Zeldovich... many others!
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(i) *Annealed bound*. By Jensen's inequality, and since $Ee^{\beta H_T(X)} = e^{\beta^2/2}$, both

$$\Psi(\kappa,\beta) \le \beta^2/2 \quad \text{and} \quad \Gamma(\beta, r) \le \beta^2/2$$

hold $(\forall \kappa, r)$. Then,

$$\Psi(\kappa,\beta) = \beta^2/2 \iff \Gamma(\beta,\kappa) = \beta^2/2,$$

and, in such a case, $\mathrm{I}_{\kappa,\beta}(r)$ has a unique minimum at $r = \kappa$.

(ii) *Weak versus strong disorder*.

$$\Psi(\kappa,\beta) = \beta^2/2 \iff \beta^2/\kappa \le \Upsilon_c,$$

for some critical value $\Upsilon_c \in [0,\infty)$ depending only on the dimension.

- $d \ge 3 \implies \Upsilon_c > 0$
- $d = 1, 2$: $\Upsilon_c = 0$ is expected, in view of results for other models: discrete models (Lacoin, Vargas,. . .) or continuous (Bertin)

Asymptotics of Lyapunov exponents

Theorem (Asymptotics of free energy)

As $\beta^2/\kappa \to \infty$,

$$\Psi(\kappa, \beta) \sim \frac{\alpha^2 \beta^2}{4 \ln(\beta^2/\kappa)}$$

Ref: R.Carmona-Koralov-Molchanov'01, Cranston-Mountford-Shiga'02; Also R.Carmona-Molchanov-Viens'96.

$$\alpha = \lim_{n\to\infty} \frac{1}{n} A_n \,, \quad A_n = \sup_{\gamma : N(n,\gamma)=n} H_n(\gamma) \,.$$

$\exists$ limit: sup of Gaussian process indexed by a set with suitably bounded entropy.

Sketch: Fix T first:

$$\lim_{\beta\to\infty} \frac{1}{\beta T} \ln E_\kappa\left[e^{\beta H_T(X)} | N(T,X) = [rT]\right] = \sup_{\gamma: N(T,\gamma)=rT} H_T(\gamma) =: A_{T,r} \,.$$

$$\begin{aligned}
\lim_{\beta\to\infty} \beta^{-1}\Gamma(\beta, r) &= \lim_{\beta\to\infty}\lim_{T\to\infty} \frac{1}{\beta T} \ln E_\kappa\left[e^{\beta H_T(X)} | N(T,X) = [rT]\right] \\
&= \lim_{T\to\infty} T^{-1} A_{T,r} \quad \text{(interchange limits)} \\
&= \sqrt{r}\lim_{T\to\infty} T^{-1} A_T \quad (\text{scaling} : A_{T,r} \stackrel{\mathcal{L}}{=} \sqrt{r} A_T) \\
&= \alpha\sqrt{r}
\end{aligned}$$

Then, for $\kappa = 1$ and $\beta \to \infty$,

$$\Psi(1, \beta) \sim \sup\left\{\alpha\beta\sqrt{r} - \mathrm{I}_1(r) :\ r \geq 0\right\} = \frac{\alpha^2\beta^2}{4\ln(\beta^2)}$$

which yields the case of general κ by scaling. □

For the maximizer $r_{\max}(\kappa, \beta)$ in Variational Formula, $r_{\max}(\kappa, \beta) \sim \frac{\alpha^2\beta^2}{4\ln^2(\beta^2/\kappa)}$.
Note : For large β^2/κ,

$$\Psi(\kappa, \beta) << \beta^2/2 \,, \quad r_{\max}(\kappa, \beta) >> \kappa$$

Outline

Two versions of the Overlap

Localization properties are derived through the overlap between two independent polymer paths $X, \tilde{X}$ sharing the same environment.
Two versions appear:

$$I_{\kappa,\beta,T} = \frac{1}{T}\int_0^T \mu^{\otimes 2}_{\kappa,\beta,t}(X(t)=\widetilde{X}(t))dt$$

$$J_{\kappa,\beta,T} = \frac{1}{T}\int_0^T \mu^{\otimes 2}_{\kappa,\beta,T}(X(t)=\widetilde{X}(t))dt = \mu^{\otimes 2}_{\kappa,\beta,T}\Big(\underbrace{\frac{1}{T}\int_0^T \mathbf{1}\{X(t)=\widetilde{X}(t)\}dt}_{\text{proportion of time together}}\Big)$$

How do they appear: (i) By Itô's formula,

$$d\ln Z_{\kappa,\beta,t} = \beta\mu_{\kappa,\beta,t}(dW_{X(t)}(t)) + \frac{\beta^2}{2}\left(1 - \mu^{\otimes 2}_{\kappa,\beta,t}(X(t)=\widetilde{X}(t))\right)dt\ ,$$

and, upon integration we get

$$\frac{1}{T}\ln Z_{\kappa,\beta,T} = \frac{1}{T}M_T + \frac{\beta^2}{2}(1 - I_{\kappa,\beta,T})$$

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Limits of overlaps and relation with free energy

As $T \to \infty$, the martingale term vanishes, so we get (i) below.

Theorem

(i) For all β and κ,

$$\widetilde{I}_{\kappa,\beta,\infty} \overset{\exists}{:=} \lim_{T\to\infty} I_{\kappa,\beta,T} = 1 - \frac{2}{\beta^2}\Psi(\kappa,\beta)$$

(j) The limit

$$\widetilde{J}_{\kappa,\beta,\infty} = \lim_{T\to\infty} E\left[J_{\kappa,\beta,T}\right]$$

exists except for exceptional values of β^2/κ, and

$$\widetilde{J}_{\kappa,\beta,\infty} = 1 - \beta^{-1}\frac{\partial}{\partial\beta}\Psi(\kappa,\beta).$$

Integration by parts

Proof of (j): • Key is the identity

$$\frac{\partial}{\partial\beta} E[\ln Z_{\kappa,\beta,T}] = \beta T \left[1 - E[J_{\kappa,\beta,T}]\right] \quad (*)$$

It comes by integration by parts, like in spin-glass models (cf. Talagrand's books). The basic, Gaussian integration by parts formula writes

$$EgF(g) = \sigma^2 EF'(g) \text{ for } \quad g \sim \mathcal{N}(0, \sigma^2).$$

With our Brownian environment, we need the integration by parts formula from Malliavin calculus is: for F a smooth function of the $(W_x(t))_{t,x}$ and $h(t,x)$ deterministic, $\|h\|_2 < \infty$,

$$E\left[F \times \int_0^T h(t,x) dW_x(t)\right] = E\left[\int_0^T h(t,x)\ D_{t,x}F\ dt\right]$$

The Malliavin derivative $D_{t,x}$ is heuristically equal to $\frac{\partial}{\partial(dW_x(t))}$.

Integration by parts

$$
\begin{aligned}
\frac{\partial}{\partial \beta} E[\ln Z_{\kappa,\beta,T}] &= E[\mu_{\kappa,\beta,T}(H_T(X))] \\
&= E\left[\mu_{\kappa,\beta,T}\left(\sum_{x\in\mathbb{Z}^d}\int_0^T dW_x(t)\delta_x(X(t))\right)\right] \\
&= \sum_{x\in\mathbb{Z}^d} E_\kappa\left[E\left[\frac{e^{\beta H_T(X)}}{Z_{\kappa,\beta,T}}\int_0^T dW_x(t)\delta_x(X(t))\right]\right]
\end{aligned}
$$

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&\overset{i.b.p.}{=} \sum_{x\in\mathbb{Z}^d} E_\kappa\left[E\int_0^T \left[D_{t,x}\frac{e^{\beta H_T(X)}}{Z_{\kappa,\beta,T}}\right]\delta_x(X(t))\right] dt
\end{aligned}
$$

$$
= \beta E_\kappa\left[E\sum_{x\in\mathbb{Z}^d}\int_0^T\left[\left(\delta_x(X_t)\frac{e^{\beta H_T(X)}}{Z_{\kappa,\beta,T}} - \frac{e^{\beta H_T(X)}}{Z_{\kappa,\beta,T}}\frac{E_\kappa[\delta_x(\tilde{X}_t)e^{\beta H_T(\tilde{X})}]}{Z_{\kappa,\beta,T}}\right)\right]\delta_x(X_t)\right] dt
$$

$$
= \beta E\sum_{x\in\mathbb{Z}^d}\int_0^T\left[\mu_{\kappa,\beta,T}(\delta_x(X(t))) - \mu_{\kappa,\beta,T}(\delta_x(X(t)))^2\right] dt = \beta T\left[1 - E[J_{\kappa,\beta,T}]\right].
$$

Hence we got the key identity

$$\frac{\partial}{\partial\beta} E[\ln Z_{\kappa,\beta,T}] = \beta T \left[1 - E[J_{\kappa,\beta,T}]\right] \quad (*)$$

• By standard convexity arguments: Ψ is differentiable in β outside an at most countable set; At such point we have

$$\begin{aligned} \widetilde{J}_{\kappa,\beta,\infty} : &\overset{\exists}{=} \lim_{T\to\infty} E\left[J_{\kappa,\beta,T}\right] \\ &= 1 - \beta^{-1}\frac{\partial}{\partial\beta}\Psi(\kappa,\beta) \end{aligned}$$

□

Outline

From (i): results in a classical formulation

The critical value corresponds to the localization transition:

$$\beta^2/\kappa > \Upsilon_c \iff \widetilde{I}_{\kappa,\beta,\infty} > 0$$

For fixed κ, β, define the favourite site $x^*(t)$ for the polymer at time t by

$$x^*(t) = \arg\max \left\{ E_\kappa \left[\exp\{\beta H_t(X)\}; X(t) = x\right] : x \in \mathbb{Z}^d \right\}$$

Theorem

$$\beta^2/\kappa > \Upsilon_c \iff \begin{cases} \exists C = C(\beta^2/\kappa) > 0 : \\ \liminf_{T\to\infty} \frac{1}{T} \int_0^T \mu_{\kappa,\beta,t}(X(t) = x^*(t))dt \geq C \quad a.s. \end{cases}$$

Pointwise result, in the style of Carmona-Hu'02, FC-Shiga-Yoshida'03.

From (j)

- The critical value is when localization starts:

$$\Upsilon_c = \inf\{\beta^2/\kappa : \widetilde{J}_{\kappa,\beta,\infty} > 0\}.$$

- Let

$$\mathcal{D}_\kappa = \{\beta > 0 : \exists \frac{\partial}{\partial\beta}\Psi(\kappa,\beta)\ ,\ \frac{\partial}{\partial\beta}\Psi(\kappa,\beta) < \beta\}$$

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Clearly, $\mathcal{D}_\kappa \subset [\kappa^{1/2}\Upsilon_c, \infty)$. Conjecture: $\mathcal{D}_\kappa = (\kappa^{1/2}\Upsilon_c, \infty)$.

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- Weaker statements on the (larger) set of increase of $\beta \mapsto \beta^2/2 - \Psi(\kappa,\beta)$: for such a β, $\exists \beta_n \to \beta$ such that

$$\lim_{T\to\infty} E\mu^{\otimes 2}_{\kappa,\beta_n,T}\left(\frac{1}{T}\int_0^T \mathbf{1}_{X(t)=\tilde{X}(t)}dt\right) > 0.$$

On the contrary, for β where the difference is locally constant, then the limit is 0 at β and around.

From (j): favourite path

For fixed κ, β, define the "favourite path" y_T^* for the polymer with time horizon T as the function

$$y_T^*(t) = \arg\max \left\{ E_\kappa \left(\exp\{\beta H_T(X)\}; X(t) = y \right); y \in \mathbb{Z}^d \right\}$$

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Theorem (FC-Cranston'11)

$$\beta \in \mathcal{D}_\kappa \Longrightarrow \liminf_{T\to\infty} E\mu_{\kappa,\beta,T} \left(\frac{1}{T} \int_0^T \mathbf{1}\{X(t) = y_T^*(t)\} dt \right) \geq C(\beta, \kappa) > 0.$$

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Provides an information on the path itself.

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Similar to above, a weaker form holds on the set of increase of the difference.

On favourite attributes

What are those favourite attributes ? x^* and y^*_T:
Both depend on κ, β (also T) and on the environment $\mathcal{W}$.
Both have long jumps: in particular the "favourite path" is not a path...
y^*_T and x^* are equal at time $t = T$, but they are not related otherwise.

Fundamental difference: The mapping $t \mapsto x^*(t)$ has oscillations at those times t when there are many maximizers: the set of jump times then looks locally like the set of zeros of Brownian motion. In contrast, from differentiability below, we see that $t \mapsto y^*_T(t)$ has no oscillations, typically.
The favourite path is much smoother than the favourite end-point.

Proposition

(i) The function $t \mapsto E_\kappa\left[\exp\{\beta H_t(X)\}\delta_x(X(t))\right]$ *is a.s. Hölder continuous of every order less than* $1/2$.
(ii) The function $t \mapsto E_\kappa\left[\exp\{\beta H_T(X)\}\delta_x(X(t))\right]$ *is almost surely of* $\mathcal{C}^1$ *class on* $[0, T]$.

□ (i) $Z(t,x) = E_\kappa\left[\exp\{\beta H_t(X)\}\delta_x(X(t))\right]$ solves the stochastic heat equation

$$dZ(t,x) = \kappa \Delta Z dt + \beta Z \circ dW_x(t).$$

It has the regularity of Brownian motion.

(ii) On the other hand,

$$E_\kappa\left[\exp\{\beta H_T(X)\}\delta_x(X(t))\right] \times \exp -\beta W_z(T)$$

is (continuously) differentiable at t when $z = x$. □

Outline

As $\boxed{\beta^2/\kappa \to \infty}$:

- Recall that $\Psi(\kappa,\beta) \sim \frac{\alpha^2\beta^2}{4\ln(\beta^2/\kappa)} \ll \beta^2/2$, and moreover, that

$$\frac{\beta^2}{2} - \Psi(\kappa,\beta) = \frac{\beta^2}{2}\widetilde{I}_{\kappa,\beta,\infty}, \quad \text{and } \frac{\partial}{\partial\beta}\left(\frac{\beta^2}{2} - \Psi(\kappa,\beta)\right) = \beta\widetilde{J}_{\kappa,\beta,\infty}.$$

- This implies that both $\widetilde{I}_{\kappa,\beta,\infty}$ and $\widetilde{J}_{\kappa,\beta,\infty}$ tend to 1 at rate $\mathcal{O}(1/\ln(\beta^2/\kappa))$. Since 1 is the maximal value, the localization is complete.
- Similarly,

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \mu_{\kappa,\beta,t}(X(t) \neq x^*(t))dt = \mathcal{O}(1/\ln(\beta^2/\kappa))$$

$$\limsup_{T\to\infty} E\mu_{\kappa,\beta,T}\left(\frac{1}{T}\int_0^T \mathbf{1}\{X(t) \neq y_T^*(t)\}dt\right) = \mathcal{O}(1/\ln(\beta^2/\kappa))$$

(No need to restrict $\beta \in \mathcal{D}_\kappa$.)

Conclusion

- When the discrepancy between quenched and annealed free energy increases, the random path sticks to the favourite one for a significant fraction of the time.
  The fraction of times grows to 1 as diffusivity $\times$ temperature^2 vanishes.
- Path localization and complete localization also hold in other models. We proved that for Parabolic Anderson model, but this works as soon as the environment is Gaussian.

  Another example: Brownian polymer in Poissonian medium (FC-Yoshida). Poisson variable have a nice integration by parts formula.