Spectral asymptotics for stable trees and the critical random graph

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Based on joint work with
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α-STABLE TREES

α-stable trees are natural objects:

- scaling limits of conditioned Galton-Watson trees (Aldous, Duquesne);
- can be described in terms of continuous state branching processes (Duquesne/Le Gall);
- can be constructed from Lévy processes (Duquesne/Le Gall);
- and fragmentation processes (Haas/Miermont).

Moreover, the Brownian continuum random tree ($\alpha = 2$) has connections with:

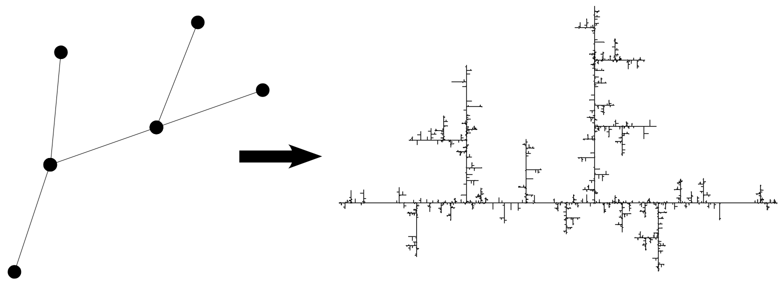
- combinatorial trees (Aldous);
- critical percolation clusters in high dimensions (Hara/Slade);
- the Erdős-Rényi random graph at criticality (Addario-Berry/Broutin/Goldschmidt).

GALTON-WATSON TREE SCALING LIMIT

Fix $\alpha \in (1,2]$. Let Z be a mean 1, aperiodic, random variable in the domain of attraction of an α-stable law, i.e. for some $a_n \to \infty$,

$$\frac{Z[n]-n}{a_n} \to \Xi,$$

where $Z[n]$ is the sum of n independent copies of Z, and Ξ satisfies $\mathbf{E}(e^{-\lambda\Xi}) = e^{-\lambda^\alpha}$. NB. $a_n = n^{\frac{1}{\alpha}}L(n)$.



Let T_n be a Galton-Watson branching process with offspring distribution Z, conditioned to have n vertices, then

$$n^{-1}a_n T_n \to \mathcal{T}^{(\alpha)}.$$

BROWNIAN MOTION ON α-STABLE TREES

Let $(X_t^{T_n})_{t\geq 0}$ be the discrete time simple random walk on T_n, then

$$\left(n^{-1}a_n X^{T_n}_{n^2 a_n^{-1} t}\right) \to \left(X_t^{\mathcal{T}^{(\alpha)}}\right)_{t\geq 0},$$

where $(X_t^{\mathcal{T}^{(\alpha)}})_{t\geq 0}$ is a strong Markov diffusion on $\mathcal{T}^{(\alpha)}$ – the Brownian motion on the α-stable tree [C.].

We will write $\Delta_{(\alpha)}$ for the generator of $(X_t^{\mathcal{T}^{(\alpha)}})_{t\geq 0}$ (i.e. its semigroup is $P_t = e^{t\Delta_{(\alpha)}}$), and the corresponding Dirichlet form as:

$$\mathcal{E}_{(\alpha)}(f,g) := -\int_{\mathcal{T}^{(\alpha)}} f\Delta_{(\alpha)} g d\mu_\alpha,$$

where μ_α is the natural probability measure on $\mathcal{T}^{(\alpha)}$ that arises as the scaling limit of uniform probability measures on Galton-Watson trees.

SPECTRUM OF Δ_α

We say λ is a (Neumann) eigenvalue for $\mathcal{E}_{(\alpha)}$ if: there exists an $f \in \mathcal{F}_{(\alpha)}$ such that

$$\mathcal{E}_{(\alpha)}(f,g) = \lambda \int_{\mathcal{T}^{(\alpha)}} fg d\mu_\alpha, \quad \forall g \in \mathcal{F}_{(\alpha)}.$$

Roughly speaking, $-\Delta_{(\alpha)} f = \lambda f$ and 0 derivative on 'boundary'.

The eigenvalue counting function is:

$$N_{(\alpha)}(\lambda) := \# \left\{ \text{eigenvalues of } \mathcal{E}_{(\alpha)} \leq \lambda \right\}.$$

From the spectral decomposition of the transition density of $X^{\mathcal{T}^{(\alpha)}}$, we have

$$\int_{\mathcal{T}^{(\alpha)}} p_t^{(\alpha)}(x,x) \mu_{(\alpha)}(dx) = \int_0^\infty e^{-\lambda t} N_{(\alpha)}(d\lambda),$$

and so if $N_{(\alpha)}(\lambda) \approx \lambda^\gamma$, then $p_t^{(\alpha)}(x,x) \approx t^{-\gamma}$.

PLAN

- Introduction to spectral asymptotics.
- Spectral asymptotics of self-similar fractals via renewal arguments.
- Adaptation to random fractals using branching processes.
- Consideration of α-stable trees.
- Application to critical Erdős-Rényi random graph.

CAN ONE HEAR THE SHAPE OF A DRUM?

Suppose D is a domain (with smooth boundary).

Consider the associated Dirichlet problem:

$$\Delta u + \lambda u = 0,$$

$$u|_{\partial D} = 0.$$

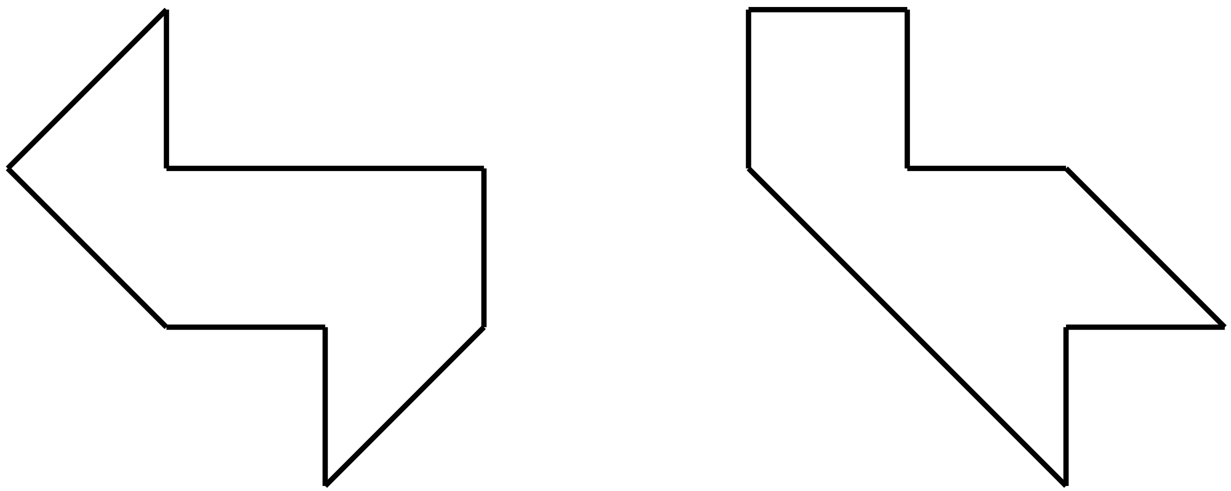
This has discrete spectrum of eigenvalues: $0 < \lambda_1 \leq \lambda_2 \leq \ldots$.

If D_1 and D_2 have same eigenvalue sequence, must they be congruent? (Kac 1966, Bochner, Bers...).

NO!

There exist isospectral pairs of manifolds (Milnor 1964, Witt).

There also exist isospectral pairs of domains in the plane, e.g.



(Gordon/Webb/Wolpert 1992, Buser/Semmler).

So does the spectrum tell us anything about the domain?

WEYL'S FORMULA

(Weyl 1911, 1912, Ivrii 1980) For a d-dimensional compact manifold with smooth boundary, the eigenvalue counting function

$$N(\lambda) := \#\{n \geq 1 : \lambda_n \leq \lambda\}$$

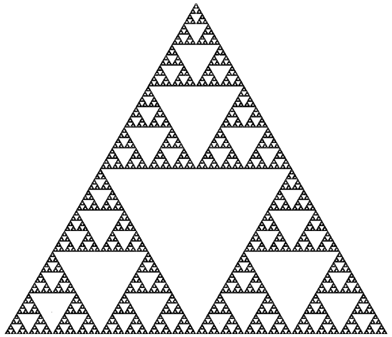
satisfies

$$N(\lambda) = c_d|D|\lambda^{d/2} - \frac{1}{4}c_{d-1}|\partial D|_{d-1}\lambda^{(d-1)/2} + o(\lambda^{(d-1)/2}).$$

NB. c_d are constants that only depend on d.

OUT OF THE CLASSICAL SETTING

Does the same result hold for (self-similar) fractal spaces?



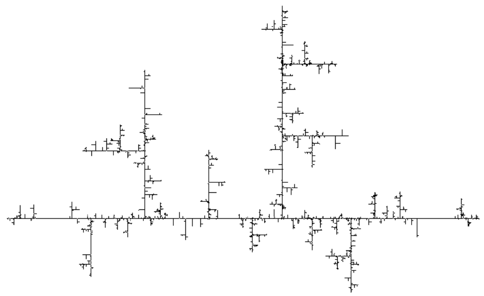
The Sierpinski gasket, for example, has Hausdorff dimension $\ln 3/\ln 2$, so is it the case that

$$N(\lambda) \sim c\lambda^{\ln 3/2\ln 2}?$$

Does boundary appear in second order term?

How about when there is randomness in the construction?

Do we see any effect of this in the spectral asymptotics, or is the disorder 'averaged' out in a law of large numbers?



RESULTS FOR THE SIERPINSKI GASKET

Rammal/Toulouse 1983, Fukushima/Shima 1992, Kigami/Lapidus 1993, Kigami 2001,

$$N(\lambda) = \lambda^{\ln 3/\ln 5} G(\ln \lambda) + O(1),$$

where G is a non-constant function with period $\ln 5$. Note that:

1. Spectral dimension

$$d_S := \lim_{\lambda\to\infty} \frac{2\ln N(\lambda)}{\ln \lambda} = \frac{\ln 9}{\ln 5}$$

is not equal to Hausdorff dimension.

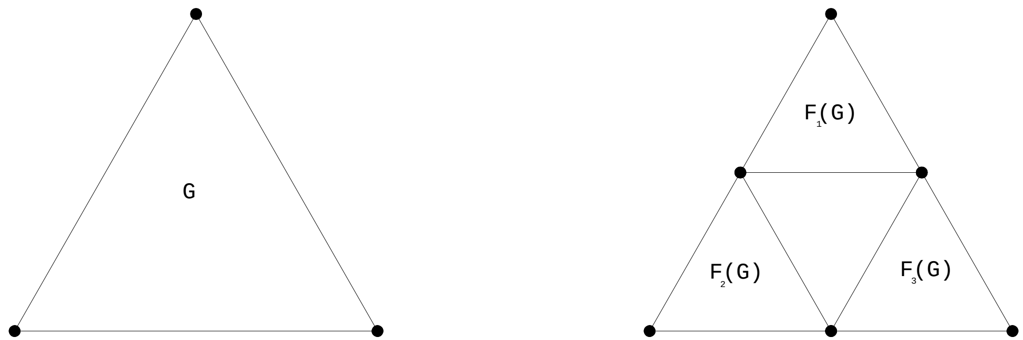
2. Weyl limit $\lim_{\lambda\to\infty} \lambda^{-d_S/2} N(\lambda)$ does not exist.

3. Size of second order term corresponds to an intrinsic boundary of the fractal, rather than a Euclidean one.

SIERPINSKI GASKET SELF-SIMILARITY

The Sierpinski gasket K satisfies

$$K = \bigcup_{i=1}^{3} F_i(K).$$



Moreover, the natural Dirichlet form on K satisfies:

$$\mathcal{E}(f,f) = \frac{5}{3} \sum_{i=1}^{3} \mathcal{E}(f \circ F_i, f \circ F_i).$$

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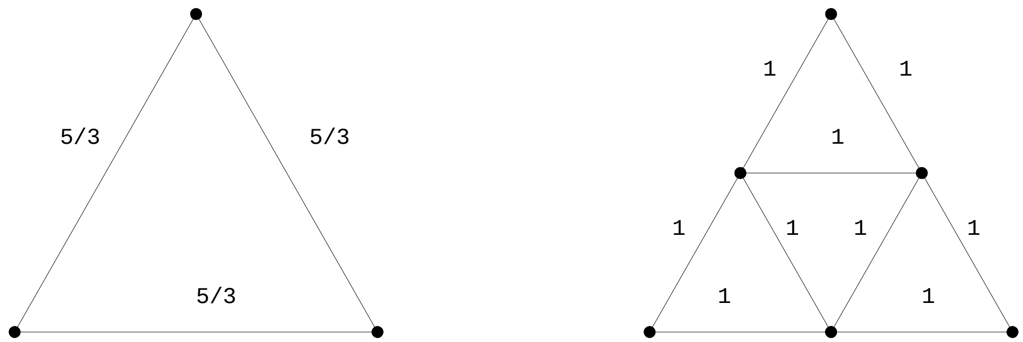

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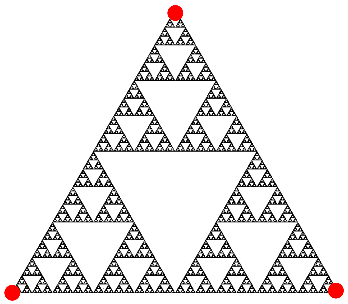
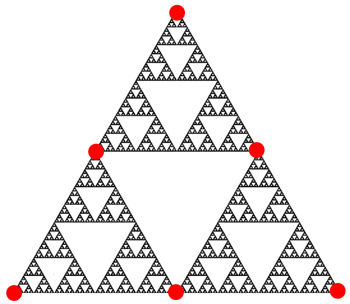
EIGENVALUE COMPARISON #1

Consider two versions of the Dirichlet problem:

$\mathcal{E}^D$  $\tilde{\mathcal{E}}$ 

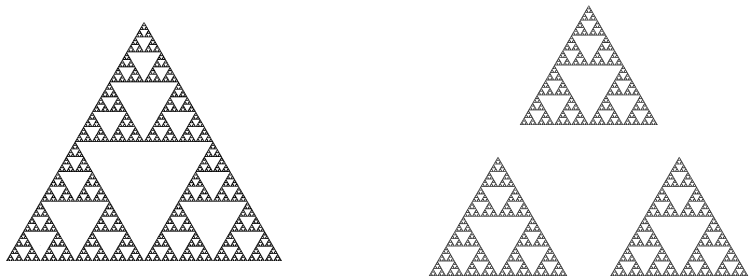
If f is an e.function of $\mathcal{E}^D$ with e.value λ, then can check

$$g := \begin{cases} f \circ F_i^{-1}, & \text{on } F_i(K), \\ 0 & \text{otherwise,} \end{cases}$$

is a e.function of $\tilde{\mathcal{E}}$ with e.value 5λ. It follows that

$$3N^D(\lambda/5) \leq \tilde{N}(\lambda) \leq N^D(\lambda).$$

EIGENVALUE COMPARISON #2



Similarly, if f is an e.function of $\mathcal{E}$ with e.value λ, then can check

$$f \circ F_i, \quad i = 1, 2, 3,$$

are e.functions of $\mathcal{E}$ with e.value $\lambda/5$. It follows that

$$N(\lambda) \leq 3N(\lambda/5).$$

'Dirichlet-Neumann bracketing' says $N^D(\lambda) \leq N(\lambda) \leq N^D(\lambda) + 3$, and so

$$3N^D(\lambda/5) \leq N^D(\lambda) \leq 3N^D(\lambda/5) + 9.$$

RENEWAL EQUATION

Write

$$m(t) := e^{-\gamma t} N^D(e^t) \text{ and } u(t) := e^{-\gamma t}\left(N^D(e^t) - 3N^D(e^t/5)\right).$$

Then

$$\begin{aligned} m(t) &= u(t) + 3e^{-\gamma \ln 5} m(t - \ln 5) \\ &= u(t) + \int_0^\infty m(t-s) 3e^{-\gamma s} \delta_{\ln 5}(ds) \\ &= u(t) + \int_0^\infty m(t-s) \delta_{\ln 5}(ds), \end{aligned}$$

where we have chosen $\gamma = \ln 3 / \ln 5$. The renewal theorem then implies

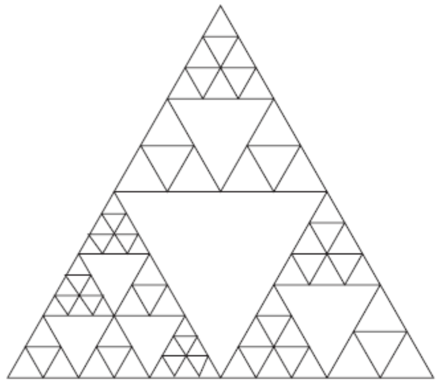
$$m(t) \sim G(t),$$

where G is a $\ln 5$-periodic function. Hence,

$$N(\lambda) \sim N^D(\lambda) \sim \lambda^{\ln 3 / \ln 5} G(\ln \lambda).$$

RANDOM SIERPINSKI GASKET [HAMBLY]

Consider a random Sierpinski gasket:

e.g. select from  and  to build .

We now have a statistical self-similarity for the Dirichlet form:

$$\mathcal{E}(f,f) = \sum_{i=1}^{M} \frac{1}{R_i} \mathcal{E}_i(f \circ F_i, f \circ F_i).$$

RENEWAL EQUATION IN MEAN

This translates to the e.value counting function:

$$\sum_{i=1}^{M} N_i^D(\lambda R_i \mu_i) \leq N^D(\lambda) \leq \sum_{i=1}^{M} N_i^D(\lambda R_i \mu_i) + C.$$

Let $m(t) := e^{-\gamma t}\mathbf{E}N^D(e^t)$ and

$$u(t) := e^{-\gamma t}\mathbf{E}\left(N^D(e^t) - \sum_{i=1}^{M} N_i^D(\lambda R_i \mu_i)\right),$$

then

$$m(t) = u(t) + \int_0^\infty m(t-s)e^{-\gamma s}\nu(ds),$$

where $\nu([0,t]) = \mathbf{E}\#\{i \leq M : -\ln R_i\mu_i \leq t\}$. Under a non-lattice condition, if γ is chosen such that $\int e^{-\gamma s}\nu(ds) = 1$, then

$$m(t) \to m(\infty) \in (0,\infty).$$

QUENCHED BRANCHING PROCESS

Let

$$X(t) := N^D(e^t) \text{ and } \eta(t) := N^D(e^t) - \sum_{i=1}^{M} N_i^D(\lambda R_i \mu_i),$$

then

$$X(t) = \eta(t) + \sum_{i=1}^{M} X_i(t + \ln R_i \mu_i).$$

Iterating, it is possible to write:

$$X(t) = \sum_{\mathbf{i} \in \Sigma} \eta_{\mathbf{i}}(t + \ln R_{\mathbf{i}} \mu_{\mathbf{i}}),$$

the sum of a random characteristic over the elements of a continuous time branching process. It follows that

$$N^D(\lambda) = \lambda^\gamma W + o(\lambda^\gamma),$$

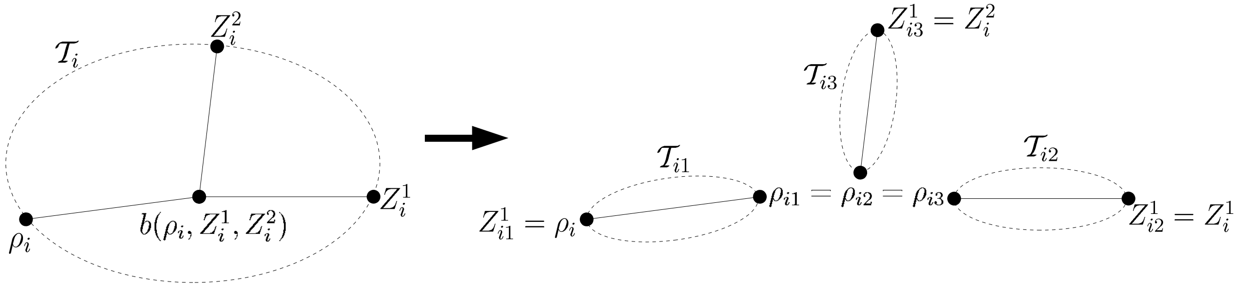
where the random variable W represents a limiting ‘mass’.

SELF-SIMILARITY OF THE BROWNIAN CRT

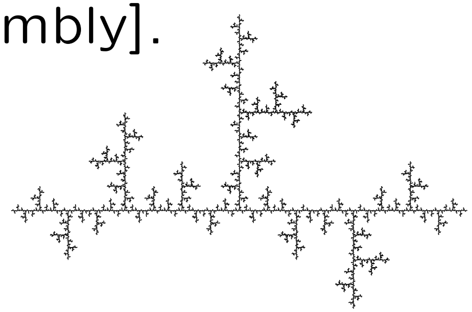
It is known that $\mathcal{T} = \mathcal{T}^{(2)}$ is self-similar (Aldous 1994):

- Pick 2 random points from Brownian CRT.
- Split tree at branch-point of root and these.

Results in a three copies of original tree with masses Dirichlet $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ distribution, and lengths scaled by square-root of these.



Can apply recursively to code Brownian CRT as a random self-similar fractal that is almost-surely homeomorphic to a deterministic fractal [C., Hambly].



MEAN AND ALMOST-SURE FIRST ORDER BEHAVIOUR [C., HAMBLY]

From the renewal equation, we are able to prove: for some deterministic $C \in (0,\infty)$,

$$\mathbf{E}N(\lambda) = C\lambda^{2/3} + O(1).$$

Using invariance under re-rooting, this implies:

$$\mathbf{E}p_t(\rho,\rho) = \mathbf{E}\int_{\mathcal{T}} p_t(x,x)\mu(dx) = C't^{-2/3} + O(1).$$

Moreover, from the branching process argument, we have that: $\mathbf{P}$-a.s.,

$$N(\lambda) = C\lambda^{2/3} + o(\lambda^{2/3}).$$

Also trace of semigroup is asymptotically smooth (cf. log-logarithmic fluctuations that occur in $p_t(\rho,\rho)$, almost surely, [C.])

SECOND ORDER BEHAVIOUR

Let $Y(t) := e^{-2t/3}X(t) - m(t)$. Can use a renewal equation to show that:

$$e^{t/3}\mathbf{E}Y(t)^2 \to y(\infty).$$

One can then use the (approximate) stochastic self-similarity,

$$Y(t) \approx \sum_{i \in \Lambda_{\varepsilon t}} D_i Y_i(t + \frac{3}{2} \ln D_i),$$

to prove a central limit theorem, saying that:

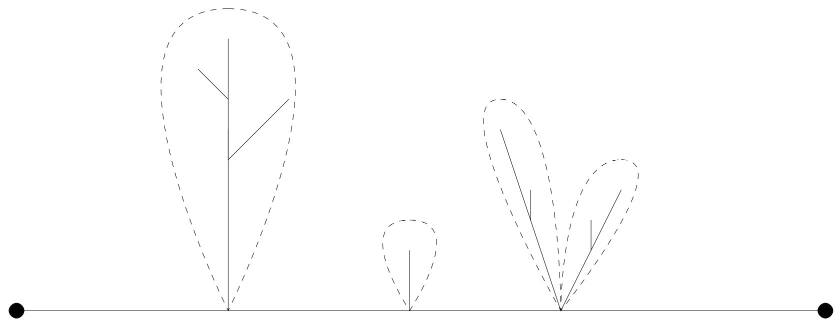
$$\frac{N_{(2)}(\lambda) - C\lambda^{2/3}}{\lambda^{1/3}} \to \mathcal{N}(0, y(\infty)).$$

Conjecture: $y(\infty) \neq 0$, which would imply second order fluctuations are of an order that is strictly bigger than the second order of $O(1)$ determined by the boundary (and which was seen in the mean behaviour).

α-STABLE TREES, $\alpha \in (1, 2)$

Spinal decomposition (Haas/Pitman/Winkel 2009):
- Choose two points at random.
- Split tree along arc between these.

Results in a countable number of copies of original tree with masses given by a Poisson-Dirichlet distribution.



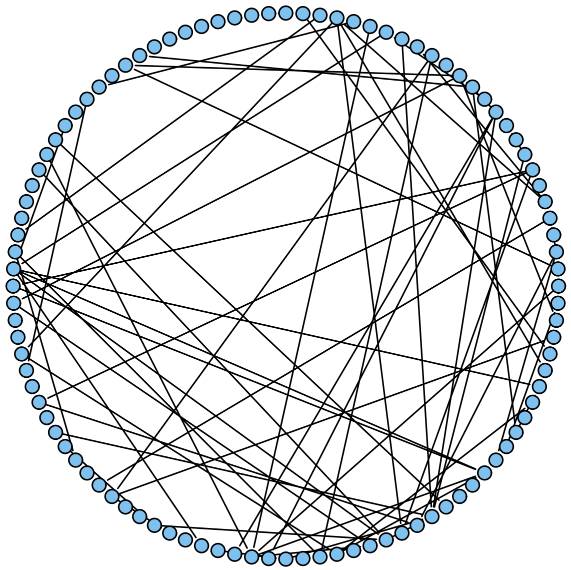
Can use this self-similarity property to deduce (C./Hambly 2010):

$$\mathbf{E}N(\lambda) = C\lambda^{\frac{\alpha}{2\alpha-1}} + O\left(\lambda^{\frac{1}{2\alpha-1}+\varepsilon}\right).$$

First term also seen $\mathbf{P}$-a.s. and second term in probability.
Imply short-time heat kernel asymptotics of order $t^{-\frac{\alpha}{2\alpha-1}}$.

CRITICAL ERDŐS-RÉNYI RANDOM GRAPH

$G(N, p)$ is obtained via bond percolation with parameter p on the complete graph with N vertices. We concentrate on critical window: $p = N^{-1} + \lambda N^{-4/3}$. e.g. $N = 100$, $p = 0.01$:



All components have:
- size $\Theta(N^{2/3})$ and surplus $\Theta(1)$ (Erdős/Rényi, Aldous),
- diameter $\Theta(N^{1/3})$ (Nachmias/Peres).

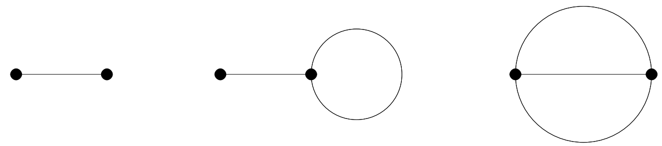
CONSTRUCTION OF SCALING LIMIT FROM BROWNIAN CRT

(Addario-Berry/Broutin/Goldschmidt) If $\mathcal{C}_1^N$ is the largest connected component of $G(N,p)$ (in the critical window), then

$$N^{-1/3}\mathcal{C}_1^N \to \mathcal{M},$$

for some random metric space $\mathcal{M}$. Moreover, associated random walks can be rescaled to a diffusion on $\mathcal{M}$ [C.].

Conditional on the number of loops, J say, and its mass, $\mathcal{M}$ can be constructed by:
- choosing 3-regular graph with $3(J-1)$ edges.



(Formula is slightly different if $J = 0, 1$.)
- placing independent Brownian CRTs along edges, with masses determined by a Dirichlet $(\frac{1}{2}, \dots, \frac{1}{2})$ distribution.

SPECTRAL ASYMPTOTICS FOR CRITICAL RANDOM GRAPH

Using Dirichlet-Neumann bracketing, can show

$$N_{\mathcal{M}}(\lambda) \sim \sum_{i=1}^{2(J-1)} N_{\mathcal{T}_i}(\lambda).$$

It follows that:

$$\mathbf{E}N_{\mathcal{M}}(\lambda) = C\mathbf{E}Z_1\lambda^{2/3} + O(1),$$

and also $\mathbf{P}$-a.s.,

$$\lambda^{-2/3}N_{\mathcal{M}}(\lambda) \to CZ_1.$$

Moreover, in distribution,

$$\frac{N_{\mathcal{M}}(\lambda) - CZ_1\lambda^{2/3}}{Z_1^{1/2}\lambda^{1/3}} \to \mathcal{N}(0, y(\infty)).$$