Warwick University, 4-8th September 2011

Finite time corrections in KPZ growth models

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arXiv:1104.2129

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Outline

- Two examples
- Some universality results in KPZ
- First order corrections

Nematic liquid crystals: stable (black) vs instable (gray) cluster

Takeuchi,Sano'10: PRL 104, 230601 (2010)

Videos: Takeuchi,Sano,Sasamoto,Spohn: Sci. Rep. 1, 34 (2011)

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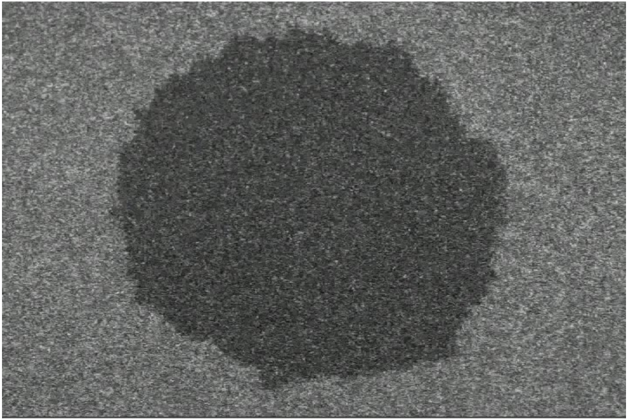

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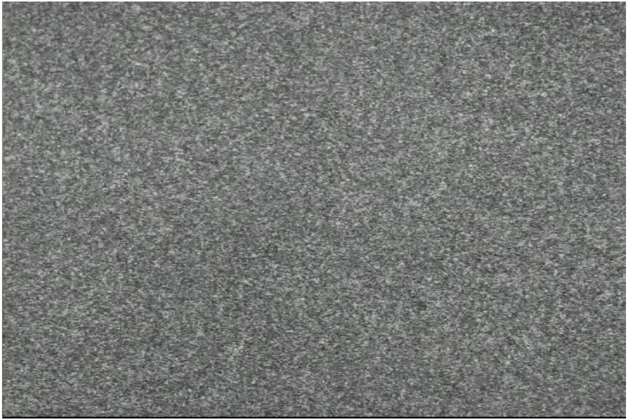

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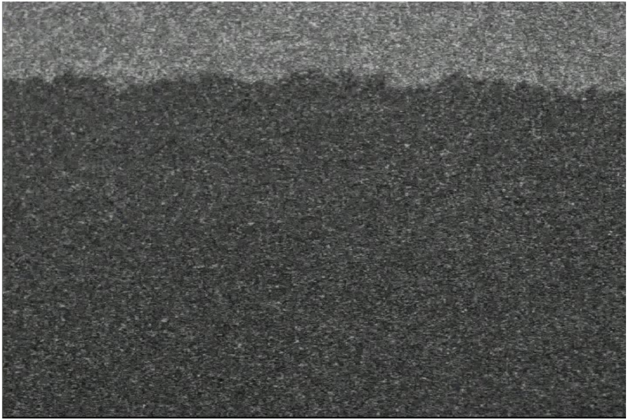

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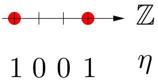
Q: Can we say something about the **interface roughness**?

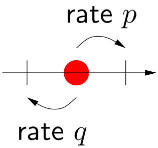
- PASEP: Partially Asymmetric Simple Exclusion Process

- **Configurations**

$$\eta = \{\eta_j\}_{j\in\mathbb{Z}},\ \eta_j = \begin{cases} 1, & \text{if } j \text{ is occupied,} \\ 0, & \text{if } j \text{ is empty.} \end{cases}$$



- **Dynamics**
  Independently, particles jump on the right site with rate p, to the left with rate $p = 1 - p$, provided the arrival site is empty.

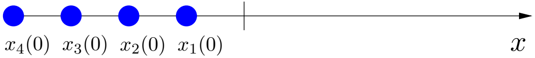


- $p = 1$: Totally asymmetric case (TASEP).

$\Rightarrow$ Particles are ordered: position of particle k is $x_k(\tau)$

- Initial condition: $x_k(0) = -k$, $k = 1, 2, \ldots$

$\Leftrightarrow$ for the height function: $h(x, t = 0) = |x|$, $x \in \mathbb{Z}$.



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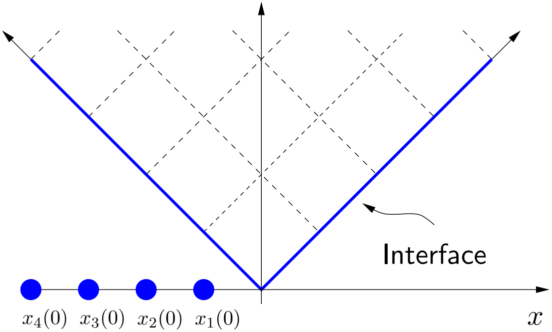

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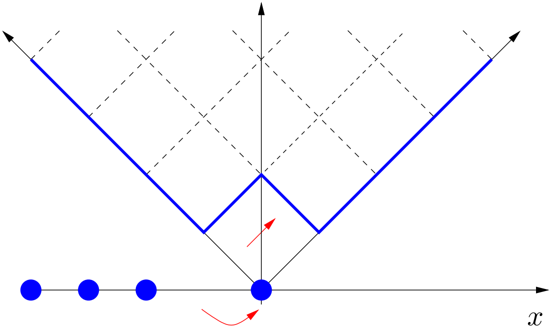

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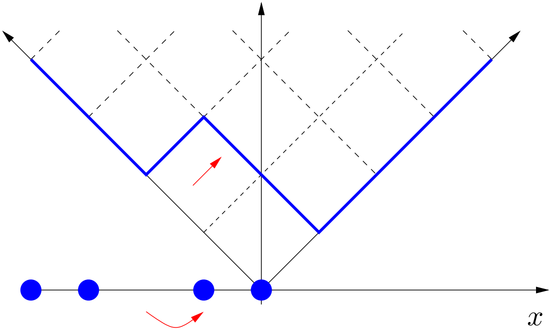

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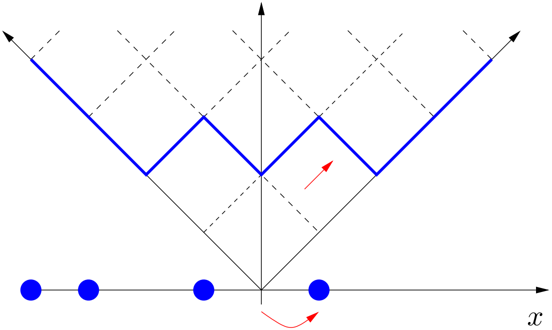
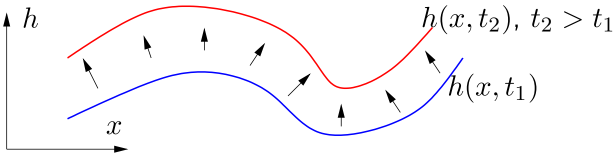


- Surface described by a height function

$$h(x,t)$$

$x \in \mathbb{R}^d$ the space, $t \in \mathbb{R}$ the time



- Models with local growth

$\Rightarrow$ macroscopic growth velocity v is a function of the slope only:

$$\frac{\partial h}{\partial t} = v(\nabla h)$$

- Random local growth + smoothening mechanics

$\Rightarrow$ deterministic macroscopic growth (for large growth time t)

- Simplest continuous equation for a irreversible, local, random and non-linear growth: the KPZ equation

Kardar,Parisi,Zhang'86

$$\frac{\partial h(x,t)}{\partial t} = \nu \frac{\partial^2 h(x,t)}{\partial x^2} + \tfrac{1}{2}\lambda \left(\frac{\partial h(x,t)}{\partial x} \right)^2 + \eta(x,t)$$

- η: space-time uncorrelated white noise
- $\nu > 0$: smoothing
- $\lambda > 0$: lateral growth, non-linear term

See review of Sasamoto, Spohn for recent developments on the equation: arXiv:1010.2691 and Amin,Corwin,Quastel'10 + ...

- Random local growth + smoothing mechanics

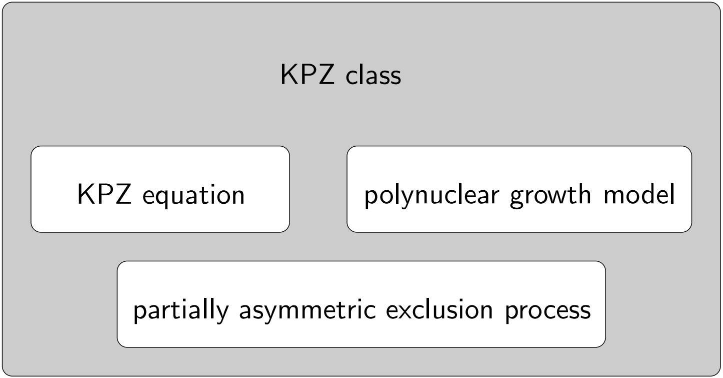
$\Rightarrow$ deterministic macroscopic growth (for large growth time t)

- If the growth velocity $v = v(u)$ has

$$v''(u) \neq 0,$$

then the model is believed to be in the KPZ universality class

Some models in the (1d) KPZ universality class



- Universality expected for large time t

- Limit shape:

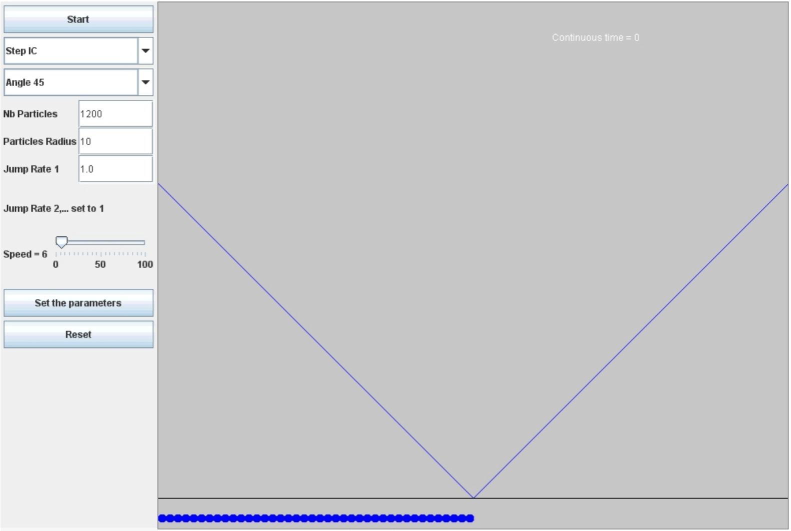
$$h_{\text{macro}}(\xi) := \lim_{t\to\infty} \frac{h(\xi t, t)}{t}$$

- Fluctuation exponent: $1/3$
- Spatial correlation exponent: $2/3$

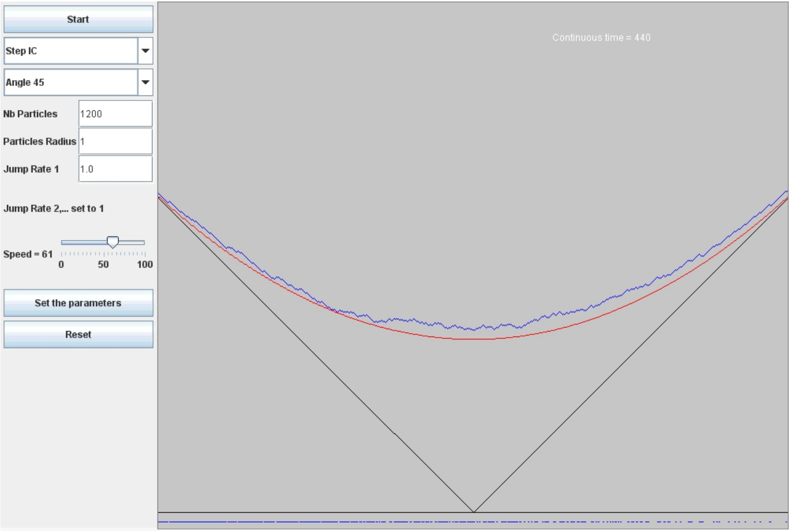
$\Rightarrow$ Rescaled height function around $x = \xi t$,

$$h_t^{\text{resc}}(u) = \frac{h(\xi t + u t^{2/3}, t) - t h_{\text{macro}}((\xi t + u t^{2/3})/t)}{t^{1/3}}$$

Limit shape = red line in the next animation.



Simulation with **step initial condition:** $x_k(0) = -k, k \in \mathbb{N}$.



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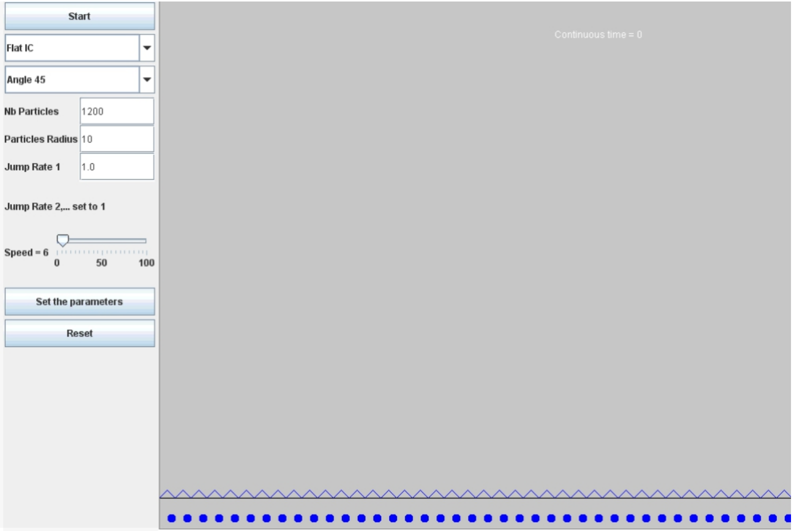
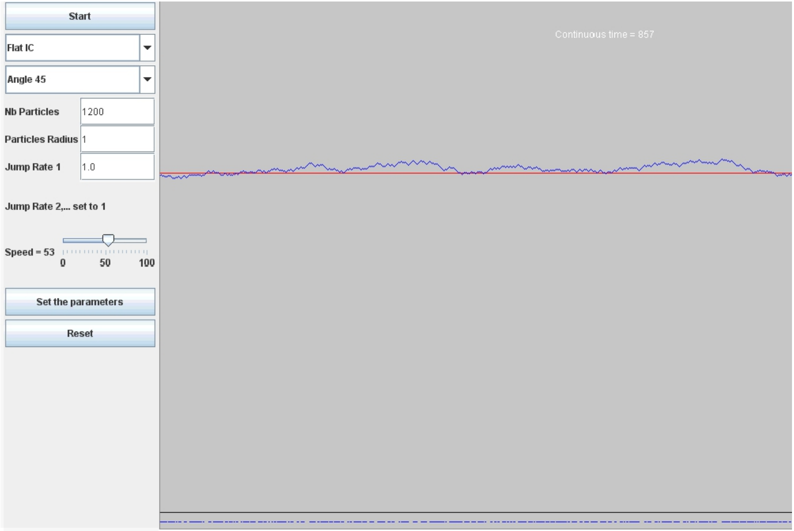


Simulation with **flat initial condition**: $x_k(0) = -2k, k \in \mathbb{Z}$.

TASEP: flat initial condition



Simulation with **flat initial condition**: $x_k(0) = -2k, k \in \mathbb{Z}$.

Expected universal results (proven for specific models)

- Curved limit shape: Airy$_2$ process $\mathcal{A}_2$,

$$\lim_{t\to\infty} h_t^{\rm resc}(u) = \lim_{t\to\infty} \frac{h(\xi t + u t^{2/3}, t) - t h_{\rm macro}((\xi t + u t^{2/3})/t)}{t^{1/3}}$$
$$= \kappa_v \mathcal{A}_2(u/\kappa_h)$$

  with κ_v, κ_h numerical coefficients, depending on ξ only.

- $\mathcal{A}_2$ is stationary, $\mathrm{Cov}(\mathcal{A}_2(0), \mathcal{A}_2(u)) \sim c/u^2$ for $u \gg 1$, $\mathbb{P}(\mathcal{A}_2(0) \leq s) = F_{\rm GUE}(s)$

1Point: Baik,Rains'00+Prähofer,Spohn'01

Multipoints: Prähofer,Spohn'02;Johansson'03;...

Expected universal results (proven for specific models)

- Flat limit shape, non-random IC: Airy$_1$ process $\mathcal{A}_1$,

$$\lim_{t\to\infty} h_t^{\rm resc}(u) = \lim_{t\to\infty} \frac{h(\xi t + ut^{2/3}, t) - t h_{\rm macro}((\xi t + ut^{2/3})/t)}{t^{1/3}}$$
$$= \kappa_v \mathcal{A}_1(u/\kappa_h)$$

  with κ_v, κ_h numerical coefficients.

- $\mathcal{A}_1$ is stationary, $\mathrm{Cov}(\mathcal{A}_1(0), \mathcal{A}_1(u)) \sim \exp(-cu^3)$ for $u \gg 1$ (numerical), $\mathbb{P}(\mathcal{A}_1(0) \leq s) = F_{\rm GOE}(2s)$

1Pt: Baik,Rains'00+Prähofer,Spohn'01;Ferrari,Spohn'05

Multi: Sasamoto'05;Borodin,Ferrari,Prähofer,Sasamoto'07;...

Expected universal results (proven for specific models)

- Flat limit shape, stationary IC; with ξ on the characteristics (otherwise trivially Brownian motion): Process $\mathcal{A}_0$,

$$\lim_{t\to\infty} h_t^{\rm resc}(u) = \lim_{t\to\infty} \frac{h(\xi t + u t^{2/3}, t) - t h_{\rm macro}((\xi t + u t^{2/3})/t)}{t^{1/3}}$$
$$= \kappa_v \mathcal{A}_0(u/\kappa_h)$$

  with κ_v, κ_h numerical coefficients.

- $\mathcal{A}_0$ is not stationary, $\mathbb{P}(\mathcal{A}_0(0) \leq s) = F_0(s)$

1Pt: Baik,Rains'00+Prähofer,Spohn'02;Imamura,Sasamoto'04

Multipt: Baik,Ferrari,Péché'10

- How fast does h_t^{resc} converge to the limit processes?
- What is the nature of the first order correction, i.e., of the microscopic ("atomic scale") correction

$$t^{1/3}(h_t^{\mathrm{resc}} - \mathcal{A}.)?$$

- For TASEP with step IC and KPZ solution (coming from WASEP) is the sign of

$$a(p) = \lim_{t\to\infty} \mathbb{E}[t^{1/3}(h_t^{\mathrm{resc}}(0) - \mathcal{A}_2(0))]$$

different. What is the value of $p_c \in (1/2, 1)$ such that $a(p_c) = 0$? (A simulation of Sasamoto indicate $p_c \simeq 0.78$)

- Setting: PASEP, step IC, parameter p.
- We focus now only at one point distribution, so consider the random variable

$$\zeta_t := \frac{h(\xi t) - t h_{\mathrm{ma}}(\xi)}{\kappa_v t^{1/3}} = [h(\xi t) - c_1 t]\,\delta_t$$

with $c_1 := h_{\mathrm{ma}}(\xi)$ and

$$\delta_t := \kappa_v^{-1} t^{-1/3}.$$

Denote: $\lim_{t\to\infty} \zeta_t = \zeta_{\mathrm{GUE}}$

- We obtain

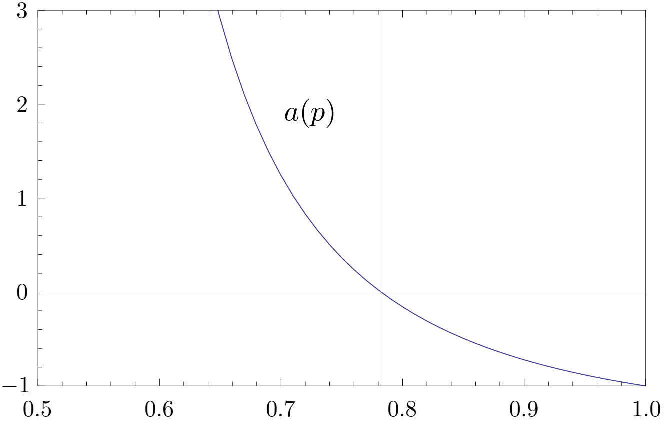
$$\lim_{t\to\infty} \mathbb{E}[t^{1/3}(\zeta_t - \zeta_{\mathrm{GUE}})] = a(p) = 2\sum_{\ell=1}^{\infty} \frac{(1-p)^\ell}{p^\ell - (1-p)^\ell} - 1$$

i.e.,

$$a(p) = 0 \quad \leftrightarrow \quad p = p_c = 0.78227\,87862\ldots$$

Ferrari,Frings'11

Zero shift - PASEP



$$a(p) = 0 \leftrightarrow p = p_c = 0.78227\,87862\ldots$$

As before, consider the random variable

$$\zeta_t := \frac{h(\xi t) - t h_{\mathrm{ma}}(\xi)}{\kappa_v t^{1/3}} = [h(\xi t) - c_1 t]\, \delta_t$$

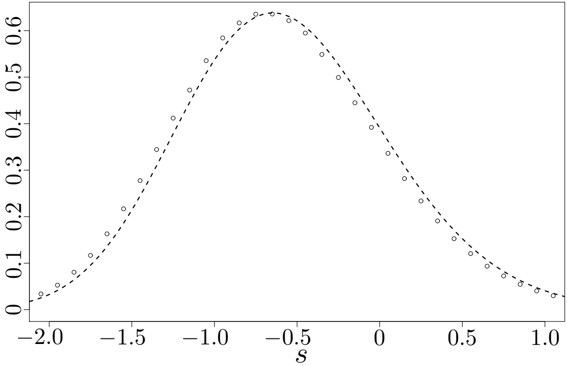
with $c_1 := h_{\mathrm{ma}}(\xi)$ and

$$\delta_t := \kappa_v^{-1} t^{-1/3}.$$

- Generically one has

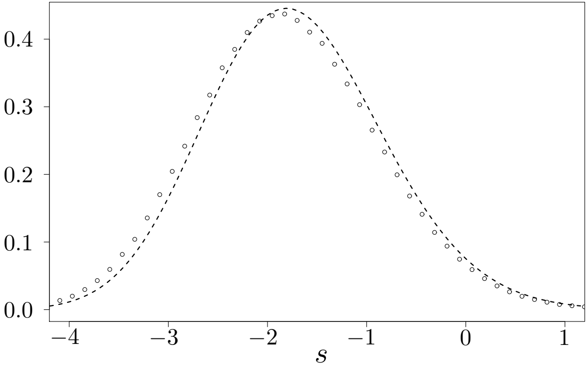
$$\zeta_t = \zeta_{\mathrm{GUE}} + a\, \delta_t + \mathcal{O}(\delta_t^2).$$

  where ζ_{GUE} and a are a priori non-independent random variables.
- $\mathbb{E}(\zeta_t) = \mathbb{E}(\zeta_{\mathrm{GUE}}) + \mathbb{E}(a)\, \delta_t + \mathcal{O}(\delta_t^2)$
- $\mathrm{Var}(\zeta_t) = \mathrm{Var}(\zeta_{\mathrm{GUE}}) + 2\, \mathrm{Cov}(a, \zeta_{\mathrm{GUE}})\, \delta_t + \mathcal{O}(\delta_t^2)$



(Totally) asymmetric exclusion process TASEP with flat IC

Dots: data; Dashed line: asymptotic density; $t = 1000$



(Totally) asymmetric exclusion process TASEP with step IC
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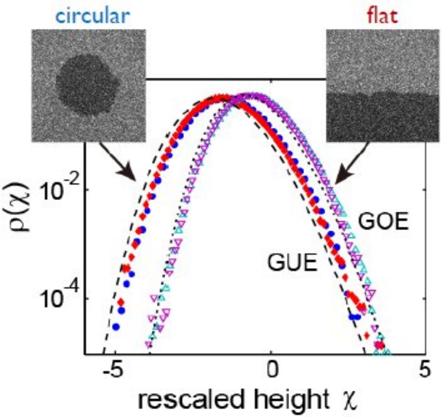
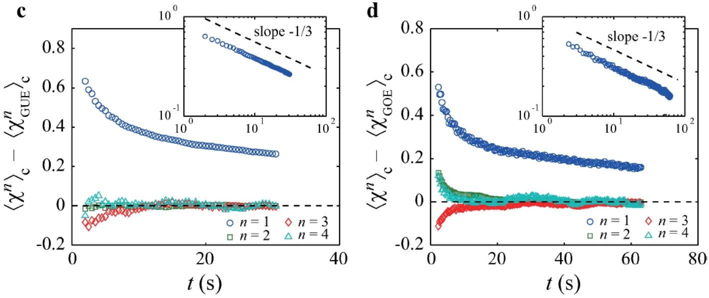


From the experiment on nematic liquid crystals. Taken from:

Takeuchi,Sano,Sasamoto,Spohn: Sci. Rep. 1, 34 (2011)

- A-priori: $\mathrm{Var}(\zeta_t) = \mathrm{Var}(\zeta_\infty) + 2\,\mathrm{Cov}(a, \zeta_\infty)\,\delta_t + \mathcal{O}(\delta_t^2)$.



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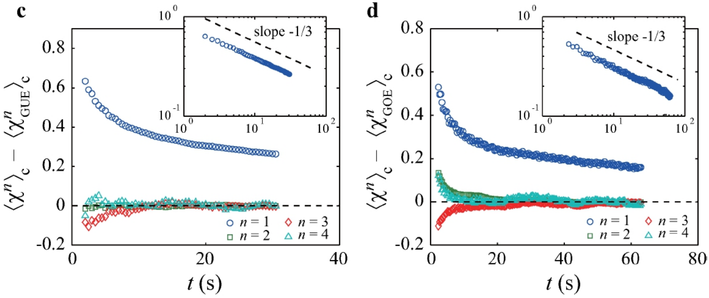
- Experiment, simulations: $\mathrm{Var}(\zeta_t) = \mathrm{Var}(\zeta_\infty) + \mathcal{O}(\delta_t^2)$ only.

$\Rightarrow$ Indicates that $\mathrm{Cov}(a, \zeta_\infty) = 0$



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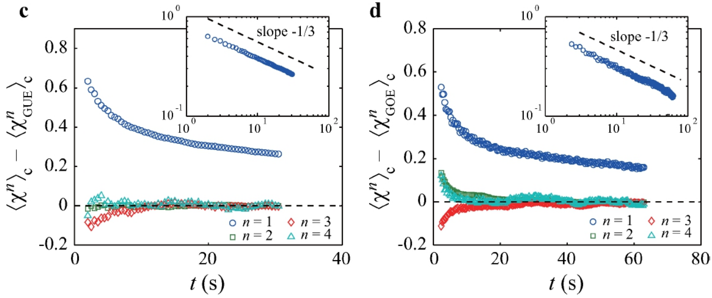
- which holds in particular if a is a constant.



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Density - PASEP step IC (and PNG ...)

- Consider PASEP with step-initial condition (similar results holds for TASEP, PNG with flat / step IC).

- Let $I_t := (\mathbb{Z} - c_1 t)\delta_t$: the lattice where ζ_t takes values.
- Then, with the constant $a = a(p) \in \mathbb{R}$, it holds

$$\mathbb{P}(\zeta_t = s)/\delta_t = F'_{\mathrm{GUE}}(s - a\delta_t) + \mathcal{O}(\delta_t^2), \quad s \in I_t.$$

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$\Rightarrow$ By a (non-universal) constant shift, the convergence of the density is $\mathcal{O}(\delta_t^2) = \mathcal{O}(t^{-2/3})$ instead of $\mathcal{O}(\delta_t) = \mathcal{O}(t^{-1/3})$.

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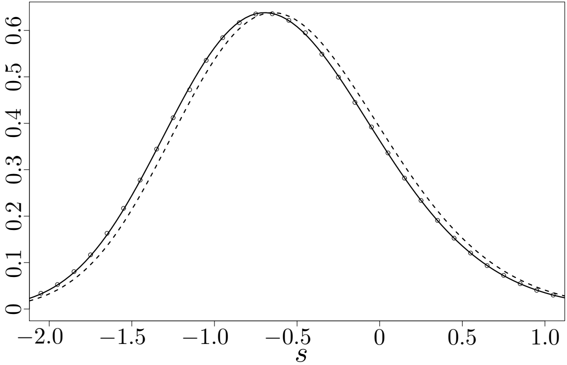
- New scaling: instead of

$$\zeta_t = (h_t(0) - c_1 t)\delta_t \quad \Rightarrow \quad \widetilde{\zeta}_t = (h_t(0) - c_1 t - a)\delta_t$$

so that

$$\mathbb{P}(\widetilde{\zeta}_t = s)/\delta_t = F'_{\rm GUE}(s) + \mathcal{O}(\delta_t^2), \quad s \in \widetilde{I}_t = (\mathbb{Z} - c_1 t - a)\delta_t.$$

Ferrari,Frings'11

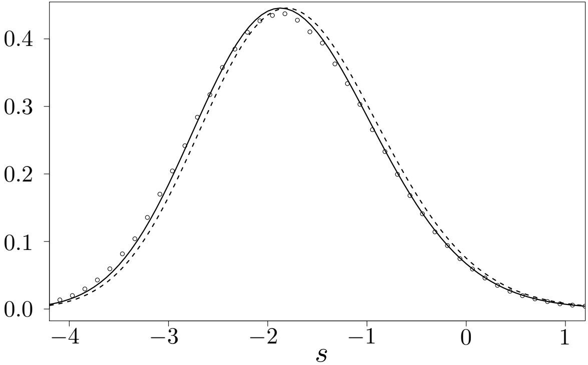


(Totally) asymmetric exclusion process TASEP with flat IC

Dots: $\widetilde{\zeta}_t = (h_t(0) - c_1 t - a)\delta_t$, $t = 1000$.

Solid line: $2F'_{\rm GOE}(2s)$; Dashed line: $2F'_{\rm GOE}(2(s - a\delta_t))$.

For curved limit shape the convergence, still $\mathcal{O}(t^{-2/3})$, is slower.



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Dots: $\widetilde{\zeta}_t = (h_t(0) - c_1 t - a)\delta_t$, $t = 1000$.

Solid line: $F'_{\rm GUE}(s)$; Dashed line: $F'_{\rm GUE}(s - a\delta_t)$.

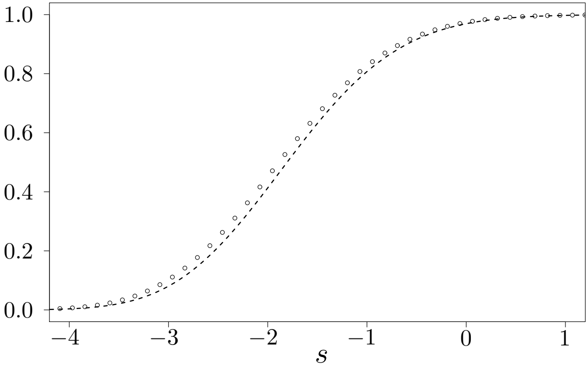
For curved limit shape the convergence, still $\mathcal{O}(t^{-2/3})$, is slower.

- With $\widetilde{\zeta}_t = (h_t(0) - c_1 t - a)\delta_t$. For any fixed $m = 1, 2, \ldots$, it holds

$$\mathbb{E}(\widetilde{\zeta}_t^m) = \int_{\mathbb{R}} s^m dF_{\mathrm{GUE}}(s) + \mathcal{O}(\delta_t^2).$$

$\Rightarrow$ By an appropriate (non-universal) constant shift, the convergence of the density and moments is $\mathcal{O}(\delta_t^2) = \mathcal{O}(t^{-2/3})$.

Ferrari,Frings'11



(Totally) asymmetric exclusion process TASEP with step IC.
Plot of $\mathbb{P}(\widetilde{\zeta}_t \leq s)$ vs $F_{\mathrm{GUE}}(s)$; $t = 1000$.

- Q: Why is the fit not good?

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- A: Because of the discreteness of the model / measurement ($\widetilde{I}_t$ has a lattice width δ_t). Ferrari,Frings'11

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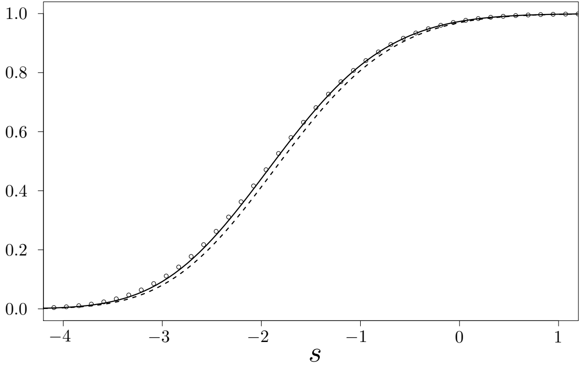
- A: Because of the discreteness of the model / measurement ($\widetilde{I}_t$ has a lattice width δ_t). Ferrari,Frings'11
- To have a good fit with the distribution function, one has to fit with the solid line, i.e.,

$$\mathbb{P}(\widetilde{\zeta}_t \leq s) \quad \text{with} \quad F_{\rm GUE}(s + \tfrac{1}{2}\delta_t).$$



(1) Show that for an appropriate constant a,

$$\mathbb{P}((h_t(0) - c_1 t - a)\delta_t \leq s) = F_{\mathrm{GUE}}(s + \tfrac{1}{2}\delta_t)(1 + \mathcal{O}(\delta_t^2))$$

for $s \in \widetilde{I}_t$.

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for $s \in \widetilde{I}_t$.

(2) Point (1) follows as a corollary of a similar statement for the distribution of tagged particles:

$$\mathbb{P}(x_{[\sigma t]}(t/(p-q)) \geq c_1(\sigma)t - s\kappa_v(\sigma)t^{1/3} - \widetilde{a}) = F_{\mathrm{GUE}}(s + \tfrac{1}{2}\delta_t)(1 + \mathcal{O}(\delta_t^2)).$$

(1) Show that for an appropriate constant a,

$$\mathbb{P}((h_t(0) - c_1 t - a)\delta_t \leq s) = F_{\mathrm{GUE}}(s + \tfrac{1}{2}\delta_t)(1 + \mathcal{O}(\delta_t^2))$$

for $s \in \widetilde{I}_t$.

(2) Point (1) follows as a corollary of a similar statement for the distribution of tagged particles:

$$\mathbb{P}(x_{[\sigma t]}(t/(p-q)) \geq c_1(\sigma)t - s\kappa_v(\sigma)t^{1/3} - \widetilde{a}) = F_{\mathrm{GUE}}(s + \tfrac{1}{2}\delta_t)(1 + \mathcal{O}(\delta_t^2)).$$

(3) To show (2) one starts with the formula of Tracy and Widom (2009).

(3) To show (2) one starts with the formula of Tracy and Widom (2009): $u = c_1(\sigma)t - s\kappa_v(\sigma)t^{1/3} - \widetilde{a}$, $n = \sigma t$, $\tau = q/p < 1$,

$$\mathbb{P}\big(x_n(t/(p-q)) \geq u\big) = \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}\mu}{\mu} (\mu;\tau)_\infty \det\big(\mathbb{1} + \mu J_\mu\big)$$

where $(\mu;\tau)_\infty = \prod_{k=0}^{\infty}(1-\mu\tau^k)$ is the q-Pochhammer symbol and the integral is taken over a circle around the origin with radius in the interval $(0,\tau)$. The operator J_μ has kernel

$$J_\mu(\eta,\eta') = \frac{1}{2\pi \mathrm{i}} \oint \mathrm{d}\zeta \, \frac{\exp\big(\frac{t\zeta}{1-\zeta}\big)(1-\zeta)^u \zeta^n}{\exp\big(\frac{t\eta'}{1-\eta'}\big)(1-\eta')^u (\eta')^{n+1}} \, \frac{f\big(\mu, \frac{\zeta}{\eta'}\big)}{\zeta - \eta},$$

where η, η' are on a circle around 0 with radius $r \in (\tau, 1)$ and ζ runs on a circle around 0 with radius in $(1, r/\tau)$. For $1 < |z| < \tau^{-1}$, the function f is given by

$$f(\mu, z) = \sum_{k=-\infty}^{\infty} \frac{\tau^k}{1-\tau^k \mu} \, z^k,$$

(4) Finally one has to do the asymptotic analysis for $t \to \infty$. At the end of the day one gets

$$\mathbb{P}\big(x_n(t/(p-q)) \geq u\big) = F_{\mathrm{GUE}}(s+\delta_t/2) \times \Big[1 - \big(\tfrac{G}{\sqrt{\sigma}} - \tfrac{1}{2} + \widetilde{a}\big)\,\mathrm{Tr}\big((\mathbb{1}-\chi_s K_{\mathcal{A}_2}\chi_s)^{-1}\chi_s(\mathrm{Ai}\otimes\mathrm{Ai})\chi_s\big)t^{-1/3} + \mathcal{O}(t^{-2/3})\Big]$$

with

$$G = \frac{1}{2\pi\mathrm{i}}\oint_{\tau<|\mu|<1}\frac{\mathrm{d}\mu}{\mu}(\mu;\tau)_\infty\left(\sum_{k=0}^{\infty}\frac{\mu\tau^k}{1-\mu\tau^k} + \sum_{k=1}^{\infty}\frac{\tau^k}{\tau^k-\mu}\right).$$

So, one sets $\widetilde{a} = \frac{1}{2} - \frac{G}{\sqrt{\sigma}}$ and finds out that

$$G = \sum_{\ell=1}^{\infty}\frac{\tau^\ell}{1-\tau^\ell}.$$