Rate of convergence estimates for the zero dissipation limit in Abelian sandpiles

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Critical sandpile model

[Bak, Tang, Wiesenfeld; 1987], [Dhar; 1990]

Configurations: $\Lambda \subset \mathbb{Z}^d$ finite, $(\eta_x)_{x\in\Lambda} \in \{0,1,\dots\}^\Lambda$

Stabilization: $\mathcal{S}_\Lambda : \{0,1,\dots\}^\Lambda \to \{0,1,\dots,2d-1\}^\Lambda$

Toppling: if $\eta_x \geq 2d$, x can topple: send one particle to each neighbour $\eta_y \to \eta_y - \Delta_{xy}$, $y \in \Lambda$ where Δ is the graph Laplacian.

Repeat as long as there is x with $\eta_x \geq 2d$.

Open boundary condition: when toppling on the boundary, some particles leave the system.

Lemma. [Dhar; 1990] $\mathcal{S}_\Lambda$ *is well-defined (Abelian property).*

Critical sandpile model

Addition operators: Let $\Omega_\Lambda = \{0, 1, \dots, 2d-1\}^\Lambda$.
We define $a_x : \Omega_\Lambda \to \Omega_\Lambda$ by $a_x\eta = \mathcal{S}_\Lambda(\eta + e_x)$, where e_x has a single particle at x, and no particles elsewhere.

Abelian property: $a_x a_y = a_y a_x$ for $x, y \in \Lambda$.

Avalanche: the sequence of topplings occurring in stabilizing $\eta + e_x$.

Markov chain: State space Ω_Λ. Pick $x \in \Lambda$ uniformly at random.
Then jump: $\eta \to a_x\eta = \mathcal{S}_\Lambda(\eta + e_x)$.

Evolution: $\eta(n) = \mathcal{S}_\Lambda(\eta(0) + \sum_{i=1}^n e_{X_i})$, where $X_1, X_2, \dots$ are i.i.d. with $\mathbf{P}(X_i = x) = |\Lambda|^{-1}$, $x \in \Lambda$.

Example: addition resulting in no toppling

$d = 2$. Possible number of particles: $0, 1, 2, 3$.

2	2	1	0
2	3	3	2
3	3	1	1
2	2	2	1

$\xrightarrow{\text{addition}}$

2	2	1	0
2	3	3	2
3	3	2	1
2	2	2	1

Example: addition resulting in a sequence of topplings

2	2	1	0
2	3	3	2
3	3	2	1
2	2	2	1

addition $\longrightarrow$

2	2	1	0
2	4	3	2
3	3	2	1
2	2	2	1

toppling $\rightsquigarrow$

2	3	1	0
3	0	4	2
3	4	2	1
2	2	2	1

two topplings $\rightsquigarrow$

2	3	2	0
3	2	0	3
4	0	4	1
2	3	2	1

two topplings $\rightsquigarrow$

2	3	2	0
4	2	1	3
0	2	0	2
3	3	3	1

toppling $\rightsquigarrow$

3	3	2	0
0	3	1	3
1	2	0	2
3	3	3	1

Critical sandpile model

Recurrent states: only one recurrent class: $\mathcal{R}_\Lambda \subset \Omega_\Lambda$
Stationary distribution ν_Λ is uniform on $\mathcal{R}_\Lambda$.

Sandpile group: $\mathcal{R}_\Lambda$ forms an Abelian group under $\eta \oplus \zeta = \mathcal{S}_\Lambda(\eta + \zeta)$, isomorphic to $\mathbb{Z}^\Lambda / \mathbb{Z}^\Lambda \Delta$. The Markov chain is a random walk on $\mathcal{R}_\Lambda$ with generators $\{a_x : x \in \Lambda\}$.

Self-organized criticality: ν_Λ has power law correlations: if $d \geq 2$ then

$$\lim_{\Lambda \to \mathbb{Z}^d} \mathrm{Cov}_{\nu_\Lambda}(I[\eta_0 = 0], I[\eta_x = 0]) \sim \mathrm{const} \cdot |x|^{-2d} \quad \text{as } |x| \to \infty.$$

The number of topplings in an avalanche, as well as other characteristics, are conjectured to follow a power law.

Combinatorial characterization of recurrent states

A recurrent configuration cannot contain

0	0

Definition. *A finite set F is called ample for a configuration ξ, if*

$$\xi_i \geq \#\{j \in F : j \sim i\} \quad \textit{for at least one } i \in F,$$

where $j \sim i$ denotes that j and i are neighbours.

For example,

		1	0
1	1	3	2
0		1	1

is not ample.

Combinatorial characterization of recurrent states

Theorem. [Majumdar, Dhar; 1992] *For $\eta \in \Omega_\Lambda$ we have:*

$$\eta \in \mathcal{R}_\Lambda \quad \Longleftrightarrow \quad \text{every } \emptyset \neq F \subset \Lambda \text{ is ample for } \eta.$$

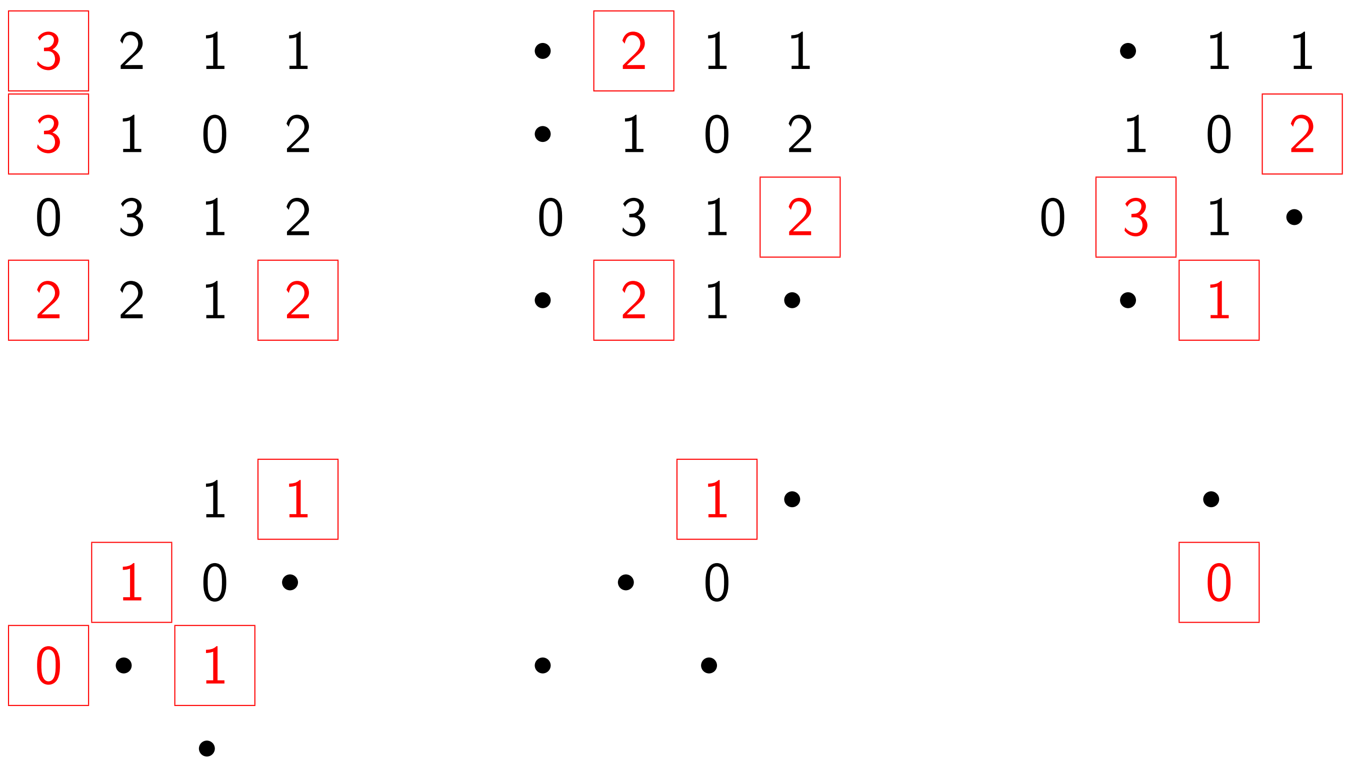
Definition. *Recurrent configurations on $\mathbb{Z}^d$:*

$$\Omega = \{0, \ldots, 2d-1\}^{\mathbb{Z}^d} = \text{set of stable configurations in } \mathbb{Z}^d.$$

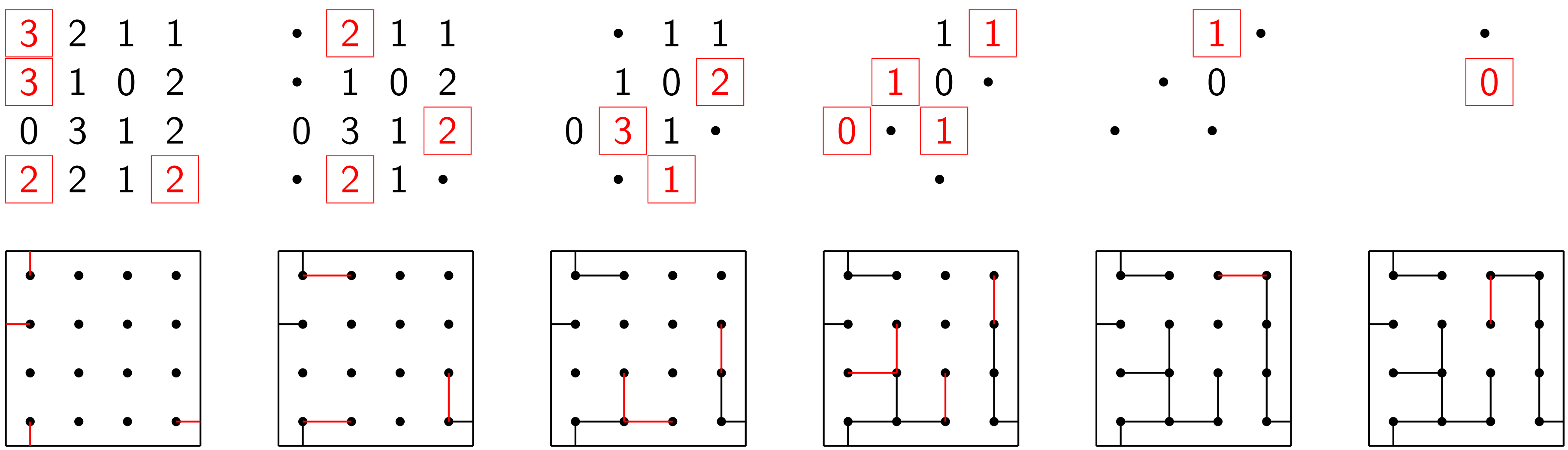
$$\mathcal{R} := \{\eta \in \Omega : \eta_\Lambda \in \mathcal{R}_\Lambda \text{ for all finite } \Lambda \subset \mathbb{Z}^d\}$$
$$= \{\eta \in \Omega : \text{every finite } \emptyset \neq F \subset \mathbb{Z}^d \text{ is ample for } \eta\}.$$

The burning test

An algorithm that checks if $\eta \in \Omega_\Lambda$ is recurrent [Dhar; 1990].



Majumdar-Dhar bijection



One-to-one correspondence:

$\mathcal{R}_\Lambda \quad \Longleftrightarrow \quad$ spanning trees on Λ with wired boundary conditions.

Sandpile with bulk dissipation

Exact calculations: [Mahieu, Ruelle; 2001] probabilities of some local configurations, and correlation functions in $d = 2$, in connection with CFT. For integer $\gamma \geq 1$, allow $0, 1, \dots, 2d + \gamma - 1$ particles on each vertex. On each toppling, γ particles are dissipated:

$$\Delta_{xy}^{(\gamma)} = \begin{cases} 2d + \gamma & \text{if } x = y; \\ -1 & \text{if } x \sim y; \\ 0 & \text{otherwise.} \end{cases}$$

They then let $\gamma \downarrow 0$ in formulas obtained.

Continuous model: [Gabrielov; 1993]
$\Omega_\Lambda^{(\gamma)} = [0, 2d + \gamma)^\Lambda$, $\gamma \geq 0$ real. Add unit height and stabilize.
$\mathcal{R}_\Lambda^{(\gamma)} \subset \Omega_\Lambda^{(\gamma)}$ — Stationary distribution $m_\Lambda^{(\gamma)}$: Lebesgue measure on $\mathcal{R}_\Lambda^{(\gamma)}$.

Sandpile with bulk dissipation

Discrete description: $m_\Lambda^{(\gamma)}$ easily understood in terms of a discrete measure.

$$\xi_x := \begin{cases} [\eta_x] & \text{if } 0 \le \eta_x < 2d; \\ 2d & \text{if } 2d \le \eta_x < 2d + \gamma. \end{cases}$$

$m_\Lambda^{(\gamma)} \longrightarrow \nu_\Lambda^{(\gamma)}$ on $\Omega_\Lambda^{\text{discr}} = \{0, 1, \ldots, 2d\}^\Lambda$.

Weights: The discrete measure $\nu_\Lambda^{(\gamma)}$ obeys the following weighting:

$$\nu_\Lambda^{(\gamma)}(\xi) = \frac{1}{Z} \gamma^{N(\xi)},$$

where $N(\xi) = \#\{x \in \Lambda : \xi_x = 2d\}$.

Extension of the Majumdar-Dhar bijection

Adding dissipative edges: Define the graph $G_\Lambda = (V_\Lambda, E_\Lambda)$, where $V_\Lambda = \Lambda \cup \{s\}$, E_Λ contains the usual edges on $\Lambda \cup \{s\}$ and one "dissipative" edge between each $x \in \Lambda$ and s.

Bijection with spanning trees: The Majumdar-Dhar bijection extends to a one-to-one correspondence between $\mathcal{R}_\Lambda^{(\gamma)}$ and spanning trees of G_Λ.

Weighted spanning trees: Give dissipative edges weight γ and other edges weight 1. The weight of a spanning tree t of G_Λ is $\prod_{e \in t} w(e)$.

Proposition. [J., Redig, Saada; 2010] *The Majumdar-Dhar bijection gives a coding of $\nu_\Lambda^{(\gamma)}$ by weighted spanning trees. That is:*

$$\nu_\Lambda^{(\gamma)}(\xi) = \frac{1}{Z} w(t(\xi)).$$

Infinite volume limit

Theorem. [J., Redig, Saada; 2010]
(i) (Stationary measure) For any $\gamma \geq 0$ *we have* $m_\Lambda^{(\gamma)} \Rightarrow m^{(\gamma)}$ *as* $\Lambda \uparrow \mathbb{Z}^d$*.*
(ii) (Dynamics) For any $\gamma > 0$ *the process* $\eta(t) = \mathcal{S}^{(\gamma)}(\eta(0) + \sum_{x \in \mathbb{Z}^d} N_x(t))$ *is well-defined a.s. (rate* 1 *Poisson additions).*
(iii) (Invariance) $m^{(\gamma)}$ *is invariant for the dynamics.*
(iv) (Zero dissipation limit) As $\gamma \downarrow 0$*,* $m^{(\gamma)} \Rightarrow m^{(0)}$*.*

Remark. The Transfer-Current Theorem **[Burton, Pemantle; 1993]** applied to the collection of dissipative edges gives:
Under $m^{(\gamma)}$, $h_x = I[\eta_x \in [2d, 2d + \gamma)]$ is a determinantal process.

Rate of convergence as $\gamma \downarrow 0$

The question: How fast does $m^{(\gamma)} \Rightarrow m^{(0)}$, as $\gamma \downarrow 0$?

Minimal configurations: a finite configuration $\xi \in \Omega_F^{\text{discr}} = \{0, 1, \dots, 2d\}^F$ is called minimal, if decreasing any of the values ξ_x makes it not ample.

Exact computations: For a minimal configuration ξ, we can express $\nu^{(\gamma)}(\xi)$ as a determinant involving the Green function of random walk in $\mathbb{Z}^d$ killed at geometric rate $\gamma/(2d+\gamma)$ **[Majumdar, Dhar; 1991]**.

Rate of convergence: **[J., Redig, Saada; 2010]** We have

$$\left| \nu^{(\gamma)}[\xi_0 = 0] - \nu^{(0)}[\xi_0 = 0] \right| \leq \begin{cases} C\gamma \log(1/\gamma) & \text{if } d = 2; \\ C\gamma & \text{if } d \geq 3. \end{cases}$$

Open question: Are these the precise rates for all finite configurations?

Power law upper bounds

Theorem. [J.; 2010] *Suppose E depends on the heights in $[-k,k]^d$.*
(1) If $d=3$, there exist positive constants C,η such that for all $0 \le \gamma < 1$

$$\left| m^{(\gamma)}(E) - m^{(0)}(E) \right| \le Ck^2\gamma^\eta + Ck^5(\log k)\gamma.$$

(2) If $d=2$, there exist positive constants c_0, C, C_0 such that for all $0 \le \gamma < c_0 k^{-C_0}$ we have

$$\left| m^{(\gamma)}(E) - m^{(0)}(E) \right| \le Ck^{21/23}\gamma^{1/46-o(1)}.$$

Ingredients of the proof: Majumdar-Dhar bijection + Wilson's algorithm. We construct a coupling that is successful with high probability when γ is small.

Wilson's algorithm

Weighted spanning forest: the weak limit of the weighted spanning tree measures as $\Lambda \uparrow \mathbb{Z}^d$. We want to sample from this measure.

Geometrically killed LERW: Consider the random walk on $\mathbb{Z}^d \cup \{s\}$ that on each step jumps to a neighbour with probability $1/(2d+\gamma)$, and jumps to s with probability $\gamma/(2d+\gamma)$. The Loop-Erased Random Walk (LERW) is obtained by chronologically removing all loops from the path.

Wilson's method. Enumerate $\mathbb{Z}^d$ as $x_1, x_2, \dots$. Put $\mathcal{F}_0 = \{s\}$. Assume $\mathcal{F}_0, \dots, \mathcal{F}_{i-1}$ have been defined ($i \geq 1$). Run a random walk $S^{(i)}$ starting at x_i until the time $T^{(i)}$ when it hits $\mathcal{F}_{i-1}$. Put $\mathcal{F}_i = \mathcal{F}_{i-1} \cup \mathrm{LE}(S^{(i)}[0, T^{(i)}])$. Finally, put $\mathcal{F} = \cup_{i \geq 0} \mathcal{F}_i$.

Theorem. [Wilson; 1996] *Regardless of the enumeration chosen, $\mathcal{F}$ has the distribution of the weighted spanning forest.*

Setup for the event E

Under the Majumdar-Dhar bijection, the occurrence or not of the event E can be determined from the spanning forest paths starting in $[-k-1, k+1]^d$. This can be generated by Wilson's method, if we start the enumeration with the vertices $\{x_1, \ldots, x_N\} = [-k-1, k+1]^d \cap \mathbb{Z}^d$.

Coupling for LERW. The coupling required for the Theorem can be constructed, if we can couple a sufficiently long initial segment of a geometrically killed LERW to the corresponding initial segment of the unkilled LERW, and give estimates on the coupling.

An easy argument for $d = 3$

Let S be simple random walk on $\mathbb{Z}^d$ starting at 0. Let $\mathbf{B}(r) := \{x \in \mathbb{Z}^d : |x| < r\}$. Let τ_r denote the first exit time from $\mathbf{B}(r)$.

Stabilization of LERW. Consider the last time σ when $\mathbf{B}(m)$ is visited. Then $\mathrm{LE}(S) \cap \mathbf{B}(m)$ can still change after time σ, due to closing of loops that started before time σ. However, after the last visit to the set $S[0, \sigma]$, $\mathrm{LE}(S) \cap \mathbf{B}(m)$ cannot change.

An easy bound is obtained by considering $m < N < n$ and the events $\{\sigma \leq \tau_N\}$ and $\{$no return to $\mathbf{B}(N)$ after $\tau_n\}$. This gives

$$\mathbf{P}[\mathrm{LERW} \cap \mathbf{B}(m) \text{ is unchanged after } \tau_n] \geq 1 - C(m/n)^{1/2}.$$

From this one can derive the $d = 3$ statement of the Theorem with an explicit exponent.

Main ideas for $d = 2$

Infinite LERW. **[Lawler; 1988]** showed that $\mathrm{LE}(S[0,\tau_n])$ converges in distribution to a random infinite self-avoiding path. Due to recurrence, there is no a.s. convergence.

Laplacian walk. Letting $\Gamma = \mathrm{LE}(S[0,\tau_n])$, we have

$$\mathbf{P}[\Gamma(k+1) = x \,|\, \Gamma[0,k] = [z_0,\ldots,z_k]] = \frac{\mathrm{Es}^n_{\Gamma[0,k]}(x)}{\sum_{y:y\sim z_k} \mathrm{Es}^n_{\Gamma[0,k]}(y)}, \quad x \sim z_k.$$

Error estimates. **[Lawler; 1988]** showed that the right hand side differs from its limit as $n \to \infty$ by a factor $(1 + O(\frac{k^2}{n} \log \frac{n}{k}))$, uniformly over paths.

Killed LERW. Let $T \sim \mathrm{Geom}(\gamma/(2d+\gamma))$. We want to give error estimates on the weak convergence of $\mathrm{LE}(S[0,T])$ to the infinite LERW as $\gamma \downarrow 0$.

Main ideas for $d = 2$

Evolution of the killed LERW. Let $\Gamma = \mathrm{LE}(S[0,T])$. Analogously to the Laplacian walk, we have

$$\mathbf{P}[\Gamma(k+1) = x \mid \Gamma[0,k] = [z_0, \ldots, z_k]] = \frac{\frac{1}{4+\gamma}\mathbf{P}^x[\xi_A > T]}{\frac{\gamma}{4+\gamma} + \frac{1}{4+\gamma}\sum_{y \notin A,\, y \sim z_k} \mathbf{P}^y[\xi_A > T]},$$

where $A = \{z_0, \ldots, z_k\}$, and ξ_A is the hitting time of A.

Technical difficulty. We cannot easily compare $\mathrm{LE}(S[0,T])$ to $\mathrm{LE}(S[0,\tau_n])$ for some n. When we consider the path $S[0,\tau_n]$, the loop-erased path cannot trap itself. It will, by definition, reach $\partial\mathbf{B}(n)$. When we consider $S[0,T]$, trapping can occur.

Trapping. Even for small γ, the term $\frac{\gamma}{4+\gamma}$ in the denominator may be significant. Hence the error estimate cannot be uniform from path to path.

Main ideas for $d = 2$

Regular paths. We need to restrict to paths that are sufficiently regular, so that the trapping effect is small. This is achieved by restricting to paths that satisfy:

$$\mathbf{P}^y[\tau_{2R} < \xi_{\Gamma[0,k]} \,|\, \Gamma[0,k]] > R^{-\beta}, \quad y \notin A,\, y \sim z_k,$$

for a suitable radius R and exponent $\beta > 0$.

Strategy. On regular paths, one can approximate the killed LERW by the Laplacian Walk, that is, replace T by an exit time τ_n for a suitable n. The estimates can be made uniform on such paths. We can also estimate the probability of bad paths (on which the coupling will not be realized).

Open problems

$d = 4$: Our approach gives only logarithmic rate of convergence. It appears challenging to improve this to a power law.

$d \geq 5$: A different approach is needed. This is related to the fact that the Uniform Spanning Forest is not connected in dimensions $d \geq 5$.

Exact value of the exponent: is the rate for all cylinder events E equal $c(E)\gamma$ for $d \geq 3$, and $c(E)\gamma \log(1/\gamma)$ for $d = 2$?

Thank you very much for your attention.