Random Walks in Dirichlet Random Environment

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The topic of the talk is

- Random walks in Dirichlet random environment (RWDE) : a special class of random walks in random environment (RWRE) with at each site independent Dirichlet random variables

or equivalently

- Directed edge reinforced random walks: linearly reinforced random walks on directed graphs with counters on directed edges

This model originally appeared in the work of Pemantle on reinforced RW on trees. These environments have a remarkable property of "stability under time reversal". I will explain some consequences on the environment viewed from the particle and ballisticity criteria.

Random Walks in Random Environment on $\mathbb{Z}^d$

Let $(e_1, \dots, e_d)$ be the canonical base of $\mathbb{Z}^d$, and $e_{j+d} = -e_j$.
$(e_1, \dots, e_{2d})$ is the set of unit vectors of $\mathbb{Z}^d$.
The set of environments is the set of (weakly) elliptic transition probabilities

$$\Omega = \{(\omega(x, x+e_i)) \in (0,1)^{\mathbb{Z}^d \times \{1,\dots,2d\}},\ \forall x \in \mathbb{Z}^d,\ \sum_{i=1}^{2d} \omega(x, x+e_i) = 1\}.$$

For $\omega \in \Omega$, the law of the Markov chain in environment ω, starting from x, is denoted by P_x^ω:

$$P_x^\omega(X_{n+1} = y + e_i | X_n = y) = \omega(y, y + e_i),$$

We define the law $\mathbb{P}$ on the environment Ω as follows: At each point x in $\mathbb{Z}^d$, we choose independently the transition probabilities

$$(\omega(x, x + e_i))_{i=1,\ldots,2d}$$

according to the same law μ; μ is a law on the set

$$\{(\omega_1, \ldots, \omega_{2d}) \in (0,1]^{2d}, \quad \sum_{i=1}^{2d} \omega_i = 1\}.$$

The annealed law is

$$\mathbb{P}_x(\cdot) = \mathbb{E}(P_x^\omega(\cdot))$$

Many results in the last 10 years mainly in the ballistic regime (Sznitman's (T) condition) or at weak disorder. But even more open questions.

Dirichlet law

The Dirichlet law is the multivariate generalization of the beta law. The Dirichlet law with positive parameters $\alpha_1, \dots, \alpha_n$, ($\mathcal{D}ir(\alpha_1, \dots, \alpha_n)$), is the law on the simplex

$$\{(p_1, \dots, p_n),\ p_i > 0,\ \sum p_i = 1\}$$

with density

$$\frac{\Gamma(\sum_{i=1}^n \alpha_i)}{\prod_{i=1}^n \Gamma(\alpha_i)} \left(\prod_{i=1}^n p_i^{\alpha_i - 1} \right) dp_1 \cdots dp_{n-1}.$$

Dirichlet laws give a natural family of laws on probabilities on finite sets. They play an important role in Bayesian statistics.

Random Walks in Dirichlet environments (RWDE)

We choose some positive weights $\alpha_1, \ldots, \alpha_{2d}$, one for each direction e_i : the Dirichlet environment corresponds to the case

$$\mu = \mathcal{D}ir(\alpha_1, \ldots, \alpha_{2d}),$$

i.e. to the case where the $(\omega(x, x + e_i))_{i=1,\ldots,2d}$ are choosen independently with the same Dirichlet law $\mathcal{D}ir((\alpha_1, \ldots, \alpha_{2d}))$.

The annealed process has the law of a directed edge reinforced RW

$$\begin{aligned} & \mathbb{P}_0(X_{n+1} = X_n + e_i | \sigma\{X_k,\ k \le n\}) \\ = \quad & \frac{\alpha_i + I_n(X_n, X_n + e_i)}{\sum \alpha_k + I_n(X_n, X_n + e_k)}, \end{aligned}$$

where $I_n(x, x + e_k) =$ number of crossings of the **directed** edge $(x, x + e_k)$ before time n.

Dirichlet environments have a property of stability under time reversal which is a key tool in proving

- Transience on transient graphs
- Directional transience
- The existence of an invariant measure for the environment viewed from the particle
- This property is also related to the explicit parameters which appear for limit theorems in 1D (Enriquez, S., Tournier, Zindy).

RWDE on directed graphs

Consider a connected directed graph $G = (V, E)$ with finite degree. We denote by $\underline{e}$, $\overline{e}$ the tail and the head of the edge $e = (\underline{e}, \overline{e})$. We consider a set of positive weights $(\alpha_e)_{e \in E}$, $\alpha_e > 0$.

- The environment set is

$$\Omega = \{(\omega_e)_{e \in E} \in (0,1]^E, \text{ s. t. } \forall x \in V, \sum_{\underline{e}=x} \omega_e = 1\}.$$

  This is the set of possible transition probabilities on the graph G.

- The random Dirichlet environment is defined as follows : at each vertex x, pick independently the exit probabilities $(\omega_e)_{\underline{e}=x}$ according to a Dirichlet law $\mathcal{D}ir((\alpha_e)_{\underline{e}=x})$.

It defines the law $\mathbb{P}^{(\alpha)}$ on Ω.

Stability under time reversal

Let $G = (V, E)$ be a finite connected directed graph. Let $\mathrm{div} : \mathbb{R}^E \mapsto \mathbb{R}^V$ be the divergence operator

$$\mathrm{div}(\theta)(x) = \sum_{\underline{e}=x} \theta(e) - \sum_{\overline{e}=x} \theta(e), \quad \forall \theta \in \mathbb{R}^E.$$

If $G = (V, E)$ we denote by $\check{G} = (V, \check{E})$ the reversed graph obtained by reversing all the edges. If ω is an environment we denoted by $\check{\omega}$ the time-reversed environment defined as usual

$$\check{\omega}_{\check{e}} = \pi(\underline{e})\omega_e \frac{1}{\pi(\overline{e})}.$$

where π is the invariant probability measure in the environment ω.

Lemma (S., 08)

Let $(\alpha_e)_{e \in E}$ be positive weights with null divergence. If (ω_e) is a Dirichlet random environment with distribution $\mathbb{P}^{(\alpha)}$ then $\check{\omega}$ is a Dirichlet random environment on $\check{G}$ with distribution $\mathbb{P}^{(\check{\alpha})}$, where $\check{\alpha}$ is defined by $\check{\alpha}_{\check{e}} = \alpha_e$.

Time reversal on the torus

On the torus

$$T_d^{(N)} = (\mathbb{Z}/N\mathbb{Z})^d,$$

if we have take some weights $(\alpha_1, \dots, \alpha_{2d})$, i.e.

$$\alpha_{(x,x+e_i)} = \alpha_i,$$

then the weights have null divergence. So if ω is distributed according to $\mathbb{P}^{(\alpha)}$, then $\check{\omega}$ is distributed according to $\mathbb{P}^{(\check{\alpha})}$ where $\check{\alpha}$ is obtained from α by symmetry:

$$\check{\alpha}_{(x,x-e_i)} = \alpha_i$$

The environment viewed from the particle

Let τ be the spatial shift on the environment

$$\tau_x(\omega)(y, y + e_i) = \omega(x + y, x + y + e_i).$$

The environment viewed from the particle is the Markov chain on state space Ω defined by

$$\overline{\omega}_n = \tau_{X_n}(\omega),$$

where (X_n) is the Markov chain in environment ω.

(Q) Does there exists a probability measure $\mathbb{Q}$ on Ω absolutely continuous with respect to $\mathbb{P}$ and invariant for $(\overline{\omega}_n)$?

Answering (Q) is a key step. When it exists it is equivalent to $\mathbb{P}$ unique and ergodic.

For general RWRE the answer is known only in a few special cases

- $d = 1$ (Kesten, Molchanov)
- Balanced environments (Lawler)
- For "non-nestling" weakly disordered environments in $d \geq 4$ (Bolthausen-Sznitman)
- In a weaker form (equivalence in half-spaces) under (T) condition (Rassoul-Agha, Rassoul-Agha Seppäläinen).

Equivalence of the static and dynamic point of view

Theorem (S., 10)

Let $d \geq 3$ and $(\alpha_1, \dots, \alpha_{2d})$ be any positive weights. Set

$$\kappa = 2\Big(\sum_{k=1}^{2k} \alpha_k\Big) - \min_{i=1,\dots,d}(\alpha_i + \alpha_{i+d})$$

i) If $\kappa \leq 1$, there does not exist any probability measure on Ω absolutely continuous with respect to $\mathbb{P}^{(\alpha)}$ and invariant for $\overline{\omega}_n$.

ii) If $\kappa > 1$, there exists a (unique ergodic) probability measure $\mathbb{Q}^{(\alpha)}$ on Ω, absolutely continuous with respect to $\mathbb{P}^{(\alpha)}$ and invariant for $\overline{\omega}_n$. Moreover $\frac{d\mathbb{Q}^{(\alpha)}}{d\mathbb{P}^{(\alpha)}}$ is in $L^p(\mathbb{P}^{(\alpha)})$ for all $p < \kappa$.

The meaning of κ

For $i = 1, \ldots, d$, set $K_i = \{0, e_i\}$ and

$$\kappa_i = 2(\sum_{k=1}^{2k} \alpha_k) - (\alpha_i + \alpha_{i+d}).$$

κ_i is the sum of the weights exiting K_i and we have that

$$\mathbb{E}_0^{(\alpha)}(T_{K_i}^s) < \infty \text{ if and only if } s < \kappa_i$$

where T_{K_i} is the exit time of K_i.

We have $\kappa = \max \kappa_i$ and if $\kappa \leq 1$ then the annealead expected time spent in one of the small traps K_i is infinite. This explains the non-existence of the absolutely continuous invariant measure.

Directional transience

Theorem (S., Tournier, 09)

Assume that $\alpha_{e_1} > \alpha_{-e_1}$. *Then for any d*

$$\mathbb{P}_0(D = \infty) \geq 1 - \frac{\alpha_{-e_1}}{\alpha_{e_1}},$$

where $D = \inf\{n,\ X_n \cdot e_1 < 0\}$ *(equality is strongly conjectured).*
Thanks to Kalikow's 0-1 law

$$\lim_{n\to\infty} |X_n \cdot e_1| = +\infty, \ \ \mathbb{P}_0 \ p.s.$$

For $d = 2$ *thanks to Zerner-Merkl 0-1 law*

$$\lim_{n\to\infty} X_n \cdot e_1 = +\infty, \ \ \mathbb{P}_0 \ p.s.$$

In dimension 1, the law of $P_0^\omega(D = \infty)$ is explicit (it is Kesten's renewal series, explicit for beta environment, cf Chamayou-Letac. It implies explicit limit laws in 1D, cf Enriquez, S., Tournier, Zindy).

Theorem

Let $d \geq 3$. *Let* $d_\alpha = \mathbb{E}_0^{(\alpha)}(X_1) = \frac{1}{\sum \alpha_k} \sum \alpha_k e_k$.

i) If $\kappa \leq 1$, *then*

$$\lim_{n \to \infty} \frac{X_n}{n} = 0, \quad \mathbb{P}_0^{(\alpha)} \ p.s.$$

ii) If $\kappa > 1$ *and* $d_\alpha = 0$ *then*

$$\lim_{n \to \infty} \frac{X_n}{n} = 0, \quad \mathbb{P}_0^{(\alpha)} \ p.s.$$

and for all $i = 1, \ldots, d$

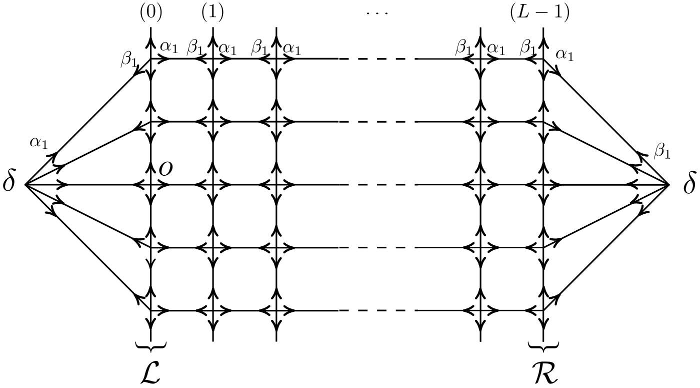
$$\liminf X_n \cdot e_i = -\infty, \ \limsup X_n \cdot e_i = +\infty, \quad \mathbb{P}_0^{(\alpha)} \ p.s.$$

iii) If $\kappa > 1$ *and* $d_\alpha \neq 0$ *then there exists* $v \neq 0$ *such that*

$$\lim_{n \to \infty} \frac{X_n}{n} = v, \quad \mathbb{P}_0^{(\alpha)} \ p.s.$$

Sketch of proof of directional transience

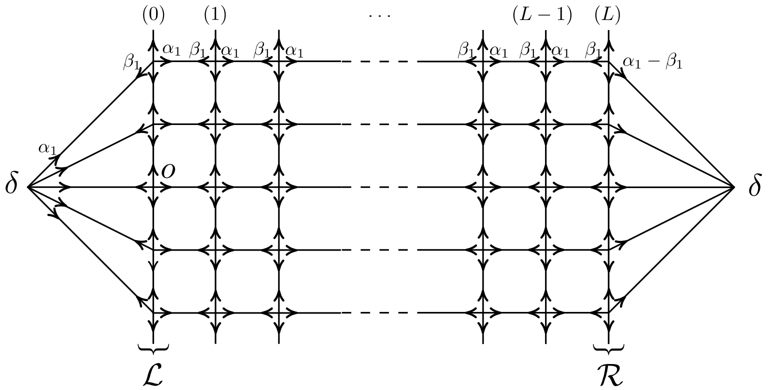
We prove a uniform estimate on cylinders of arbitrary length and width.



$\mathbb{P}^{(\alpha,L-1)}(\text{left} \to \text{right})$?

Sketch of proof of directional transience

We sligthly modify the graph and the weights



$\mathbb{P}^{(\alpha,L-1)}(\text{left} \to \text{right}) \geq \mathbb{P}^{(\tilde{\alpha},L)}(\text{left} \to \text{right}) = \frac{\alpha_1 - \beta_1}{\alpha_1}$

Sketch of proof of the existence of the invariant measure

For $N > 0$ let $T_N = (\mathbb{Z}/NZ)^d$ be the d-dim torus. For $\omega \in \Omega_N$, let π_N^ω be the invariant probability measure. We set

$$f_N(\omega) = N^d \pi_N^\omega(0), \quad \mathbb{Q}_N^{(\alpha)} = f_N \cdot \mathbb{P}_N^{(\alpha)}.$$

We prove that for all $s < \kappa$

$$\sup_N \mathbb{E}^{(\alpha)}\left(f_N^s\right) < \infty.$$

We have

$$(f_N)^s = \left(\frac{\pi(0)}{\frac{1}{N^d} \sum_{x \in T_N} \pi(x)} \right)^s \leq \prod_{x \in T_N} \left(\frac{\pi(0)}{\pi(x)}\right)^{s/N^d}.$$

Let $\check{\omega}$ be the time reversed environment on T_N

$$\check{\omega}(x, x+e_i) = \frac{\pi^\omega(x+e_i)}{\pi^\omega(x)}\omega(x+e_i, x).$$

By the time reversal lemma, $\check{\omega}$ is a Dirichlet environment with the reversed weights.
Moreover, for all $\theta : E_N \mapsto \mathbb{R}$ (where E_N is the set of edges of T_N), simple computation gives

$$\frac{\check{\omega}^{\check{\theta}}}{\omega^\theta} = \pi^{\mathrm{div}\theta},$$

where $\check{\theta}$ is defined by $\check{\theta}(x,y) = \theta(y,x)$ and $\omega^\theta = \prod_{E_N} \omega(e)^{\theta(e)}$ and $\pi^{\mathrm{div}\theta} = \prod_{T_N} \pi(x)^{\mathrm{div}\theta(x)}$.

If $\theta_N : E_N \mapsto \mathbb{R}_+$ satisfies

$$\mathrm{div}(\theta_N) = \frac{s}{N^d} \sum_{x \in T_N} (\delta_0 - \delta_x)$$

then,

$$\begin{array}{rcl} \mathbb{E}^{(\alpha)}(f_N^s) & \leq & \mathbb{E}^{(\alpha)}(\prod (\frac{\pi(0)}{\pi(x)})^{s/N^d}) \\ & = & \mathbb{E}^{(\alpha)}(\pi^{\mathrm{div}(\theta_N)}) \\ & = & \mathbb{E}^{(\alpha)}(\frac{\check{\omega}^{\check{\theta}_N}}{\omega^{\theta_N}}) \\ & \leq & \mathbb{E}^{(\alpha)}(\check{\omega}^{q\check{\theta}_N})^{1/q} \mathbb{E}^{(\alpha)}(\omega^{-p\theta_N})^{1/p} \end{array}$$

for all $1/p + 1/q = 1$.

$$\mathbb{E}^{(\alpha)}(f_N^s) \leq \mathbb{E}^{(\alpha)}(\check{\omega}^{q\check{\theta}_N})^{1/q}\mathbb{E}^{(\alpha)}(\omega^{-p\theta_N})^{1/p}$$

- The right hand side is finite when $p\theta_N(e) < \alpha_e$ for all e. Hence, we need that $\theta_N(e) < (1-\epsilon)\alpha_e$ for some $\epsilon > 0$. Then, we can find p small enough such that the right hand term is finite.
- Thanks to the time reversal property the transition probabilities $\check{\omega}$ are independent at each sites and everything can be computed. By Taylor expansion, there exists $c > 0$ s.t.

$$\mathbb{E}^{(\alpha)}(f_N^s) \leq \exp(c \sum \theta_N(e)^2).$$

Hence, we need that the L_2-norm of θ_N is bounded.

Lemma

For all $N > 0$, there exists $\tilde{\theta}_N : E_N \mapsto \mathbb{R}_+$ such that

$$div(\tilde{\theta}_N) = \frac{\kappa}{N^d} \sum_{x \in T_N} (\delta_0 - \delta_x)$$

and

$$\tilde{\theta}_N(e) \leq \alpha_e \tag{1}$$

$$\sum_{E_N} \tilde{\theta}_N(e)^2 \leq C \tag{2}$$

where $C > 0$ is a constant not depending on N.

Then, $\theta_N = \frac{s}{\kappa}\tilde{\theta}_N$ makes the job.
Sketch of proof:
(1) comes from the Max-Flow Min Cut theorem.
(2) comes from $d \geq 3$.
(1) and (2) at the same time needs more work.