A replica analysis of the one-dimensional KPZ equation

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(Based on a collaboration with T. Imamura)

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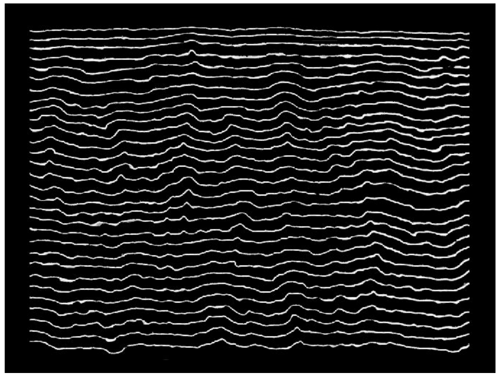
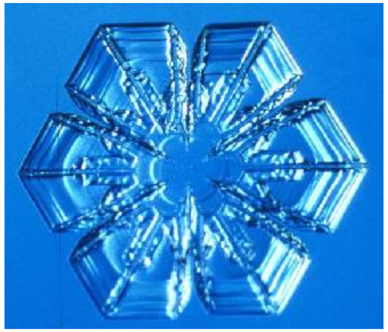
References: arxiv:1105.4659, 1108.2118

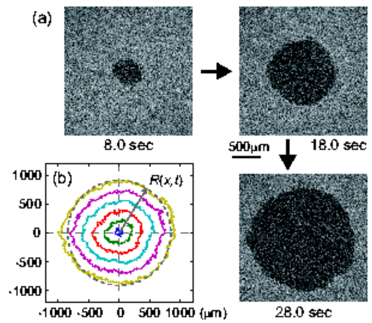
1. Introduction: 1D surface growth

- Paper combustion, bacteria colony, crystal growth, liquid crystal turbulence (2010 Takeuchi Sano)
- Non-equilibrium statistical mechanics
- Stochastic interacting particle systems
- Integrable systems



(Myllys et al)





Kardar-Parisi-Zhang(KPZ) equation

1986 Kardar Parisi Zhang

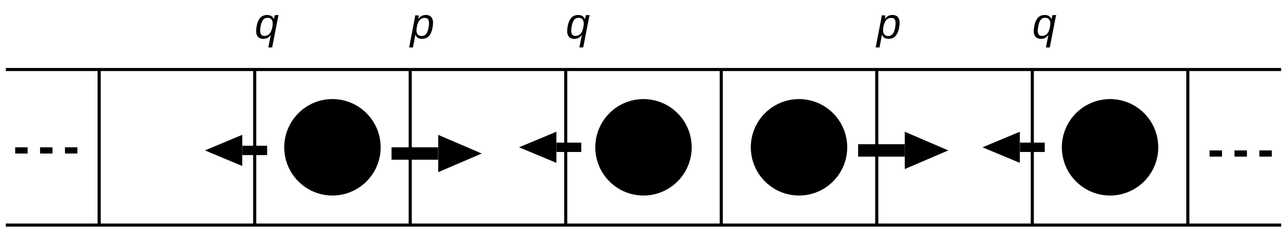
$$\partial_t h(x,t) = \frac{1}{2}\lambda(\partial_x h(x,t))^2 + \nu\partial_x^2 h(x,t) + \sqrt{D}\eta(x,t)$$

where η is the Gaussian noise with covariance
$\langle\eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$

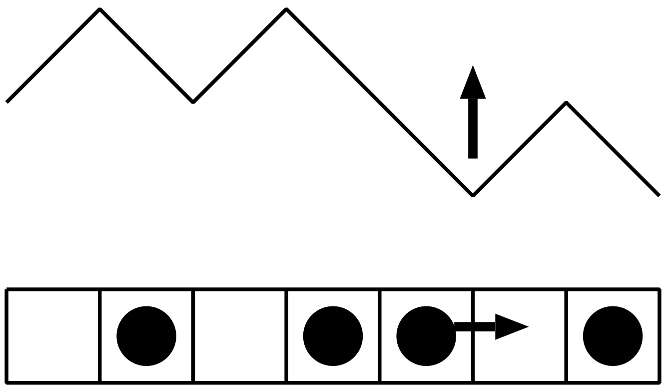
- The Brownian motion is stationary.
- Dynamical RG analysis: $h(x=0,t) \simeq vt + c\xi t^{1/3}$
  KPZ universality class
- Now revival: New analytic and experimental developments

A discrete model: ASEP as a surface growth model

ASEP(asymmetric simple exclusion process)

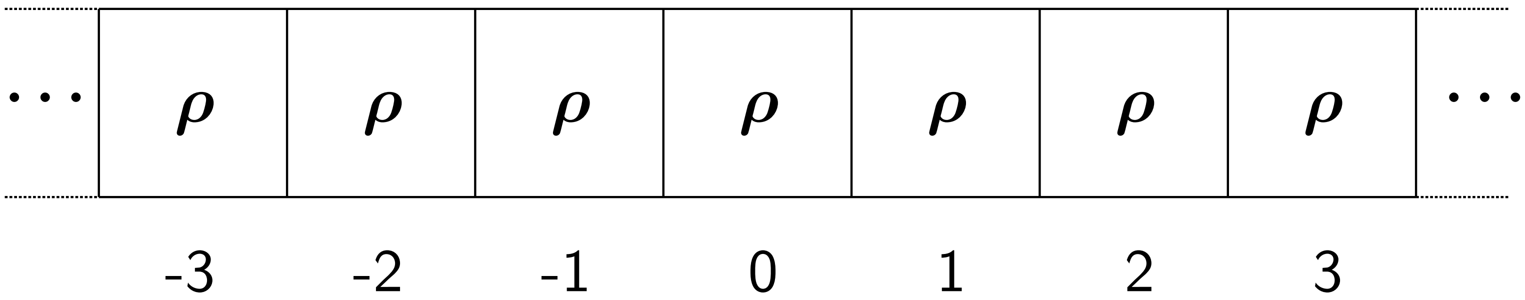


Mapping to surface growth



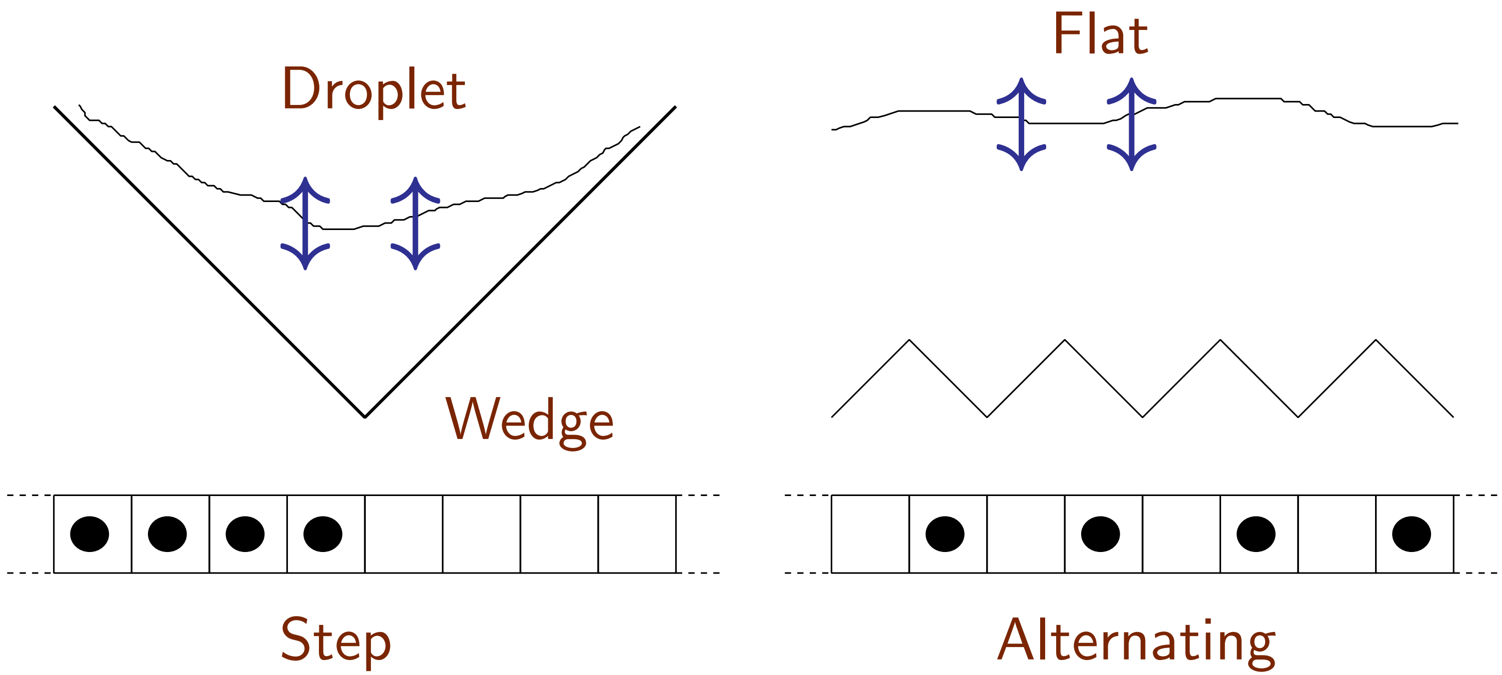
Stationary measure

ASEP $\cdots$ Bernoulli measure: each site is independent and occupied with prob. ρ $(0 < \rho < 1)$. Current is $\rho(1-\rho)$.



Surface growth $\cdots$ Random walk height profile

Surface growth and 2 initial conditions besides stationary



Integrated current $N(x,t)$ in ASEP $\Leftrightarrow$ Height $h(x,t)$ in surface growth

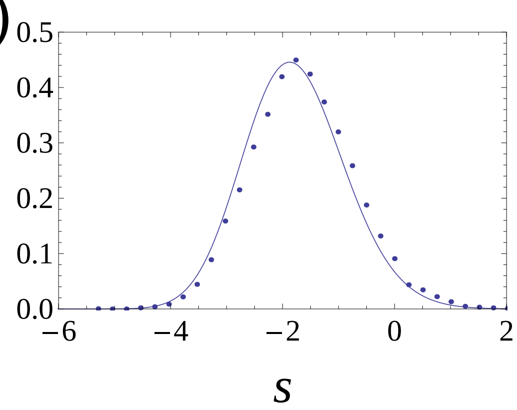
Current distributions for ASEP with wedge initial conditions

2000 Johansson (TASEP) 2008 Tracy-Widom (ASEP)

$$N(0, t/(q-p)) \simeq \frac{1}{4}t - 2^{-4/3} t^{1/3} \xi_{\mathrm{TW}}$$

Here $N(x=0,t)$ is the integrated current of ASEP at the origin and ξ_{TW} obeys the GUE Tracy-Widom distributions;

$$F_{\mathrm{TW}}(s) = \mathbb{P}[\xi_{\mathrm{TW}} \le s] = \det(1 - P_s K_{\mathrm{Ai}} P_s)$$



where K_{Ai} is the Airy kernel

$$K_{\mathrm{Ai}}(x,y) = \int_0^\infty d\lambda \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)$$

Current Fluctuations of ASEP with flat initial conditions: GOE TW distribution

More generalizations: stationary case: F_0 distribution, multi-point fluctuations, etc

Can they be measured experimentally?

What about the KPZ equation?

Random matrix theory

GUE (Gaussian Unitary Ensemble) hermitian matrices

$$A = \begin{bmatrix} u_{11} & u_{12} + iv_{12} & \cdots & u_{1N} + iv_{1N} \\ u_{12} - iv_{12} & u_{22} & \cdots & u_{2N} + iv_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ u_{1N} - iv_{1N} & u_{2N} - iv_{2N} & \cdots & u_{NN} \end{bmatrix}$$

$$u_{jj} \sim N(0, 1/2) \quad u_{jk}, v_{jk} \sim N(0, 1/4)$$

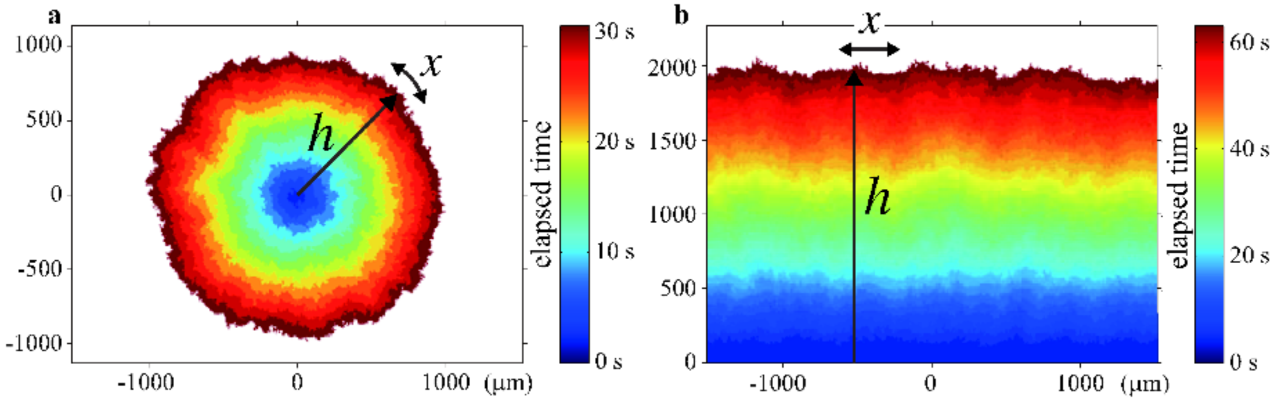
The largest eigenvalue x_{max} $\cdots$ GUE TW distribution

GOE (Gaussian Orthogonal Ensemble) real symmetric matrices

$\cdots$ GOE TW distribution

Experiments by liquid crystal turbulence

2010-2011 Takeuchi Sano



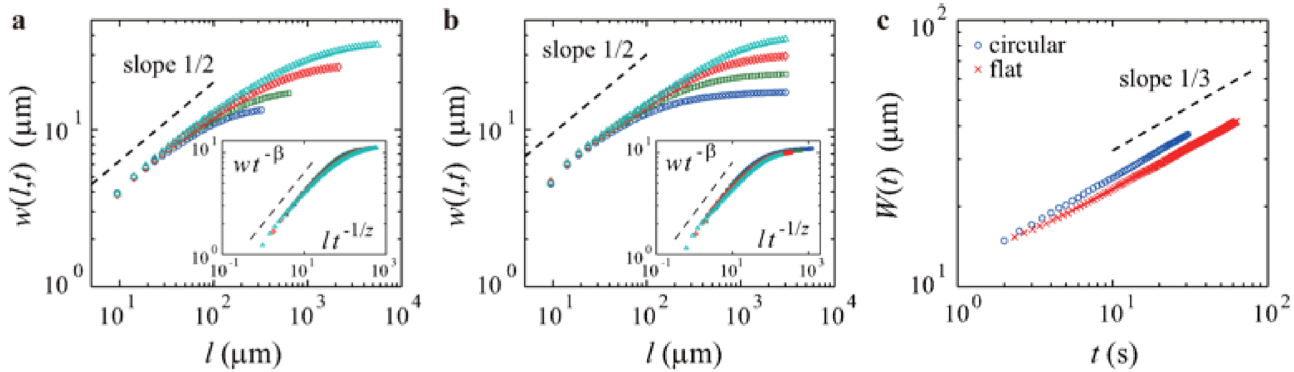


Figure 2 | Family-Vicsek scaling. a,b, Interface width $w(l, t)$ against the length scale l at different times t for the circular (a) and flat (b) interfaces. The four data correspond, from bottom to top, to $t = 2.0$ s, 4.0 s, 12.0 s and 30.0 s for the panel a and to $t = 4.0$ s, 10.0 s, 25.0 s and 60.0 s for the panel b. The insets show the same data with the rescaled axes. c, Growth of the overall width $W(t) \equiv \sqrt{\langle [h(x,t) - \langle h \rangle]^2 \rangle}$. The dashed lines are guides for the eyes showing the exponent values of the KPZ class.

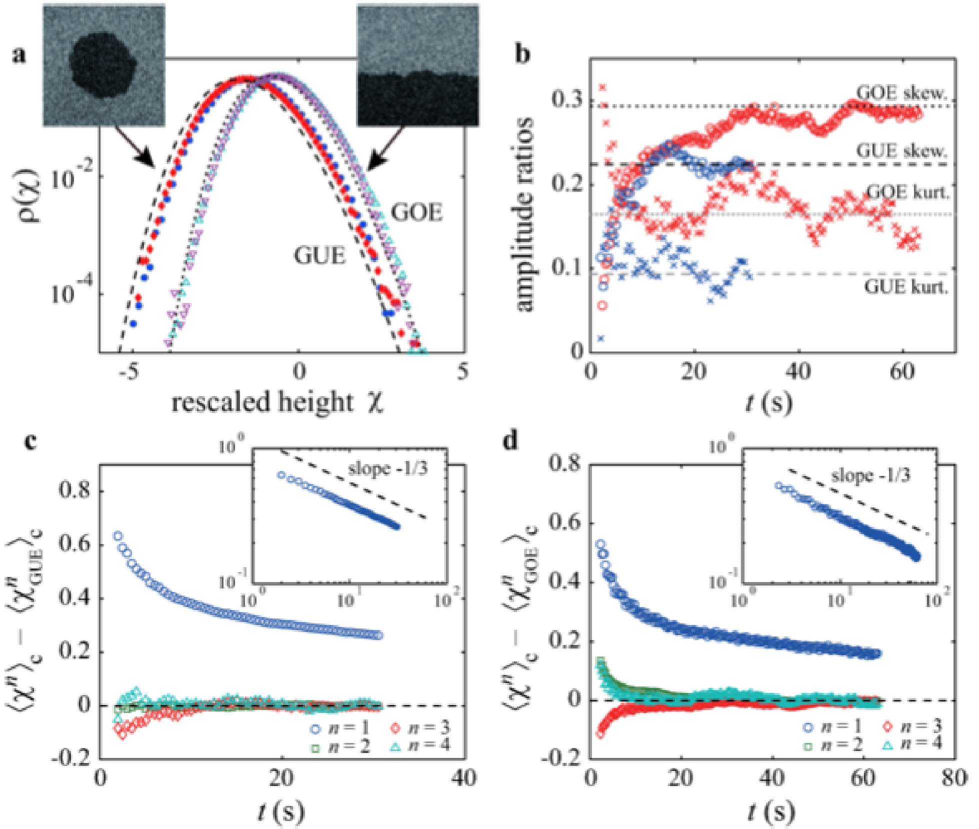


Figure 3 | **Universal fluctuations.** a, Histogram of the rescaled local height $\chi = (h - v_\infty t)/(\Gamma t)^{1/3}$. The blue and red solid symbols show the histograms for the circular interfaces at $t = 10$ s and 30 s; the light blue and purple open symbols are for the flat interfaces at $t = 20$ s and 60 s, respectively. The dashed and dotted curves show the GUE and GOE TW distributions, respectively. Note that for the GOE TW distribution χ is multiplied by $2^{-2/3}$ in view of the theoretical prediction[31]. b, The skewness (circle) and the kurtosis (cross) of the distribution of the interface fluctuations for the circular (blue) and flat (red) interfaces. The dashed and dotted lines indicate the values of the skewness and the kurtosis of the GUE and GOE TW distributions[31]. c, d, Differences in the cumulants between the experimental data $\langle \chi^n \rangle_c$ and the corresponding TW distributions $\langle \chi^n_{GUE} \rangle_c$ for the ciruclar interfaces (c) and $\langle \chi^n_{GOE} \rangle_c$ for the flat interfaces (d). The insets show the same data for $n = 1$ in logarithmic scales. The dashed lines are guides for the eyes with the slope $-1/3$.

See Takeuchi Sano Sasamoto Spohn, Sci. Rep. 1,34(2011)

Solution for the KPZ equation

2010 Sasamoto Spohn, Amir Corwin Quastel

- Narrow wedge initial condition
- Based on (i) the fact that the weakly ASEP is KPZ equation (1997 Bertini Giacomin) and (ii) a formula for step ASEP by 2009 Tracy Widom
- The explicit distribution function for finite t
- The KPZ equation is in the KPZ universality class

Before this

2009 Balaźs, Quastel, and Seppäläinen

The $1/3$ exponent for the stationary case

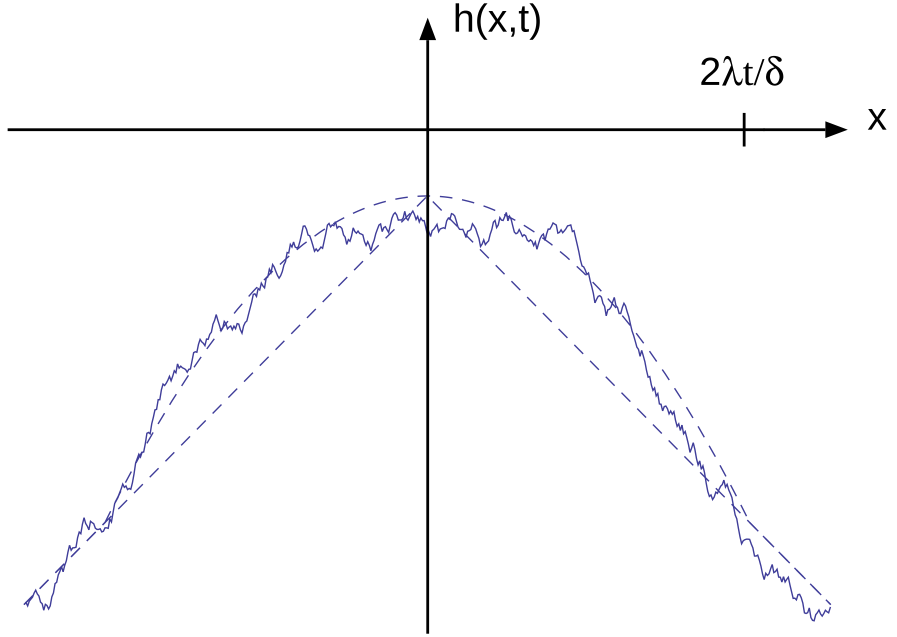
Narrow wedge initial condition

We consider the droplet growth with macroscopic shape

$$\boldsymbol{h(x,t)} = \begin{cases} -\boldsymbol{x^2/2\lambda t} & \text{for } |\boldsymbol{x}| \le \boldsymbol{\lambda t/\delta}\,, \\ (\boldsymbol{\lambda/2\delta^2})\boldsymbol{t} - |\boldsymbol{x}|/\boldsymbol{\delta} & \text{for } |\boldsymbol{x}| > \boldsymbol{\lambda t/\delta} \end{cases}$$

which corresponds to taking the following narrow wedge initial conditions:

$$\boldsymbol{h(x,0)} = -|\boldsymbol{x}|/\boldsymbol{\delta}\,, \quad \boldsymbol{\delta \ll 1}$$


h(x,t)
2λt/δ
x

Distribution

$$(\lambda/2\nu)h(x,t/2\nu) = -x^2/2t - \tfrac{1}{12}\gamma_t^3 + 2\log\alpha + \gamma_t \xi_t$$

Here $\gamma_t = 2^{-1/3}\alpha^{4/3}t^{1/3}\,,\quad \alpha = (2\nu)^{-3/2}\lambda D^{1/2}$.

The cumulative distribution of ξ_t

$$F_t(s) = \mathbb{P}[\xi_t \le s] = 1 - \int_{-\infty}^{\infty} \exp\big[-\mathrm{e}^{\gamma_t(s-u)}\big]$$

$$\times\big(\det(1 - P_u(B_t - P_{\mathrm{Ai}})P_u) - \det(1 - P_u B_t P_u)\big)\mathrm{d}u$$

where $P_{\mathrm{Ai}}(x,y) = \mathrm{Ai}(x)\mathrm{Ai}(y)$.

P_u is the projection onto $[u, \infty)$ and the kernel B_t is

$$B_t(x, y) = K_{\mathrm{Ai}}(x, y) + \int_0^\infty \mathrm{d}\lambda (\mathrm{e}^{\gamma_t \lambda} - 1)^{-1} \times \big(\mathrm{Ai}(x + \lambda)\mathrm{Ai}(y + \lambda) - \mathrm{Ai}(x - \lambda)\mathrm{Ai}(y - \lambda)\big) .$$

Developments (not all!)

- 2010 Calabrese Le Doussal Rosso, Dotsenko Replica
- 2010 Corwin Quastel Half-BM by step Bernoulli ASEP (2009 Tracy Widom)
- 2010 O'Connell A directed polymer model related to quantum Toda lattice
- 2010 Prolhac Spohn Multi-point distributions by replica
- 2011 Corwin Quastel Renormalization fixed point
- 2011 Calabrese Le Dossal Flat case by replica
- 2011 O'Connell Warren Multi-layer picture
- 2011 Imamura Sasamoto Half-BM case by replica

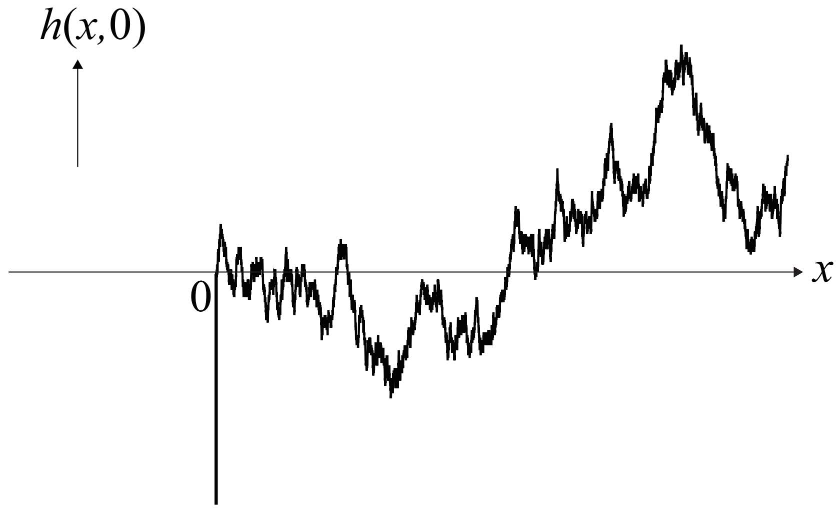
Morikazu Toda (1917-2010)

- Passed away last year at the age of 93.
- The Toda lattice is well known as a solition equation but the solition solution was not obtained in the first paper.
- His orignal motivation was to consider a fundamental aspect of statistical mechanics from a dynamical point of view. A numerical simulation suggested that aproximation scheme was not very useful.
- He discovered his lattice by matching the equation of motion with a formula of elliptic functions. The solution was periodic in space.
- The discovery was during a summer vacation in 1966. He brought only a few books including a consise book of mathematical formulas.

2. Results: Half Brownian motion initial condition

$$\frac{\lambda}{2\nu} h(x, t = 0) = \begin{cases} x/\delta, \ \delta \to 0, & x < 0, \\ \alpha B(x), & x \geq 0. \end{cases}$$

with $\alpha = (2\nu)^{-3/2} \lambda D^{1/2}$.



Macroshape at t

$$h(x,t) \sim \begin{cases} -x^2/2\lambda t, & x \leq 0, \\ 0, & 0 < x. \end{cases}$$

Here we focus on the crossover region around the origin.

- One of the few solvable initial conditions so far.
- A step toward the stationary case.

Cole-Hopf transformation

1997 Bertini and Giacomin

$$h_{\nu,\lambda,D}(x,t) = \frac{2\nu}{\lambda} \log\left(Z_{\nu,\lambda,D}(x,t)\right)$$

$Z_{\nu,\lambda,D}(x,t)$ is the solution of the stochastic heat equation,

$$\frac{\partial Z_{\nu,\lambda,D}(x,t)}{\partial t} = \nu \frac{\partial^2 Z_{\nu,\lambda,D}(x,t)}{\partial x^2} + \frac{\lambda\sqrt{D}}{2\nu}\eta(x,t) Z_{\nu,\lambda,D}(x,t).$$

The partition function $Z_{\nu,\lambda,D}(x,t)$ can be considered as a directed polymer in random potential η.

Feynmann-Kac and Scaling

Feynmann-Kac expression for the partition function,

$$Z(x,t) = \mathbb{E}_x \left(\exp \left[\frac{\lambda\sqrt{D}}{2\nu} \int_0^t \eta\left(b(2\nu s), t-s\right) ds \right] Z(b(t),0) \right)$$

Using this we can establish

$$\frac{\lambda}{2\nu} h_{\nu,\lambda,D}\left(x, \frac{t}{2\nu}\right) = h_{\frac{1}{2},1,1}(\alpha^2 x, \alpha^4 t).$$

In the following, we set $\nu = \frac{1}{2}, \lambda = 1, D = 1, \alpha = 1$ and consider $h(x,t) = h_{\frac{1}{2},1,1}(x,t)$.

Generating function

We are interested in the distibution of $h = \log Z$, which is difficult to handle. Instead we consider the moments $\langle Z^N \rangle$, or the generating function of them.

We introduce the scaled height $\tilde{h}_t(X)$ by

$$h\left(2\gamma_t^2 X, t\right) = -\frac{\gamma_t^3}{12} - \gamma_t X^2 + \gamma_t \tilde{h}_t(X)$$

where $\gamma_t = \left(\frac{t}{2}\right)^{\frac{1}{3}}$, $\;x = 2\gamma_t^2 X$ and consider

$$G_{\gamma_t}(s; X) = \sum_{N=0}^{\infty} \frac{\left(-e^{-\gamma_t s}\right)^N}{N!} \left\langle Z^N \left(2\gamma_t^2 X, t\right)\right\rangle e^{N\frac{\gamma_t^3}{12} + N\gamma_t X^2}$$

$$= \langle e^{-e^{\gamma_t(\tilde{h}_t(X) - s)}} \rangle.$$

Result

$G_{\gamma_t}(s;X)$ is expressed as the Fredholm determinant

$$G_{\gamma_t}(s;X) = \det\left(1 - P_0 K_X P_0\right)$$

Here P_s represents the projection onto (s,∞) and the kernel of K_X is given by

$$\begin{aligned} & K_X(\xi_j,\xi_k) \\ & = \int_{\mathbb{R}} dy \mathrm{Ai}^{\Gamma}\left(\xi_j + y, \frac{1}{\gamma_t}, -\frac{X}{\gamma_t}\right) \mathrm{Ai}_{\Gamma}\left(\xi_k + y, \frac{1}{\gamma_t}, -\frac{X}{\gamma_t}\right) \frac{e^{\gamma_t y}}{e^{\gamma_t y} + e^{\gamma_t s}}. \end{aligned}$$

Here $\mathrm{Ai}^{\boldsymbol{\Gamma}}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$, $\mathrm{Ai}_{\boldsymbol{\Gamma}}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$ are deformed Airy functions

$$\mathrm{Ai}^{\boldsymbol{\Gamma}}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{\pi}} \int_{\boldsymbol{\Gamma}_{i\frac{c}{b}}} \boldsymbol{dz e^{iza+i\frac{z^3}{3}}} \boldsymbol{\Gamma}\,(\boldsymbol{ibz+c}),$$

$$\mathrm{Ai}_{\boldsymbol{\Gamma}}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{\pi}} \int_{-\infty}^{\infty} \boldsymbol{dz e^{iza+i\frac{z^3}{3}}} \frac{\mathbf{1}}{\boldsymbol{\Gamma}\,(\boldsymbol{-ibz+c})}.$$

Height distribution

$$F_{\gamma_t}(s;X) = 1 - \int_{-\infty}^{\infty} du e^{-e^{\gamma_t(s-u)}} g_{\gamma_t}(u;X).$$

Here $g_{\gamma_t}(u;X)$ is expressed as a difference of two Fredholm determinants,

$$g_{\gamma_t}(u;X) = \det\left(1 - P_u(B^{\Gamma}_{\gamma_t} - P^{\Gamma}_{\mathsf{Ai}})P_u\right) - \det\left(1 - P_u B^{\Gamma}_{\gamma_t} P_u\right),$$

where $\boldsymbol{P}^{\boldsymbol{\Gamma}}_{\mathsf{Ai}}(\boldsymbol{\xi_1},\boldsymbol{\xi_2}) = \mathsf{Ai}^{\boldsymbol{\Gamma}}\left(\boldsymbol{\xi_1},\frac{\mathbf{1}}{\boldsymbol{\gamma_t}},-\frac{\boldsymbol{X}}{\boldsymbol{\gamma_t}}\right)\mathsf{Ai}_{\boldsymbol{\Gamma}}\left(\boldsymbol{\xi_2},\frac{\mathbf{1}}{\boldsymbol{\gamma_t}},-\frac{\boldsymbol{X}}{\boldsymbol{\gamma_t}}\right)$ and

$$\begin{aligned}
\boldsymbol{B}^{\boldsymbol{\Gamma}}_{\boldsymbol{\gamma_t}}(\boldsymbol{\xi_1},\boldsymbol{\xi_2}) &= \int_{\mathbf{0}}^{\infty} \boldsymbol{dy}\,\mathsf{Ai}^{\boldsymbol{\Gamma}}\left(\boldsymbol{\xi_1}+\boldsymbol{y},\frac{\mathbf{1}}{\boldsymbol{\gamma_t}},-\frac{\boldsymbol{X}}{\boldsymbol{\gamma_t}}\right)\mathsf{Ai}_{\boldsymbol{\Gamma}}\left(\boldsymbol{\xi_2}+\boldsymbol{y},\frac{\mathbf{1}}{\boldsymbol{\gamma_t}},-\frac{\boldsymbol{X}}{\boldsymbol{\gamma_t}}\right) \\
&+ \int_{\mathbf{0}}^{\infty} \boldsymbol{dy}\frac{\mathbf{1}}{\boldsymbol{e}^{\boldsymbol{\gamma_t y}}-\mathbf{1}}\left(\mathsf{Ai}^{\boldsymbol{\Gamma}}\left(\boldsymbol{\xi_1}+\boldsymbol{y},\frac{\mathbf{1}}{\boldsymbol{\gamma_t}},-\frac{\boldsymbol{X}}{\boldsymbol{\gamma_t}}\right)\mathsf{Ai}_{\boldsymbol{\Gamma}}\left(\boldsymbol{\xi_2}+\boldsymbol{y},\frac{\mathbf{1}}{\boldsymbol{\gamma_t}},-\frac{\boldsymbol{X}}{\boldsymbol{\gamma_t}}\right)\right. \\
&\left. -\mathsf{Ai}^{\boldsymbol{\Gamma}}\left(\boldsymbol{\xi_1}-\boldsymbol{y},\frac{\mathbf{1}}{\boldsymbol{\gamma_t}},-\frac{\boldsymbol{X}}{\boldsymbol{\gamma_t}}\right)\mathsf{Ai}_{\boldsymbol{\Gamma}}\left(\boldsymbol{\xi_2}-\boldsymbol{y},\frac{\mathbf{1}}{\boldsymbol{\gamma_t}},-\frac{\boldsymbol{X}}{\boldsymbol{\gamma_t}}\right)\right).
\end{aligned}$$

Long time limit

In general $\text{Prob}(X \leq s) = \mathbb{E}(\Theta(s - X))$, where $\Theta(y)$ is the step function, $\Theta(y) := 1 \ (y \geq 0), \ 0 \ (y < 0)$. Noticing $\lim_{a\to\infty} \exp[-e^{-ax}] = \Theta(x)$, we take the $t \to \infty$ limit and see

$$\lim_{\gamma_t\to\infty} \text{Prob}(\tilde{h}_t \leq s) = \lim_{\gamma_t\to\infty} G_{\gamma_t}(s; X) = \det\left(1 - P_s \mathcal{K}_X P_s\right)$$

the kernel is given by

$$\mathcal{K}_X(\xi_j, \xi_k) = \int_0^\infty dy \text{Ai}(\xi_j + y)\text{Ai}(\xi_k + y) \\ + \text{Ai}(\xi_k)\left(e^{-\frac{X^3}{3} + X\xi_j} - \int_0^\infty dy e^{-Xy}\text{Ai}(\xi_j + y)\right).$$

This appeared in GUE and TASEP with external source.

Multi-point distribution

$$G_{\gamma_t}(\{s\}_n, \{X\}_n) = \left\langle e^{-\sum_{j=1}^{n} e^{\gamma_t(\tilde{h}_t(X_j)-s_j)}} \right\rangle,$$

where we abbreviated $s_1, \cdots, s_n$ and $X_1, \cdots, X_n$ as $\{s\}_n$ and $\{X\}_n$ respectively and we set $X_1 < X_2 < \cdots < X_n$.

Using the "factorization approximation" by Prolhac-Spohn, we get

$$G^{\sharp}_{\gamma_t}(\{s\}_n, \{X\}_n) = \det(1 - Q),$$

and the kernel $Q(x, y)$ is given by

$$Q(u_1, u_{n+1}) = \int_{-\infty}^{\infty} du_2 \cdots du_n \langle u_1 | e^{(X_1 - X_2)H} | u_2 \rangle \cdots \times \langle u_n | e^{(X_n - X_1)H} L_1 | u_{n+1} \rangle \Phi(\{u - s\}_n),$$

H is the Airy Hamiltonian $H = -\frac{\partial^2}{\partial u^2} + u$, and

$$\Phi(\{x\}_n) = \frac{\sum_{j=1}^{n} e^{-\gamma_t x_j}}{1 + \sum_{j=1}^{n} e^{-\gamma_t x_j}},$$

$$L_j(x, y) = \int_0^\infty dw \mathrm{Ai}_\Gamma \left(w + x, \frac{1}{\gamma_t}, -\frac{X_j}{\gamma_t} \right) \mathrm{Ai}^\Gamma \left(w + y, \frac{1}{\gamma_t}, -\frac{X_j}{\gamma_t} \right),$$

3. Replica analysis: δ-Bose gas

Taking the Gaussian average over the noise η, one finds that the replica partition function can be written as

$$\langle Z^N(x,t)\rangle$$

$$= \prod_{j=1}^{N} \int_0^\infty dy_j \int_{x_j(0)=y_j}^{x_j(t)=x} D[x_j(\tau)] \exp\left[-\int_0^t d\tau \left(\sum_{j=1}^{N} \frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 \right.\right.$$

$$\left.\left. - \sum_{j\neq k=1}^{N} \delta\left(x_j(\tau) - x_k(\tau)\right)\right)\right] \times \left\langle \exp\left(\sum_{k=1}^{N} B(y_k)\right)\right\rangle$$

$$= \langle x| e^{-H_N t} |\Phi\rangle.$$

H_N is the Hamiltonian of the δ-Bose gas,

$$H_N = -\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} - \frac{1}{2}\sum_{j\neq k}^{N}\delta(x_j - x_k),$$

$|\Phi\rangle$ represents the state corresponding to the initial condition. We compute $\langle Z^N(x,t)\rangle$ by expanding in terms of the eigenstates of H_N,

$$\langle Z(x,t)^N\rangle = \sum_z \langle x|\Psi_z\rangle\langle\Psi_z|\Phi\rangle e^{-E_z t}$$

where E_z and $|\Psi_z\rangle$ are the eigenvalue and the eigenfunction of H_N: $H_N|\Psi_z\rangle = E_z|\Psi_z\rangle$.

The state $|\Phi\rangle$ can be computed as

$$\begin{aligned}
&\langle x_1, \cdots, x_N | \Phi \rangle \\
&= \frac{1}{N!} \sum_{P \in S_N} \left\langle \exp\left(\sum_{k=1}^{N} B(x_{P(k)}) \right) \right\rangle \\
&= \sum_{P \in S_N} \prod_{j=1}^{N} e^{\frac{1}{2}(2N-2j+1) x_{P(j)}} \Theta\left(x_{P(j)} - x_{P(j-1)} \right).
\end{aligned}$$

Bethe states

By the Bethe ansatz, the eigenfunction is given as

$$\langle x_1, \cdots, x_N | \Psi_z \rangle = C_z \sum_{P \in S_N} \mathrm{sgn} P$$

$$\times \prod_{1 \le j < k \le N} \left(z_{P(j)} - z_{P(k)} + i \mathrm{sgn}(x_j - x_k) \right) \exp \left(i \sum_{l=1}^{N} z_{P(l)} x_l \right)$$

N momenta z_j $(1 \le j \le N)$ are parametrized as

$$z_j = q_\alpha - \frac{i}{2} (n_\alpha + 1 - 2 r_\alpha), \quad \text{for } j = \sum_{\beta=1}^{\alpha-1} n_\beta + r_\alpha.$$

$(1 \le \alpha \le M$ and $1 \le r_\alpha \le n_\alpha)$. They are divided into M groups where $1 \le M \le N$ and the αth group consists of n_α quasimomenta $z_j' s$ which shares the common real part q_α.

$$C_z = \left(\frac{\prod_{\alpha=1}^{M} n_\alpha}{N!} \prod_{1 \le j < k \le N} \frac{1}{|z_j - z_k - i|^2} \right)^{1/2}$$

$$E_z = \frac{1}{2} \sum_{j=1}^{N} z_j^2 = \frac{1}{2} \sum_{\alpha=1}^{M} n_\alpha q_\alpha^2 - \frac{1}{24} \sum_{\alpha=1}^{M} \left(n_\alpha^3 - n_\alpha \right).$$

Expanding the moment in terms of the Bethe states, we have

$$\begin{aligned} &\langle Z^N(x,t) \rangle \\ &= \sum_{M=1}^{N} \frac{N!}{M!} \prod_{j=1}^{N} \int_{-\infty}^{\infty} dy_j \left(\int_{-\infty}^{\infty} \prod_{\alpha=1}^{M} \frac{dq_\alpha}{2\pi} \sum_{n_\alpha=1}^{\infty} \right) \delta_{\sum_{\beta=1}^{M} n_\beta, N} \\ &\quad \times e^{-E_z t} \langle x | \Psi_z \rangle \langle \Psi_z | y_1, \cdots, y_N \rangle \langle y_1, \cdots, y_N | \Phi \rangle. \end{aligned}$$

There is a question of completeness of Bethe states.

After deforming the contour, we can perform the integrations of y_j $(1 \le j \le N)$ before those of q_α $(1 \le \alpha \le M)$. We see

$$\langle \Psi_z | \Phi \rangle = C_z \sum_{P \in S_N} \mathrm{sgn} P \prod_{l=1}^{N} \int_{y_{l-1}}^{\infty} dy_l e^{-i\left(z^*_{P(l)} - \frac{1}{2}(2N-2l+1)\right) y_l}$$

$$\times \prod_{1 \le j < k \le N} \left(z^*_{P(j)} - z^*_{P(k)} + i \right)$$

$$= N! C_z \sum_{P \in S_N} \mathrm{sgn} P \prod_{1 \le j < k \le N} \left(z^*_{P(j)} - z^*_{P(k)} + i \right)$$

$$\times \prod_{l=1}^{N} \frac{1}{-i(z^*_{P(N)} + \cdots + z^*_{P(N-l+1)}) + l^2/2}.$$

Combinatorial identities

(1)

$$\sum_{P\in S_N} \mathrm{sgn} P \prod_{1\le j<k\le N} \big(w_{P(j)} - w_{P(k)} + i f(j,k)\big)$$

$$= N! \prod_{1\le j<k\le N} (w_j - w_k)$$

(2)For any complex numbers w_j $(1 \le j \le N)$ and a,

$$\sum_{P \in S_N} \mathrm{sgn} P \prod_{1 \le j < k \le N} \left(w_{P(j)} - w_{P(k)} + a \right)$$

$$\times \prod_{m=1}^{N} \frac{1}{w_{P(N)} + \cdots + w_{P(N-m+1)} + m^2 a/2}$$

$$= \prod_{1 \le j < k \le N} (w_j - w_k) \prod_{m=1}^{N} \frac{1}{w_m + a/2}.$$

[Similar identity for step Bernoulli ASEP by Tracy-Widom]

After some computations, we see that $G_{\gamma_t}(s;X)$ is expressed as the Fredholm determinant

$$G_{\gamma_t}(s;X) = \det\left(1 - P_0 K_X P_0\right)$$

Here P_s represents the projection onto (s,∞) and the kernel of K_X is given by

$$K_X(\omega_j,\omega_k) = \sum_{n=1}^{\infty}(-1)^{n-1}\int_{\mathbb{R}-ic_n}\frac{dq}{\pi}e^{-n(\omega_j+\omega_k)-2iq(\omega_j-\omega_k)} \times e^{-\gamma_t^3 nq^2+\frac{\gamma_t^3}{12}n^3-\gamma_t ns}\frac{\Gamma\left(iq-\frac{X}{\gamma_t}-\frac{n}{2}\right)}{\Gamma\left(iq-\frac{X}{\gamma_t}+\frac{n}{2}\right)},$$

where $\Gamma(x)$ is the gamma function and c_n satisfies $c_n > X/\gamma_t + n/2$.

Lemma

(a) We set $a \in \mathbb{R}$ and $m,\ n \geq 0$. When $\mathbf{Im}\, q < -n/2 + a$, we have

$$\frac{\Gamma\left(iq + a - \frac{n}{2}\right)}{\Gamma\left(iq + a + \frac{n}{2}\right)} e^{\frac{m^3 n^3}{3}} = \int_{-\infty}^{\infty} dy \mathrm{Ai}_{\Gamma}^{\Gamma}\left(y, \frac{1}{2m}, iq + a\right) e^{mny},$$

where

$$\mathrm{Ai}_{\Gamma}^{\Gamma}(a, b, c) = \frac{1}{2\pi} \int_{\Gamma_{i\frac{c}{b}}} dz e^{iaz + iz^3/3} \frac{\Gamma(ibz + c)}{\Gamma(-ibz + c)}$$

(b) For $u, v, x \in \mathbb{R}$ and $w \geq 0$, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dp \mathrm{Ai}_{\Gamma}^{\Gamma}\left(p^2+v, w, iwp+u\right) e^{ipx}$$

$$= \frac{1}{2^{\frac{1}{3}}}\mathrm{Ai}^{\Gamma}\left(2^{-\frac{2}{3}}(v+x), 2^{\frac{1}{3}}w, u\right)\mathrm{Ai}_{\Gamma}\left(2^{-\frac{2}{3}}(v-x), 2^{\frac{1}{3}}w, u\right).$$

Using this we get our results for the generating function.

Deformed Airy functions

Let $H = -\partial^2/\partial x^2 + x$. We have the following relations.

(i) The deformed Airy function representation of the propagator

$$\langle x|e^{tH}|y\rangle = \int_{-\infty}^{\infty} dz e^{-tz} \mathrm{Ai}_{\Gamma}\left(x+z, b, c-bt\right) \mathrm{Ai}^{\Gamma}\left(y+z, b, c\right).$$

Biorthogonality relation

$$\int_{-\infty}^{\infty} dw \mathrm{Ai}_{\Gamma}(x+w, b, c) \mathrm{Ai}^{\Gamma}(y+w, b, c) = \delta(x-y).$$

(ii) "Time evolution" by the Airy Hamiltonian

$$e^{tH}\mathrm{Ai}^{\Gamma}(x+w,b,c) = e^{-tw}\mathrm{Ai}^{\Gamma}(x+w,b,c-bt),$$
$$e^{tH}\mathrm{Ai}_{\Gamma}(x+w,b,c) = e^{-tw}\mathrm{Ai}_{\Gamma}(x+w,b,c+bt).$$

Summary

- Experiments for both circular and flat initial conditions.
- Half-Brownian motion initial condition by replica
- Similar structure to the narrow wedge case with Airy function replaced by the deformed Airy functions
- Multi-point distriubtions based on the factorization approximation
- An extension to the stationary case is now under way.

Stationary Case

Generalized initial condition

$$\boldsymbol{h(x,0)} = \begin{cases} \boldsymbol{B_{-,v_-}(-x) = \tilde{B}(-x) + v_- x,} & \boldsymbol{x < 0,} \\ \boldsymbol{B_{+,v_+}(x) = B(x) - v_+ x,} & \boldsymbol{x > 0,} \end{cases}$$

Combinatorial identity

Overall initial height is inverse gamma distributed.