# The exactly solvable log-gamma polymer

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Collaborators: Nicos Georgiou (Utah), Ivan Corwin (Microsoft/MIT), Neil O'Connell (Warwick), Nikos Zygouras (Warwick)

**Context.** Among directed polymer models there are currently two $1+1$ dimensional exactly solvable cases:

- polymer in a Brownian environment with continuous-time random walk paths, discovered by O'Connell-Yor (2001), subsequently worked on by O'Connell, O'C–Moriarty, and Valkó–S.

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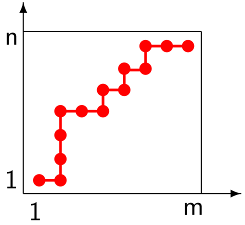
This talk is about the log-gamma polymer.

# 1+1 dimensional lattice polymer with fixed endpoints



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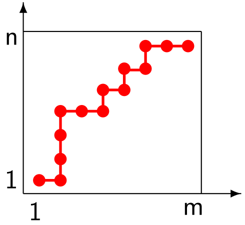


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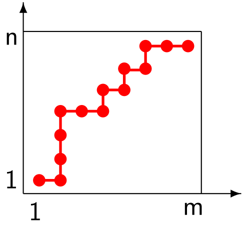


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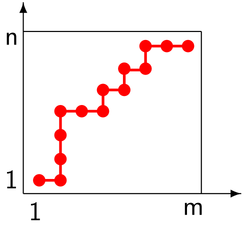


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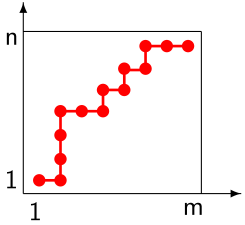


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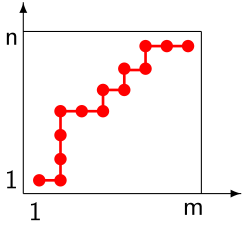


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**Key quantities again:**

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In this talk focus is on $\log Z$.

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**What is special about this choice?**

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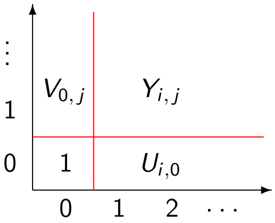

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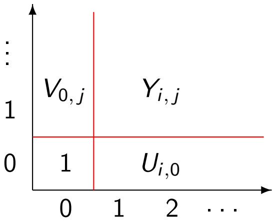


$$Y_{i,j}^{-1} \sim \text{Gamma}(\mu)$$

$$U_{i,0}^{-1} \sim \text{Gamma}(\theta)$$

$$V_{0,j}^{-1} \sim \text{Gamma}(\mu - \theta)$$

In 2-parameter model, compute $Z_{m,n}$ $\forall\,(m,n) \in \mathbb{Z}_+^2$ and define

$$U_{m,n} = \frac{Z_{m,n}}{Z_{m-1,n}} \qquad V_{m,n} = \frac{Z_{m,n}}{Z_{m,n-1}} \qquad X_{m,n} = \Big(\frac{Z_{m,n}}{Z_{m+1,n}} + \frac{Z_{m,n}}{Z_{m,n+1}}\Big)^{-1}$$

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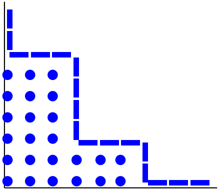

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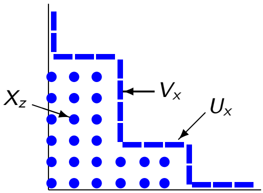


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# Burke property

**Theorem**

Variables $\{T_{f_k}, X_z : k \in \mathbb{Z},\, z \in \mathcal{I}\}$ are independent with marginals

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"Burke property" because the analogous property for last-passage percolation with exponential weights is a generalization of Burke's Theorem for M/M/1 queues.

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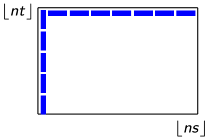


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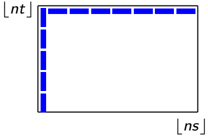


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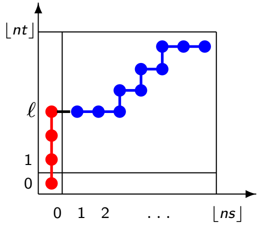

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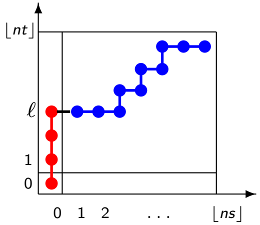


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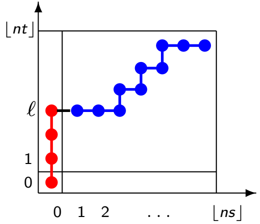


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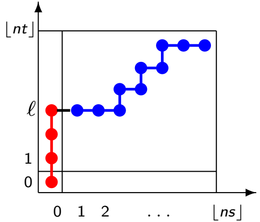


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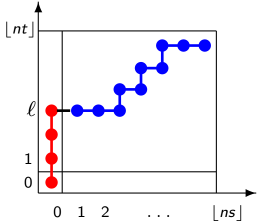


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$$p(s,t) = \inf_{0<\theta<\mu} \{-s\Psi_0(\theta) - t\Psi_0(\mu-\theta)\}$$

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- Upper bounds hold for i.i.d. model without boundaries.

# Explicit large deviations for $\log Z$

L.m.g.f. of $\log Y$, $Y \sim \Gamma^{-1}(\mu)$:

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**Theorem.**

$$\Lambda_{s,t}(\xi) = \begin{cases} p(s,t)\xi & \xi < 0 \\ \inf\limits_{\theta \in (\xi,\mu)} \big\{ tM_\theta(\xi) - sM_{\mu-\theta}(-\xi) \big\} & 0 \le \xi < \mu \\ \infty & \xi \ge \mu. \end{cases}$$

- $\Lambda_{s,t}$ linear on $\mathbb{R}_-$ because for $r < p(s,t)$

$$\lim_{n\to\infty} n^{-1} \log \mathbb{P}\{\log Z_{ns,nt} \le nr\} = -\infty.$$

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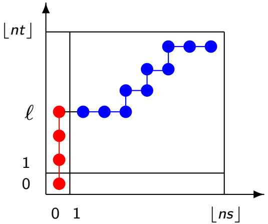
- Proof of formula for $\Lambda_{s,t}$ goes by first finding $J_{s,t}$ and then convex conjugation.

# Starting point for proof of large deviations




$$Z^{\theta}_{ns,nt} = \sum_{\ell=1}^{nt} \left( \prod_{j=1}^{\ell} V_{0,j} \right) Z_{(1,\ell),(ns,nt)} + \sum_{k=1}^{ns} \left( \prod_{i=1}^{k} U_{i,0} \right) Z_{(k,1),(ns,nt)}$$

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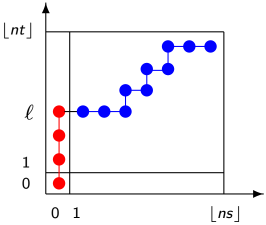


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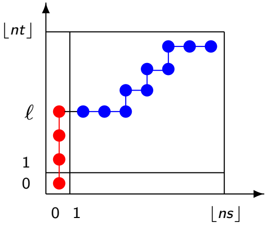


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Now we know LDP for log(l.h.s) and can extract $\log Z$ from the r.h.s.

# Combinatorial approach to the log-gamma polymer

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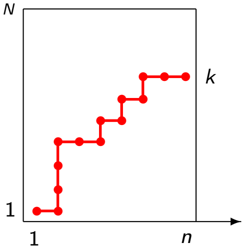

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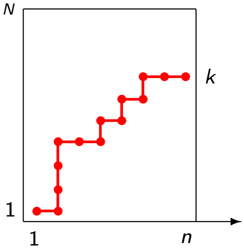


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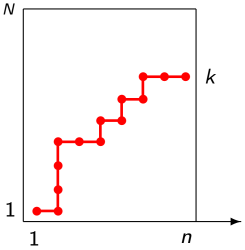


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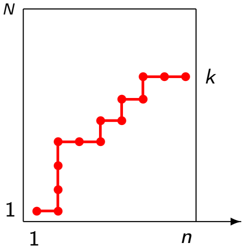


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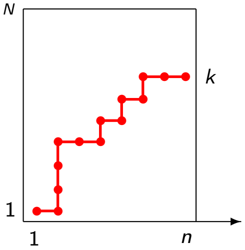


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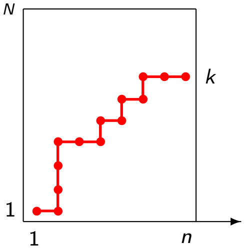


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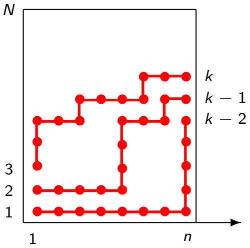


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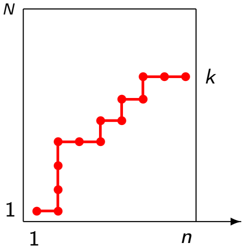


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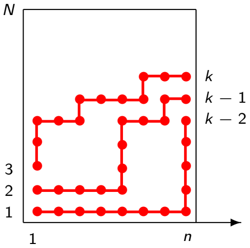


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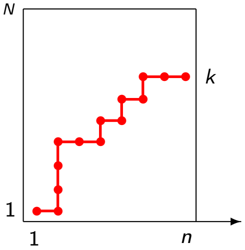


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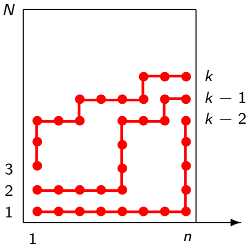


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$N = 4$ array

$$\begin{array}{ccccccc} & & & z_{11}(n) & & & \\ & & z_{22}(n) & & z_{21}(n) & & \textit{polymer} \\ & z_{33}(n) & & z_{32}(n) & & z_{31}(n) & \\ z_{44}(n) & & z_{43}(n) & & z_{42}(n) & & z_{41}(n) \end{array}$$

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|---|---|---|---|---|---|---|
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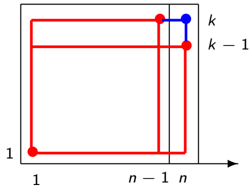

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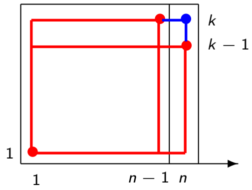


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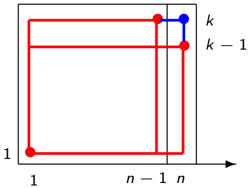


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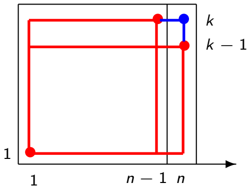


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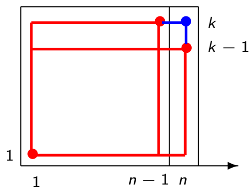


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After this, transformed weights are passed on to diagonal $z_2 = (z_{22} \ldots, z_{N2})$ and that diagonal is updated. And so on.

Let $1 \le \ell \le N$. **Geometric row insertion**

of the word $b = (b_\ell, \dots, b_N)$ into the word $\xi = (\xi_\ell, \dots, \xi_N)$

produces two new words

$$\xi' = (\xi'_\ell, \dots, \xi'_N) \quad \text{and } b' = (b'_{\ell+1}, \dots, b'_N).$$

Notation and definition:

$$\xi \xrightarrow[b']{b} \xi' \quad \text{where} \quad \begin{cases} \xi'_\ell = b_\ell \xi_\ell \\ \xi'_k = b_k(\xi'_{k-1} + \xi_k) & \ell+1 \le k \le N \\ b'_k = b_k \dfrac{\xi_k \xi'_{k-1}}{\xi_{k-1}\xi'_k} & \ell+1 \le k \le N. \end{cases}$$

Words have strictly positive real entries.

Let $z$ be an array with diagonals $z_1, \dots, z_N$, and $b \in (0, \infty)^N$ a word.

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Let $a_1 = b$, and then

$$\begin{array}{ccc} & a_1 & \\ z_1 & \longrightarrow\!\!\!\!\!\!\!\downarrow\;\; & z_1' \\ & a_2 & \\ z_2 & \longrightarrow\!\!\!\!\!\!\!\downarrow\;\; & z_2' \\ & a_3 & \\ & \vdots & \\ & a_N & \\ z_N & \longrightarrow\!\!\!\!\!\!\!\downarrow\;\; & z_N' \end{array}$$

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Diagonals $z_1', \dots, z_N'$ make up the new array $z'$.

$$
\begin{array}{ccccccccc}
 & a_1(1) & & a_1(2) & & a_1(3) & & \\
z_1(0) & \overset{}{\longrightarrow}\!\!\!\!\!\!\!\downarrow & z_1(1) & \longrightarrow\!\!\!\!\!\!\!\downarrow & z_1(2) & \longrightarrow\!\!\!\!\!\!\!\downarrow & z_1(3) & \cdots \\
 & a_2(1) & & a_2(2) & & a_2(3) & & \\
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\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
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\end{array}
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Evolution of the array $z(n)$ over time $n = 0, 1, 2, \ldots$

$$
\begin{array}{ccccccccc}
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Initial state $z(0)$ is on the left edge in terms of diagonals $z_1(0), \ldots, z_N(0)$.

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Time $n$ input from weight matrix: $a_1(n) = Y^{[n]} = (Y_{n,1}, \ldots, Y_{n,N})$.

At each time, geometric row insertion is iterated $N$ times to update each diagonal.

The previous process is for evolving the full array. We also need the variant for starting with an empty array $z(0) = \emptyset$.

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 & & e_1^{(N-1)} & \longrightarrow\!\!\!\!\!\!\downarrow & z_2(2) & \longrightarrow\!\!\!\!\!\!\downarrow & z_2(3) & \cdots \\
 & & & & & a_3(3) & & \\
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\end{array}
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Resulting array: $z(n) = \emptyset \leftarrow Y^{[1]} \leftarrow Y^{[2]} \leftarrow \cdots \leftarrow Y^{[n]}$.

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Theory of Markov functions shows this.

# Markov functions idea

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∃ Markov kernel Π for $z(n)$ on space $T$.

$$\begin{array}{c} T \\ \Pi \big\downarrow \\ T \end{array}$$

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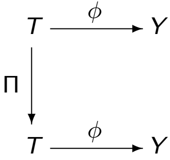

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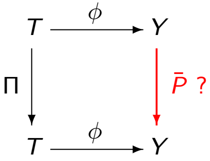

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$$\begin{array}{ccc} T & \xrightarrow{\ \phi\ } & Y \\ \Pi \downarrow & & \downarrow \bar{P}\ ? \\ T & \xrightarrow{\ \phi\ } & Y \end{array}$$

**Sufficient condition.** Suppose $\exists$ (positive but not necessary stochastic) kernels $P : Y \to Y$ and $K : Y \to T$ such that

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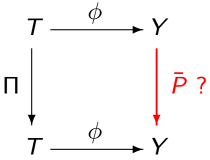


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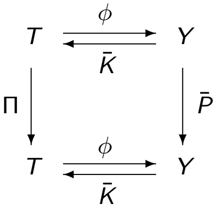
Set $w(y) = K(y, T)$. Intertwining gives $Pw = w$. Define stochastic kernels

$$\bar{K}(y, dz) = \frac{1}{w(y)} K(y, dz) \quad \text{and} \quad \bar{P}(y, d\tilde{y}) = \frac{w(\tilde{y})}{w(y)} P(y, d\tilde{y})$$

# Markov functions idea, continued

Then $\bar{K} \circ \Pi = \bar{P} \circ \bar{K}$



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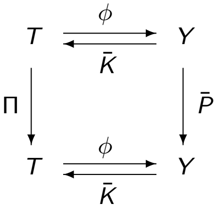


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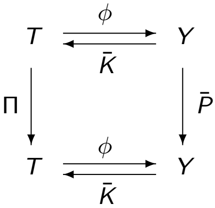


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Furthermore

$$E\left[\, f(z(n)) \,\middle|\, y(0), \ldots, y(n-1),\, y(n) = y \,\right] \;=\; \bar{K} f(y)$$

(Rogers and Pitman, 1981)

# Application of Markov functions

Spaces: $\mathbb{T}_N =$ space of arrays of size $N$,

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$$P_n(y, d\tilde{y}) = \prod_{i=1}^{N-1} \exp\left\{-\frac{\tilde{y}_{i+1}}{y_i}\right\} \prod_{j=1}^{N} \left( \Gamma(\gamma_{n,j})^{-1} \left(\frac{y_j}{\tilde{y}_j}\right)^{\gamma_{n,j}} \exp\left\{-\frac{y_j}{\tilde{y}_j}\right\} \frac{d\tilde{y}_j}{\tilde{y}_j} \right)$$

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and intertwining kernel $K : \mathbb{Y}_N \to \mathbb{T}_N$ by

$$K(y, dz) = \prod_{1\le\ell\le k<N} \left(\frac{z_{k,\ell}}{z_{k+1,\ell}}\right)^{\theta_{k+1}-\theta_\ell}$$
$$\times \exp\left(-\frac{z_{k,\ell}}{z_{k+1,\ell}} - \frac{z_{k+1,\ell+1}}{z_{k,\ell}}\right) \frac{dz_{k,\ell}}{z_{k,\ell}} \prod_{\ell=1}^{N} \delta_{y_\ell}(dz_{N,\ell})$$

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Now a closer look at the eigenfunctions we have found. All the previous makes sense also for complex parameters. This is beneficial because then we can use known special functions to diagonalize the transition kernel.

# Whittaker functions

$GL(N,\mathbb{R})$-Whittaker function for $y \in \mathbb{Y}_N$, with $\lambda \in \mathbb{C}^N$

$$\Psi_\lambda(y) = \prod_{i=1}^{N} y_i^{-\lambda_i} \int_{\mathbb{T}_N} K_\lambda(y, dz)$$

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Intertwining works also with complex parameters and gives

$$\int_{(0,\infty)^N} \frac{\Psi_{\theta+\lambda}(\tilde{y})}{\Psi_\theta(\tilde{y})} \bar{P}_n(y, d\tilde{y}) \;=\; \left( \prod_{i=1}^{N} \frac{\Gamma(\gamma_{n,i} + \lambda_i)}{\Gamma(\gamma_{n,i})} \right) \frac{\Psi_{\theta+\lambda}(y)}{\Psi_\theta(y)}$$

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**Step 1.** Row insertion reveals that if we start with bottom row

$$y^M = (e^{M(i-(k+1)/2)})_{1 \le i \le N}\,,$$

let initial array have distribution $\bar{K}(y^N, dz)$, and let $M \to \infty$, the distribution of the array $z(n)$ converges to the array started from the empty array $z(0) = \emptyset$.

So this limit recovers the distribution of the partition function from the path construction.

**Step 2.** Invert the eigenfunction relation to write an expression for the distribution of bottom row $y(n)$ when started from $y \in (0,\infty)^N$. Involves analytical properties of Whittaker functions (analogous to Fourier analysis). Take $y = y^M$ and let $M \to \infty$.

The result is a formula for the Laplace transform of the partition function:

$$\mathbb{E}(e^{-s\, z_{N,1}(n)}) = \int_{\iota\mathbb{R}^N} s^{\sum_{i=1}^N(\theta_i - \lambda_i)} \prod_{1\le i,j\le N} \Gamma(\lambda_i - \theta_j) \times \prod_{m=1}^{n}\prod_{i=1}^{N} \frac{\Gamma(\lambda_i + \hat\theta_m)}{\Gamma(\theta_i + \hat\theta_m)}\, s_N(\lambda)\, d\lambda$$

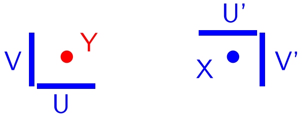
where the Sklyanin measure is given by

$$s_N(\lambda) = \frac{1}{(2\pi\iota)^N N!} \prod_{j\ne k} \Gamma(\lambda_j - \lambda_k)^{-1}$$

**A future goal: asymptotics for distribution of $\log z_{N,1}(n)$.**

# Proof of Burke property

Induction on $\mathcal{I}$ by flipping a growth corner:

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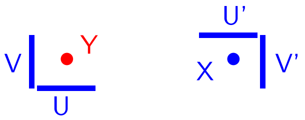


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**Lemma.** Given that $(U, V, Y)$ are independent positive r.v.'s, $(U', V', X) \stackrel{d}{=} (U, V, Y)$ iff $(U, V, Y)$ have the gamma distr's.

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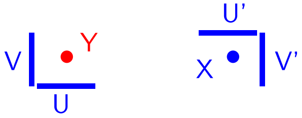


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This gives all $(z_k)$ with finite $\mathcal{I}$. General case follows.

## Combinatorial RSK (Robinson-Schensted-Knuth correspondence)

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$$x_{k,1} + \cdots + x_{k,\ell} \;=\; \max_{\pi_1,\dots,\pi_\ell \text{ disjoint}} \sum_{(i,j)\,\in\,\pi_1 \cup \cdots \cup \pi_\ell} Y_{i,j}$$

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Compare this with tropical formula

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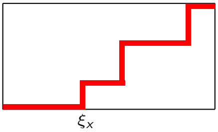
Difference is $(+\,,\,\cdot\,)$ vs. $(\max\,,+)$.

back

# Consequences of Burke property: variance identity




Exit point of path from $x$-axis

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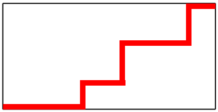


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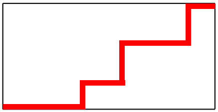
$\xi_x = \max\{k \geq 0 : x_i = (i, 0) \text{ for } 0 \leq i \leq k\}$

For $\theta, x > 0$ define positive function

$$L(\theta, x) = \int_0^x \big(\Psi_0(\theta) - \log y\big) x^{-\theta} y^{\theta-1} e^{x-y}\, dy$$

# Consequences of Burke property: variance identity




Exit point of path from $x$-axis

$\xi_x = \max\{k \geq 0 : x_i = (i, 0) \text{ for } 0 \leq i \leq k\}$

For $\theta, x > 0$ define positive function

$$L(\theta, x) = \int_0^x \big(\Psi_0(\theta) - \log y\big) x^{-\theta} y^{\theta-1} e^{x-y}\, dy$$

**Theorem.** For the stationary case,

$$\mathbb{V}\mathrm{ar}\big[\log Z_{m,n}\big] = n\Psi_1(\mu - \theta) - m\Psi_1(\theta) + 2\, E_{m,n}\Big[\sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1})\Big]$$

# Variance identity leads to fluctuation bounds for log $Z$

With $0 < \theta < \mu$ fixed and $N \nearrow \infty$ assume

$$|\, m - N\Psi_1(\mu - \theta)\,| \le CN^{2/3} \quad \text{and} \quad |\, n - N\Psi_1(\theta)\,| \le CN^{2/3} \tag{1}$$

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**Theorem: Variance bounds along characteristic**

For $(m, n)$ as in (1), $\quad C_1 N^{2/3} \le \mathbb{V}\mathrm{ar}(\log Z_{m,n}) \le C_2 N^{2/3}$ .

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**Theorem: Off-characteristic CLT**

Suppose $n = \Psi_1(\theta)N$ and $m = \Psi_1(\mu - \theta)N + \gamma N^{\alpha}$ with $\gamma > 0$, $\alpha > 2/3$. Then

$$N^{-\alpha/2}\Big\{\log Z_{m,n} - \mathbb{E}\big(\log Z_{m,n}\big)\Big\} \Rightarrow \mathcal{N}\big(0, \gamma\Psi_1(\theta)\big)$$