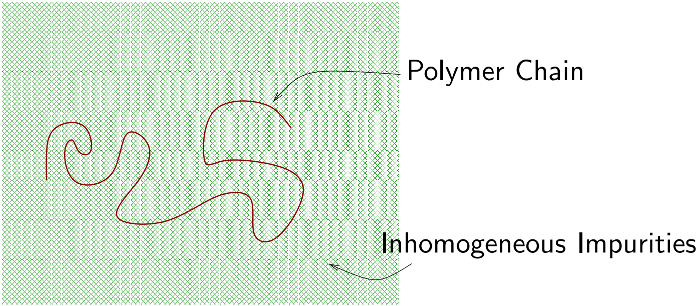
Enhanced superdiffusivity for directed polymer in correlated random environment

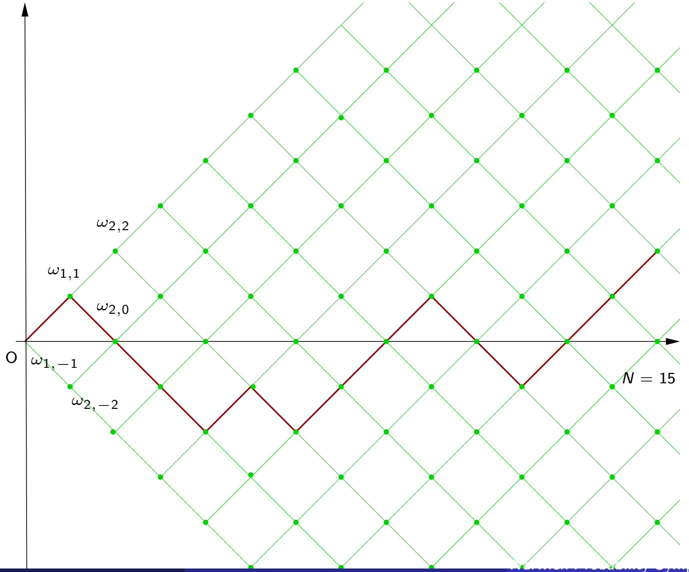
Hubert Lacoin

CNRS and Université Paris Dauphine (CEREMADE)

Warwick Probability Sympsium on Disordered Media, 5th-9th September 2011


Polymer Chain
Inhomogeneous Impurities

Definition of the most common DPRE (discrete)



Definition of the most common DPRE

- Let $(\omega_{n,z})_{n\in\mathbb{N},z\in\mathbb{Z}^d}$, be an i.i.d. Gaussian field (law of $\omega_{n,z} = \mathcal{N}(0,1)$) (law $\mathbb{P}$).
- Let Γ_n of N step NN paths in $\mathbb{Z}^d$.

We define the energy of a path $S \in \Gamma_N$ to be

$$H_N^{(\omega)}(S) := \sum_{n=1}^{N} \omega_{n,S_n}. \tag{1}$$

For $N \in \mathbb{N}$ and $\beta > 0$ fixed, the polymer measure $\mu_N^{(\beta,\omega)}$ is defined by

$$\mu_N(S) := \frac{1}{Z_N^{(\beta,\omega)}} \exp\left(\beta H_N^{(\omega)}(S)\right). \tag{2}$$

where

$$Z_N^{(\beta,\omega)} := \sum_{S\in\Gamma_N} \exp\left(\sum_{n=1}^{N} \beta\omega_{n,S_n}\right). \tag{3}$$

Main concern

We want to understand the typical behavior of $(S_n)_{n\in[1,N]}$ under μ_N when N is large.

- Comparison to SRW ($\beta = 0$): Do we have invariance principle, is the trajectory Brownian-like?
- Does the trajectory localizes around some corridors with a very favorable environment (high values for ω)?
- Do the trajectories go further that $\sqrt{N}$ to reach favorable environment (superdiffusivity)?

Universality class

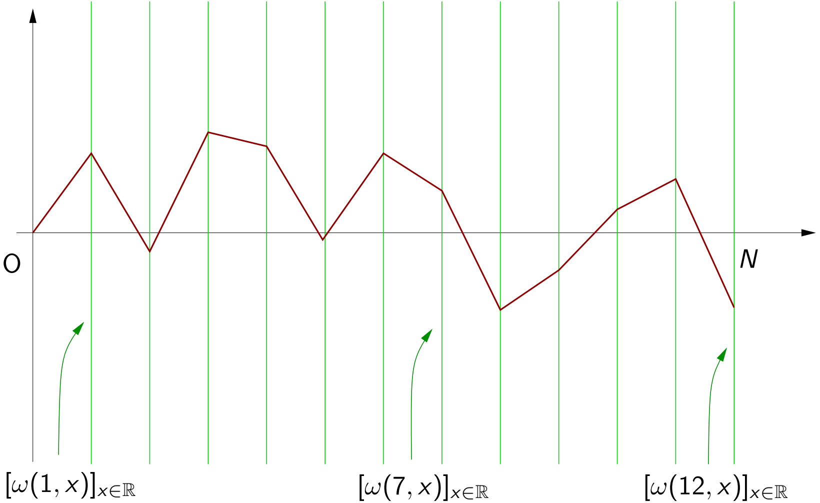
We investigate those question not exactly for the above mentioned model but also for models that are believed to have the same properties when looked at large scales.
We are interested in giving the effect of long range correlation in the environment on the trajectories properties.

1. Introduction

2. GRW Directed Polymer
   - Model with compactly supported correlation (Petermann, Méjane)
   - Model with long-range correlation

3. Brownian Motion in Poissonian environment
   - Model with no correlation (Wühtrich)
   - Model with correlation in the environment

Gaussian Random Walk Directed Polymer Model (Petermann '00, Méjane '04)



Gaussian Random Walk Polymer Model (Petermann '00, Méjane '04)

- S be a random walk on $\mathbb{R}^d$ with i.i.d. centered standard Gaussian increments (associated proba $\mathbf{P}$).
- Let $(\omega_{n,x})_{n\in\mathbb{N},x\in\mathbb{R}^d}$ be a Gaussian field (law $\mathbb{P}$) with covariance function $\mathbb{E}\left[\omega_{n,x}\omega_{n,y}\right] := \delta_{n,n'} Q(x-y)$. (a sequence of independent translation invariant Gaussian fields with covariance function Q) where $Q(0)=1$, $Q(x)>0$, $Q(x)\to 0$ as $\|x\|\to\infty$.

For $N\in\mathbb{N}$ and $\beta>0$ fixed, we define the polymer measure μ_N on paths S with its Radon-Nikodyn derivative

$$\frac{\mathrm{d}\mu_N^\beta}{\mathrm{d}\mathbf{P}}(S) := \frac{1}{Z_N}\exp\left(\beta\sum_{n=1}^N \omega_{n,S_n}\right). \tag{4}$$

where

$$Z_N := \mathbf{E}\left[\exp\left(\beta\sum_{n=1}^N \omega_{n,S_n}\right)\right]. \tag{5}$$

Theorem (Imbrie, Spencer '88, Bolthausen; '88, Albeverio, Zhou 96'; Song, Zhou 96'; Comets, Yoshida '04)

When $d \geq 3$, Q has compact support and $\beta \leq \beta_c$ (for which one can find explicit bounds), the law of $\left(\frac{S_{\lceil tN \rceil}}{\sqrt{N}}\right)_{t\in[0,1]}$ *under* μ_N, *converge to the law of a Brownian motion* $(B_t)_{t\in[0,1]}$.

Strong Disorder

When either β is large or $d = 1, 2$ this does not holds and one has localization of the trajectories, we say that **Strong Disorder** holds [Carmona, Hu '02, Comets, Shiga, Yoshida '03, Comets Vargas '06, L'10].

Superdiffusivity

It is predicted by physicists that under μ_N when **strong disorder holds** there exists $\xi > 1/2$ such that:

$$\max_{n\in[0,N]} S_n \approx N^{\xi}, \tag{6}$$

(superdiffusivity phenomenon).
Moreover it is believed that this exponent is related to the variation of $\log Z_N$ i.e.

$$\log Z_N - \mathbb{E} \log Z_N \approx N^{\chi} \tag{7}$$

with the **scaling relation** $\chi = 2\xi - 1$.

- For $d = 1$ it is predicted that $\xi = 2/3$, $\chi = 1/3$.
- For $d \geq 2$, there is no predicted value for χ and ξ. It is believed to be independant of β in the strong disorder phase.

(Lack of) Results for polymer in $\mathbb{N} \times \mathbb{Z}^d$

To prove the existence of ξ and χ for this model seems out of reach with actual tools and very few results are proved rigorously for the discrete polymer model. Some more convincing results have been obtained for models that should belong to the same *universality class*

- Licea and Newman ('95) Licea, Newman and Piza ('96) proved that for last-passage percolation in dimension 2, $3/5 \leq \xi \leq 3/4$.
- Johansson ('00) proved that for directed last-passage percolation with exponential variable $\xi = 2/3$ for transversal dimension $d = 1$.
- Seppalainen ('09) obtained $\xi = 2/3$ for a special model of directed polymer.
- Balasz, Quastel and Seppalainen ('09) computed the scaling exponent for the solution of KPZ equation finding $\xi = 2/3$, $\chi = 1/3$ using correspondence with WASEP.

For the GRW model, one can use an entropy/energy competition reasoning to get some results.

Superdiffusivity results for rapidly decaying Q and $d = 1$

Theorem (Petermann '00)

When $d = 1$ *and* Q *compactly supported, for any* $\beta > 0$ *and* $\alpha < 3/5$,

$$\lim_{N\to\infty} \mathbb{P}\mu_N \left\{ \max_{n\in[0,N]} S_n \geq N^\alpha \right\} = 1. \tag{8}$$

Informally this says that $\xi \geq 3/5$ for $d = 1$ is compatible with the conjecture.

Upper bound on the volume exponent

Theorem (Méjane 2004)

For any $\alpha > 3/4$, *in any dimension, one has*

$$\lim_{N\to\infty} \mathbb{P}\mu_N \left\{ \max_{n\in[1,N]} S_n \geq N^{\alpha} \right\} = 0. \tag{9}$$

Informally this says, $\xi \leq 3/4$ in all dimension which is compatible with the conjecture for $d = 1$.

Universality and Non-Universality

When correlations in the environment decay sufficiently fast it is natural to conjecture that the general behavior of the model is unchanged.

We place ourselves in the case when $Q(x) \asymp \|x\|^{-\theta}$ ($c_1\|x\|^{-\theta} \leq Q(x) \leq c_2\|x\|^{-\theta}$ for some $\theta > 0$). Note that there is no correlation along the dimension along which the polymer is directed.
We wonder

- If the conditions for diffusivity/localization found in the discrete model are changed?
- We also investigate the effect of these power-law decaying correlation on superdiffusivity properties.

Condition for diffusivity and localization

Theorem (L10)

For $d \geq 3$ and $\theta > 2$ then diffusivity holds for $\beta < \beta_c$ (with scaling to BM).
On the other hand if $d \geq 3$ $\beta \geq \bar{\beta}_c$, **strong disorder** *holds.*
If either $d \leq 2$ or $\theta \leq 2$ then **strong disorder** *holds for all β.*

When $\theta < 2$, $d \geq 3$, there is no-phase transition and this absence is due to the presence of correlation.

Superdiffusivity results $Q(x) \asymp \|x\|^{-\theta}$ and d arbitrary

Theorem (L10)

When $d = 1, \theta < 1$, *or when* $d \geq 2$, $\theta < 2$,

$$\lim_{\varepsilon \to 0} \lim_{t \to \infty} \mathbb{P}\mu_t \left\{ \max_{n \in [0,N]} S_n \geq \varepsilon N^{\frac{3}{4+\theta}} \right\} = 1. \quad (10)$$

When $d = 1$, $\theta > 1$ *the same result holds with an exponent* $3/5$.

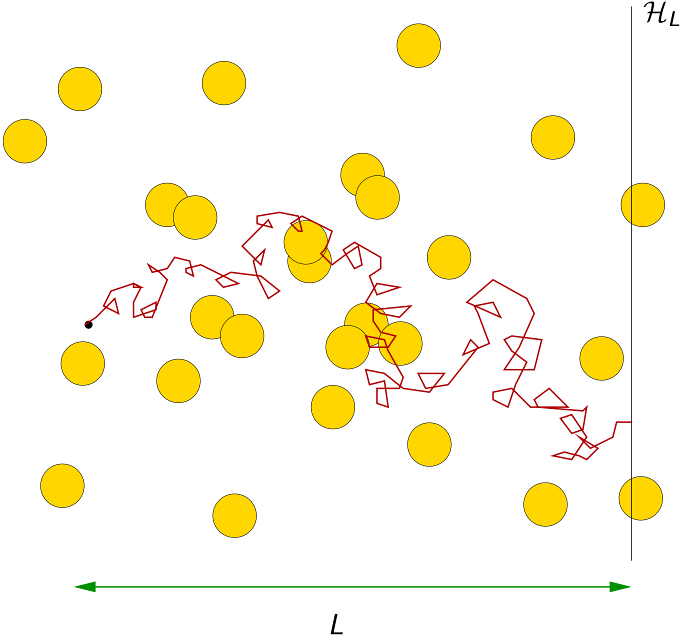
This informally translates into $\xi \geq 3/(4+\theta)$.

Some conclusion

- The property of the systems seem to be crucially changed when $\theta < 1$ for $d = 1$ or when $\theta < 2$ for $d \geq 2$.
- In the other cases, the model seems to be in the same universality class that the discrete model.
- In order to get $\xi \leq 3/4$ one has to use explicitely decorelation in the environment.

Limits

The previous results only give the impact of transversal correlation in the environment but it seems some how more natural to study environment that are isotropically correlated. We do it for another model which is Not Directed, and for which the environment is Not Gaussian.


$\mathcal{H}_L$
L

The model

- Let $\omega := \{\omega_i \in \mathbb{R}^d, i \in \mathbb{N}\}$ be an homogeneous Poisson Point Process on $\mathbb{R}^d$.
- The potential V one $\mathbb{R}^d$ is defined by

$$V^\omega(x) := \sum_{i \in \mathbb{N}} \mathbf{1}_{|\omega_i - x| \le 1}. \tag{11}$$

- Given $\lambda > 0$, we study trajectory of the Brownian Motion starting from the origin, killed with inhomogeneous rate , $\lambda + V^\omega$ conditioned to survive up to the hitting time of $\mathcal{H}_L = \{L\} \times \mathbb{R}^{d-1}$ an hyperplane at distance L.

The model

The probability of survival is given by (**P** and **E** are the law and expectation of Standard BM starting from the origin)

$$Z_L^\omega := \mathbf{E}\left[e^{-\int_0^{\tau_{\mathcal{H}_L}} \lambda + V(B_t)\,\mathrm{d}t}\right] \tag{12}$$

The paths measure conditioned to survival is given by μ_L^ω whose Radon-Nikodym derivative with respect to **P**,

$$\frac{\mathrm{d}\mu_L^\omega}{\mathrm{d}\mathbf{P}} = \frac{1}{Z_L^\omega} e^{-\int_0^{\tau_{\mathcal{H}_L}} \lambda + V(B_t)\,\mathrm{d}t}, \tag{13}$$

we study the asymptotic properties of μ_L^ω for large L.

Link with directed polymer

- The killing rate penalizes trajectories that spend to much time before hitting $\mathcal{H}_L$ and in fact, trajectory that survive hit $\mathcal{H}_L$ in a time of order L. Killing induces an effective drift in the trajectories and our model behave as "semi-directed" (polymer can backtrack but is drifted toward one direction).
- The model in dimension d is therefore very similar to directed polymer in dimension $1+(d-1)$. And prediction for the directed polymer should be valid also for this model. ($\xi = 2/3$ in dimension 2, etc...)

Related discrete model

Replace BM by SRW in $\mathbb{Z}^d$ and let V be a realization of IID random variable. In that case results concerning diffusivity and localization have been proved:

- Equality of quenched and annealed free energy at high-temperature for $d \geq 4$ [Flury '08, Zygouras '09],
- Diffusivity and invariance principle at high-temperature for $d \geq 4$ [Ioffe, Velenik '10],
- Localization of the trajectories for all temperature for $d = 2$ and $d = 3$. [Zygouras '10].

We believe that these results also hold for Brownian Motion and that they can be proved by using similar methods, but this needs some work to adapt them.

Superdiffusivity

To investigate superdiffusivity property we have to look at the probability that trajectories have to stay in a tube of diameter L^{ξ}
Set

$$\mathcal{C}_L^{\xi} := \{x \in \mathbb{R}^d, d(x, \mathbb{R}e_1) \leq L^{\xi}\} \tag{14}$$

we look at the probability of the event

$$\forall t < \tau_{\mathcal{H}_L}, B_t \in \mathcal{C}_L^{\xi} \tag{15}$$

under the polymer measure μ_L^{ω}

Superdiffusivity

Recall

$$\mathcal{C}_L^{\xi} := \{x \in \mathbb{R}^d, d(x, \mathbb{R}e_1) \leq L^{\xi}\} \tag{16}$$

Theorem (Wütrich '98)

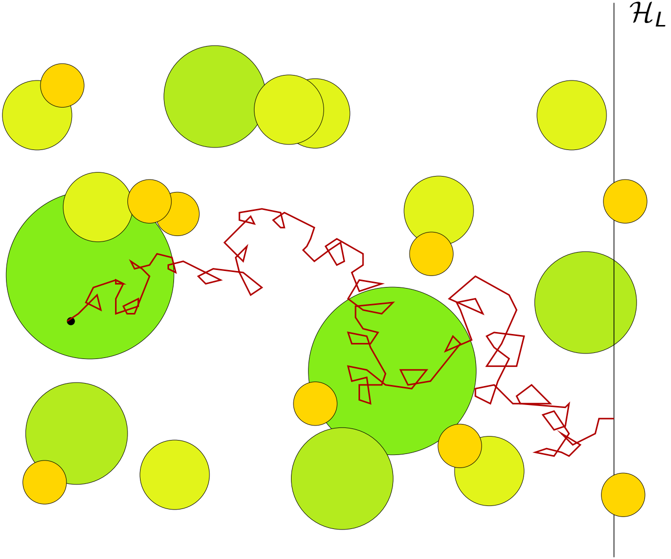
When $d = 2$, for all $\xi < 3/5$,

$$\lim_{L\to\infty} \mathbb{E}\mu_L^{\omega}\left(\forall t < \tau_{\mathcal{H}_L}, B_t \in \mathcal{C}_L^{\xi}\right) = 0. \tag{17}$$

For all d, for all $\xi > 3/4$,

$$\lim_{L\to\infty} \mathbb{E}\mu_L^{\omega}(\forall t < \tau_{\mathcal{H}_L}, B_t \in \mathcal{C}_L^{\xi}) = 1. \tag{18}$$

Model with correlations



We modify the model by introducing traps of random radius and intensity (two parameters α and γ are introduced)

$$\omega := \{(\omega_i, r_i) \in \mathbb{R}^d \times [1, \infty), i \in \mathbb{N}\} \tag{19}$$

Is a Poisson point process on $\mathbb{R}^d \times \mathbb{R}_+$ of intensity $\mathcal{L} \otimes \nu$, where ν is a probability measure defined by

$$\nu([r, \infty)) = r^{-\alpha}. \tag{20}$$

Define the potential V^ω by

$$V^\omega(x) := \sum_{i \in \mathbb{N}} r_i^{-\gamma} \mathbf{1}_{\{|x - \omega_i| \leq r_i\}}. \tag{21}$$

It is almost surely everywhere finite if $\alpha + \gamma - d > 0$. There are long ranges correlation in V and

$$\mathrm{cov}(V^\omega(0), V^\omega(x)) \asymp \|x - y\|^{d - \alpha - \gamma}. \tag{22}$$

Lower bound on the volume exponent

Recall

$$\mathcal{C}_L^{\xi} := \{x \in \mathbb{R}^d, d(x, \mathbb{R}e_1) \leq L^{\xi}\} \tag{23}$$

Theorem (L '11)

Set $\xi_1 := \min\left(\frac{1}{1+\alpha-d}, \frac{3}{3+\alpha+2\gamma-d}\right)$ *then for all* d *and all* $\xi < \xi_1$.

$$\lim_{L\to\infty} \mathbb{E}\mu_L^{\omega}(\forall t < \tau_{\mathcal{H}_L}, B_t \in \mathcal{C}_L^{\xi}) = 0. \tag{24}$$

If $\alpha - d$ and γ are small, then $\xi > 1/2$, and therefore we have example of superdiffusive behavior in any dimension.
In some cases $\xi_1 > 3/4$, which means that the upper bound ($\xi < 3/4$) for the model with no correlation is not valid here in general.

Upper on the volume exponent

Recall

$$\mathcal{C}_L^{\xi} := \{x \in \mathbb{R}^d, d(x, \mathbb{R}e_1) \le L^{\xi}\} \tag{25}$$

Theorem (L '11)

Set $\xi_2 := \max\left(\frac{1}{1+\alpha-d}, \min(\frac{1}{1+\gamma}, \frac{2+d}{2\alpha})\right)$ *then for all* d*, if* $\alpha > d$ *then for all* $\xi > \xi_2$*.*

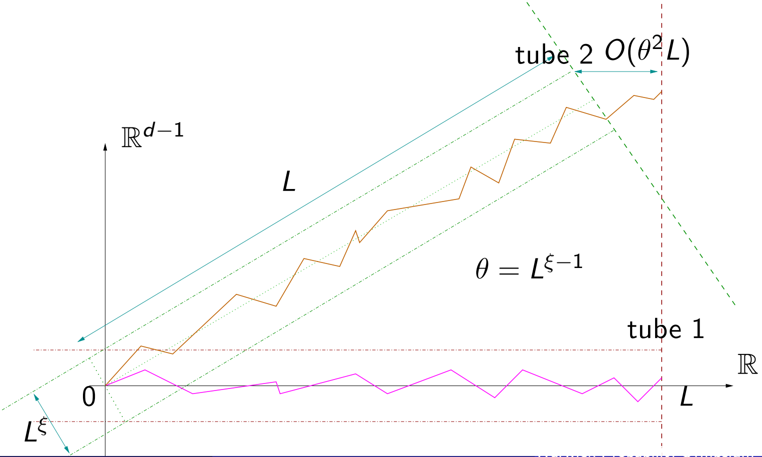
$$\lim_{L\to\infty} \mathbb{E}\mu_L^{\omega}(\forall t < \tau_{\mathcal{H}_L}, B_t \in \mathcal{C}_L^{\xi}) = 1. \tag{26}$$

Corollary: A tight result in some special cases

When $\gamma = d - \alpha < 1/3$ the upper bound matches the lower bound. (both equal to $1/(1+\gamma)$).

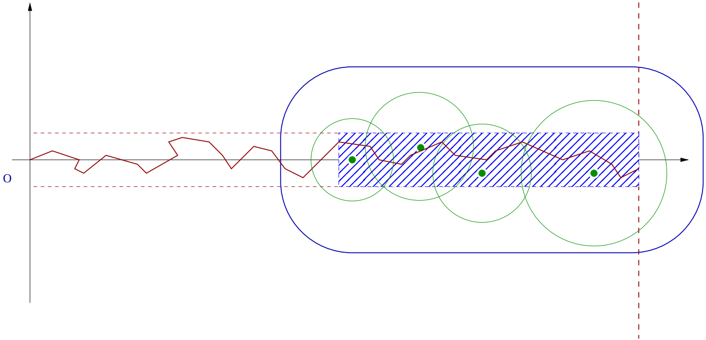
Ideas of the proof (lower bound)

First one use a geometrical argument to get that with probability at least one half the probability of going in the tube number 2 instead of tube number 1 is larger than $e^{-L^{2\xi-1}}$.



Ideas of the proof (lower bound)

Then we modify a little bit the law of the environment $\omega \to \widetilde{\omega}$ by adding some large trap (that cover the full tube 1 but do not touch tube 2). This lowers the weight of trajectories in tube 1 and leaves the weight of trajectories in tube 2 unchanged.



There is two condition on the parameters for this approach to be successful

- One must be able to add large traps (of diameter L^{ξ}) without changing to much the law of ω (distribution of ω and $\widetilde{\omega}$ must be very close in total variation (this gives the condition $1 + (d - 1 - \alpha)\xi > 0$)
- The fact of adding new traps must lower the weight of trajectories from the tube 1 at least by a factor $e^{-L^{2\xi-1}}$ (this gives the condition $2\xi - 1 < \frac{1}{2}(1 + (d + 1 - \alpha - 2\gamma)\xi)$

These two condition give the lower bound.

Ideas of proof (upper bound)

- The approach is similar to Wütrich is the sence that one uses concentration results on $\log Z$.
- Finding the "optimal" concentration result requires a multiscale analysis in order to treat traps on different scales seperatly.