The Airy line ensemble: continuum statistics and Gibbs property

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Outline

Introducing the Airy line ensemble.

Part 1: Continuum statistics (with J. Quastel, D. Remenik):

- $\mathbb{P}(\mathrm{Airy}_2(x) \le g(x) \text{ for } x \in [a,b])$,

Part 2: Non-intersecting Brownian Gibbs property for Airy line ensemble (with A. Hammond):

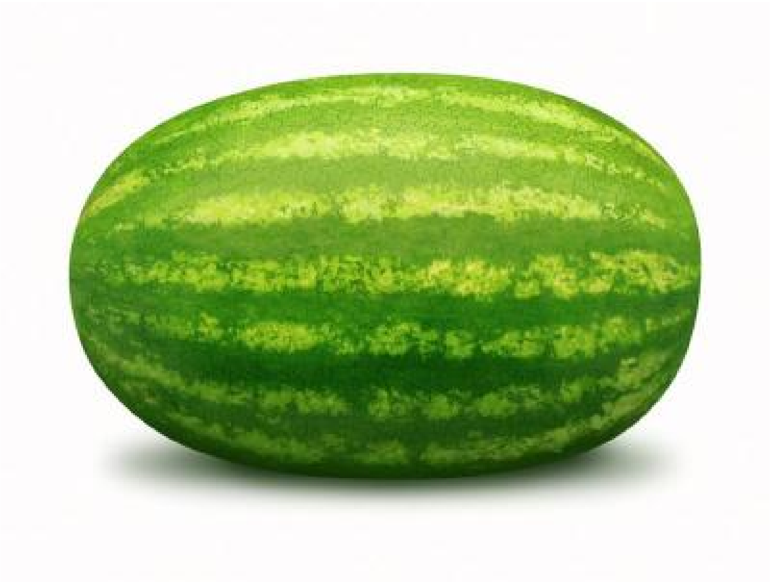
- Absolute continuity w.r.t. Brownian motion,
- Johansson's conjecture: unique argmax of $\mathrm{Airy}_2(x) - x^2$.

Part 3: KPZ line ensemble and H-Brownian Gibbs property (with A. Hammond):

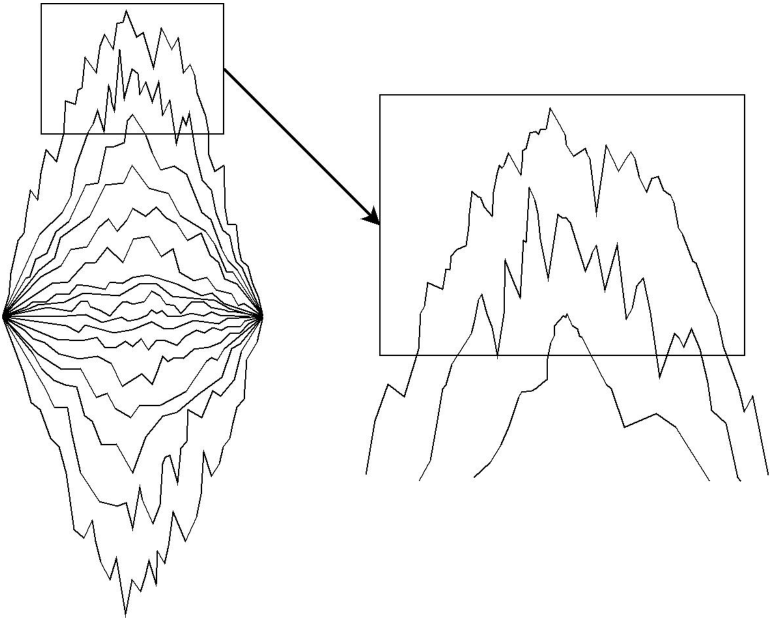
- Continuum statistics? (One point distribution)
- Hopf-Cole solution to KPZ equation with narrow-wedge initial data is absolutely continuity w.r.t. Brownian motion.

Global/local information using exactly solvable/probabilistic means.

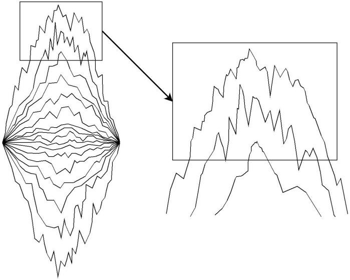
My favorite fruit: Watermelon



Non-intersecting Brownian bridges



Non-intersecting Brownian bridges



- N Brownian Bridges $B_i : [-N, N] \to \mathbb{R}$ with $B_i(-N) = B_i(N) = 0$.
- Condition on $B_i(x) \neq B_j(x)$ for all $i \neq j$ and $x \in (-N, N)$.
- B_1 top line to B_N bottom line.
- KPZ (1/3, 2/3) scaling around the edge:

$$\mathcal{D}_i^N(x) = \frac{B_i(N^{2/3}x) - 2N}{N^{1/3}}$$

Multi-line Airy process

- For each fixed x, $\mathcal{D}^N(x) = \{\mathcal{D}_i^N(x)\}_{i=1}^N$ is a (determinantal) point process.
- For any $x_1, \dots x_\ell$, (effectively Prähofer-Spohn '01)

$$\left(\mathcal{D}^N(x_1), \dots, \mathcal{D}^N(x_\ell)\right) \overset{N\to\infty}{\Longrightarrow} \left(\mathcal{A}(x_1) - x_1^2, \dots, \mathcal{A}(x_\ell) - x_\ell^2\right).$$

- $\mathcal{A}(x) = \{\mathcal{A}_i(x)\}_{i\in\mathbb{N}}$ is called the multi-line Airy process.
- Specified by finite dimensional distributions $(x_1, \dots x_\ell)$; stationary extended determinantal point process.

Is there a continuous version of this process in which these $\mathbb{N}$-indexed points form continuous lines?

Airy line ensemble – a continuous version

Theorem (C, Hammond)

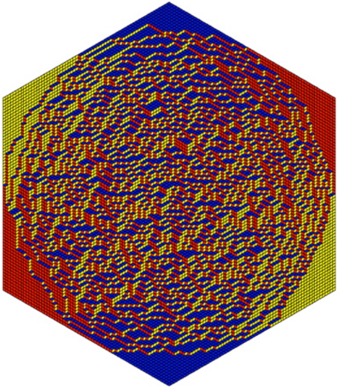
There exists a unique Airy line ensemble $\mathcal{A} = \{\mathcal{A}_i\}_{i\in\mathbb{N}}$*:*

- $\mathbb{N}$*-indexed lines* $\mathcal{A}_i : \mathbb{R} \to \mathbb{R}$ *which are continuous, non-intersecting,*
- $(\mathcal{A}(x_1), \ldots, \mathcal{A}(x_\ell))$ *distributed according to the multi-line Airy process.*

For any $k \in \mathbb{N}$ *and any* $T > 0$*,* $\{\mathcal{D}_i(\cdot)\}_{i=1}^k \Rightarrow \{\mathcal{A}_i(\cdot) - (\cdot)^2\}_{i=1}^k$ *as a process on* $[1, \ldots, k] \times [-T, T]$*.*

Johansson '02 showed that there existed a continuous version of top line $\mathcal{A}_1(\cdot)$ – the *Airy*$_2$ *process*: tightness of geometric LPP/PNG via pre-asymptotic application of Kolmogorov continuity criterion (uses exact solvability in essential way).

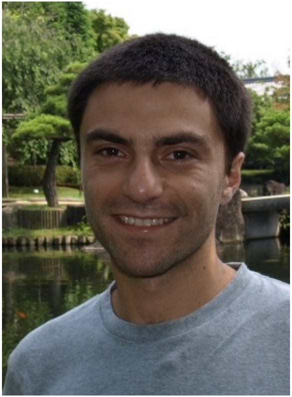
A universal edge scaling limit



- Line-ensembles conditioned on non-intersection (invariance principle?), multi-layer PNG model
- Tiling / Dimer problems
- Random matrix theory, Dyson Brownian motion
- Polymer free energy / last passage percolation

Part 1: Continuum statistics

with J. Quastel and D. Remenik



Continuum statistics for the Airy_2 process

- Focus on top curve $\mathcal{A}_1(x)$ – a.k.a. $\text{Airy}_2(x)$.
- Compute concise and clean formula for:

$$\mathbb{P}(\text{Airy}_2(x) \leq g(x) \text{ for } x \in [a, b]).$$

- Application: compute asymptotic fluctuation distribution for point-to-line last passage percolation

$$\mathbb{P}(\text{Airy}_2(x) - x^2 \leq m \text{ for } x \in (-\infty, \infty)) = F_{\text{GOE}}(4^{1/3} m)$$

- The argmax above should be the law of the polymer end-point. Johansson conjectured a unique argmax (proof by C, Hammond in part 2).
- Above is preview for some of Jeremy's talk – stay tuned.

Part 2: Non-intersecting Brownian Gibbs property for Airy line ensemble

With A. Hammond

Non-intersecting Brownian Gibbs property

Theorem

The Airy line ensemble minus a parabola (i.e., $\mathcal{A}(\cdot) - (\cdot)^2$) has the non-intersecting Brownian Gibbs property.

Definition: A line ensemble $\mathcal{L}$ has the *non-intersecting Brownian Gibbs property* if for any k and any interval $[a, b]$, the law of $\mathcal{L}_k$ on $[a, b]$ given the rest of the line ensemble is distributed according to the law of a Brownian bridge from $(a, \mathcal{L}_k(a))$ to $(b, \mathcal{L}_k(b))$ conditioned not to intersect line $\mathcal{L}_{k-1}$ or $\mathcal{L}_{k+1}$ on $[a, b]$.

- $\mathcal{D}^N$ has this property by construction.
- Think about uniform measure on paths of non-intersecting random walks (or PNG model) – has discrete version of this Gibbs property.
- Can "soften" the non-intersecting conditioning – *H-Brownian Gibbs property*.

Proof: Airy line ensemble existence and Gibbs property

Three steps to proving existence and Gibbs property:

1. Find a finite system with the Gibbs property (or at least approximately the Gibbs property).
2. Show weak convergence (as a line ensemble – i.e. measure on lines) of rescaled finite system to desired infinite system.
3. Show that Gibbs property is preserved in limit.

Step 1 (easy in this case): non-intersecting Brownian bridges have the Gibbs property by construction!

Step 3: Proof by coupling

- Skorohod representation thm to couple $\mathcal{D}^N$ to converge in L^∞;
- Rephrases Gibbs property in terms of invariance under rejection sampling of free Brownian bridges;
- Couple Brownian bridge samples to prove limit line ensemble has same invariance – hence the Gibbs property.

Step 2: Proof of weak convergence

Focus on top k curves in $[-T, T]$ interval.

Convergence of finite dimensional distributions: $\mathcal{D}^N(\cdot)$ known to converge to multi-line Airy process minus parabola.

Tightness:

- This is the hard part – why?
  - Gap between curves could go to 0 at random locations,
  - Lower curves could cause large spikes.
- Study *re-sampling acceptance probability* for top k curves on $[-T, T]$. Show this acceptance probability remains uniformly bounded from below with high probability as $N \to \infty$, hence can establish tightness by comparing to Brownian bridges.
- Uses finite system Gibbs property and monotone couplings.
- Purely probabilistic methods – no exact formulas necessary.
- Applied to Airy-like line ensembles (e.g. wanderers...)

Corollary: Local Brownian absolute continuity

For $k \in \mathbb{N}$, $x \in \mathbb{R}$, $y > 0$, the measure on functions from $[0, y] \to \mathbb{R}$ given by

$$\mathcal{A}_k(\cdot + x) - \mathcal{A}_k(x)$$

is absolutely continuous w.r.t. standard Brownian motion on $[0, y]$. Alternatively, the measure on functions from $[0, y] \to \mathbb{R}$ given by

$$\mathcal{A}_k(\cdot + x) - \left(\frac{y - \cdot}{y}\mathcal{A}_k(x) + \frac{\cdot}{y}\mathcal{A}_k(x + y)\right)$$

is absolutely continuous w.r.t. standard Brownian bridge on $[0, y]$.

- Conjectured / predicted by Prähofer and Spohn based off two-point covariance calculation in short distance scale.
- Hägg showed finite dimensional distribution convergence to Brownian motion from exact formulas.

Corollary: Proof of Johansson's argmax conjecture

Almost surely there exists a unique x at which $\mathcal{A}iry_2(x) - x^2$ is maximized over $x \in \mathbb{R}$.

Proof: For $x \in [-T, T]$ uniqueness follows from Brownian absolute continuity.

$$\text{Localize:} \quad \lim_{t \to \pm\infty} \mathrm{Airy}_2(x) - x^2 = -\infty \text{ almost surely.}$$

- For all $m \in \mathbb{R}$, show $\liminf_{x \to \pm\infty} \mathrm{Airy}_2(x) - x^2 \leq m$.
- At deterministic locations (e.g., $x \in \mathbb{Z}$) have good control over decay of probability. Want to use Borel-Cantelli.
- Issue is to control wiggles on intervals $x \in [i, i+1]$.
- If $\mathrm{Airy}_2(x^*) - (x^*)^2 \geq m$ at a random location $x^* \in [i, i+1]$, then by re-sampling on $[x^*, i+2]$, find that with proportional probability $\mathrm{Airy}_2(x) - x^2$ is large at $x = i+1$. However, this is known to be unlikely, hence control over wiggles.

Review of Airy line ensemble case

- Non-intersecting Brownian bridge (scaled) line ensemble $\mathcal{D}^N$ has Brownian Gibbs property.
- Take an "edge" scaling limit and use probabilistic methods to prove tightness (on top of exactly solvable finite dimensional distribution convergence).
- Airy line ensemble $\mathcal{A} : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$, top curve Airy_2 process.
- From exact methods find $\mathbb{P}(\mathrm{Airy}_2(x) \leq g(x) \text{ for } x \in [a, b])$.
- From probabilistic methods prove Gibbs property of limit, hence Brownian absolute continuity, Johansson's argmax uniqueness conjecture.
- Airy line ensemble universal scaling limit.
- **Uniqueness conjecture**: (Up to vertical additive shifts) the Airy line ensemble is the unique x-invariant, $\mathbb{N}$-indexed line ensemble such that $\mathcal{A}(\cdot) - (\cdot)^2$ has the non-intersecting Brownian Gibbs property.

Part 3: KPZ line ensemble and H-Brownian Gibbs property

with A. Hammond

Absolute continuity of KPZ equation

Hopf-Cole solution to KPZ equation
$\partial_t \mathcal{H} = \frac{1}{2}\partial_x^2 \mathcal{H} + \frac{1}{2}(\partial_x \mathcal{H})^2 + \dot{\mathcal{W}}$ via stochastic heat equation (SHE):

$$\mathcal{H} = \log \mathcal{Z}, \quad \text{where} \quad \partial_t \mathcal{Z} = \frac{1}{2}\partial_x^2 \mathcal{Z} + \mathcal{Z}\dot{\mathcal{W}}.$$

Narrow-wedge initial data $\mathcal{Z}(t=0,x) = \delta_{x=0}$.
$\mathcal{H} \leftrightsquigarrow$ free energy of continuum directed random polymer (CDRP).

Theorem

Fix $t > 0$, then as a process in x, the Hopf-Cole solution to KPZ with narrow-wedge initial data $\mathcal{H}(t,\cdot)$ is absolutely continuous w.r.t. Brownian motion (likewise for $\mathrm{Airy}_2^t(\cdot)$).

(Complements equil. KPZ comparison result of J. Quastel, D. Remenik)
Conjecture: (Stationary) crossover $\mathrm{Airy}_2^t(x)$ process defined by

$$\mathcal{H}(t,x) = -\frac{x^2}{2t} + \log(\sqrt{2\pi t}) + t^{1/3}\mathrm{Airy}_2^t(t^{2/3}x)$$

converges to $\mathrm{Airy}_2(x)$ as $t \to \infty$.

KPZ line ensemble

J. Warren and N. O'Connell define a multi-layer extension to SHE (CDRP partition function hierarchy):

Definition: Set $\mathcal{Z}_0(t,x) \equiv 1$. For $n \in \mathbb{N}$, $t \geq 0$ and $x \in \mathbb{R}$ set

$$\mathcal{Z}_n(t,x) = p(t,x)^n \sum_{k=0}^{\infty} \int_{\Delta_k(t)} \int_{\mathbb{R}^k} R_k^{(n)} \left(\{(t_i, x_i)\}_{i=1}^{k} \right) \prod_{i=1}^{k} \dot{\mathcal{W}}(dt_i dx_i).$$

Fact: For any $t > 0$, with probability 1, for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$, $\mathcal{Z}_n(t,x) > 0$.

The above fact justifies taking logarithms:

Definition: For $t > 0$ fixed, the *KPZ*^{t} *line ensemble* is a continuous $\mathbb{N}$-indexed line ensemble $\mathcal{H}^t = \{\mathcal{H}_i^t\}$ given by

$$\mathcal{H}_i^t(x) = \log\left(\frac{\mathcal{Z}_i(t,x)}{\mathcal{Z}_{i-1}(t,x)} \right).$$

H-Brownian Gibbs property

Theorem

The KPZ^t line ensemble has the H-Brownian Gibbs property with $H(x) = e^{\lambda x}$ and $\lambda > 0$ a function of t ($t \to \infty$ implies $\lambda \to \infty$).

Definition: Fix a *Hamiltonian* $H : \mathbb{R} \to [0, \infty)$. A line ensemble $\mathcal{L}$ has the *H-Brownian Gibbs property* if for any k and any interval $[a, b]$, the law of $\mathcal{L}_k$ on $[a, b]$ given the rest of the line ensemble is distributed according to the law of a Brownian bridge from $(a, \mathcal{L}_k(a))$ to $(b, \mathcal{L}_k(b))$ weighted by

$$\exp\left\{-\int_a^b H(\mathcal{L}_k(x) - \mathcal{L}_{k-1}(x))dx - \int_a^b H(\mathcal{L}_{k+1}(x) - \mathcal{L}_k(x))dx\right\}.$$

For $H(x) = e^{\lambda x}$ this is a "softer" version of non-intersection and as $\lambda \to \infty$ becomes non-intersecting Brownian Gibbs property.

Three step approach

1. Finite system: N. O'Connell's quantum Toda lattice diffusion for free energy of O'Connell-Yor polymer plays role of "soft" Dyson Brownian motion:

$$\psi_0^{-1}\mathfrak{H}\psi_0, \qquad \mathfrak{H} = \tfrac{1}{2}\Delta - \sum_{i=1}^{N-1} e^{x_{i+1}-x_i}.$$

   This has the H-Brownian Gibbs property, $\lambda = 1$.
2. Weak convergence: J. Quastel, G. Moreno Flores prove O'Connell-Yor partition function hierarchy converges to multi-layer extension of SHE (then take logarithms).
3. Use couplings to prove Gibbs property preserved in limit.

Reflections

- Even if you only care about the top line, there is good reason to consider the whole ensemble: RSK/ tropical RSK correspondence, polymer partition function hierarchy, PNG model, TASEP, KPZ equation.
- Some canonical solvable models (e.g. ASEP) do not presently reveal such a line ensemble structure though.
- Gibbs property and probabilistic coupling methods provide an effective method to complement exactly solvable systems techniques which have been widely used previously.
- The combination of techniques enables us to describe both local and global properties of this universal interface / free energy model (continuum statistics, Brownian absolute continuity).