Warwick, March 19-23th 2012

Non-intersecting random walk in the neighborhood of a symmetric tacnode

Patrik L. Ferrari
with Mark Adler and Pierre van Moerbeke (arXiv:1007.1163)
and Bálint Vető (arXiv:1112.5002)

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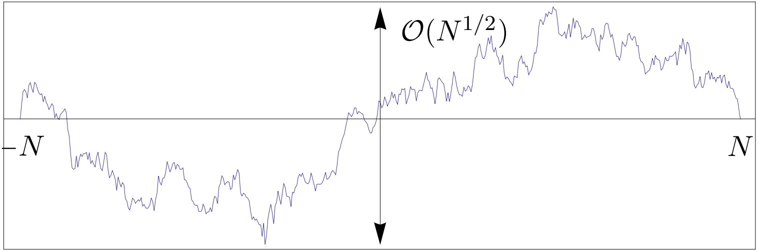
- From Brownian Bridge to Tacnode process
- Idea of the construction, key steps

- Let $X(t)$ a Brownian Bridge with $X(-N) = X(N) = 0$.



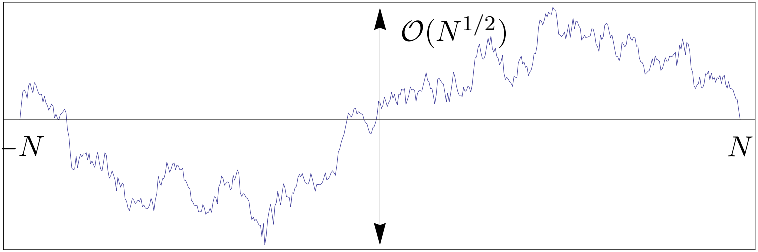
- Macroscopic shape

$$h(\alpha) := \frac{\mathbb{E}(X(\alpha N))}{N} = 0.$$

- Fluctuations scale

$$\sqrt{\mathrm{Var}(X(\alpha N))} = \sqrt{N\alpha(1-\alpha)}.$$

- Let $X(t)$ a Brownian Bridge with $X(-N) = X(N) = 0$.

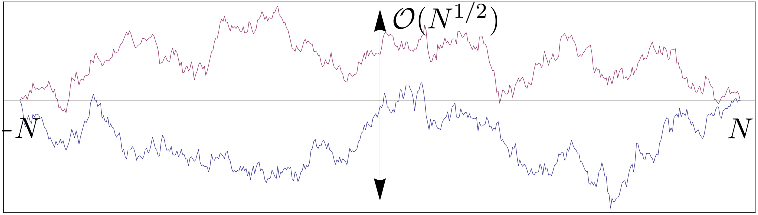


- Distribution law

$$\frac{X(\alpha N)}{\sqrt{N\alpha(1-\alpha)}} = \mathcal{N}(0,1) : \quad \text{Gaussian RW}$$

- Local process: Brownian Motion

- Let $X_1(t) \geq X_2(t) \geq \cdots \geq X_m(t)$ be m non-intersecting Brownian Bridges with $X_k(-N) = X_k(N) = 0,\ k = 1, \ldots, m$.



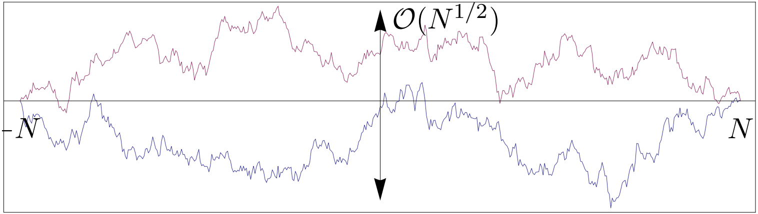
- Macroscopic shape

$$h_1(\alpha) := \lim_{N\to\infty} \frac{\mathbb{E}(X_1(\alpha N))}{N} = 0.$$

- Fluctuations scale

$$\sqrt{\mathrm{Var}(X_1(\alpha N))} = \mathcal{O}(N^{1/2}).$$

- Let $X_1(t) \geq X_2(t) \geq \cdots \geq X_m(t)$ be m non-intersecting Brownian Bridges with $X_k(-N) = X_k(N) = 0$, $k = 1, \ldots, m$.

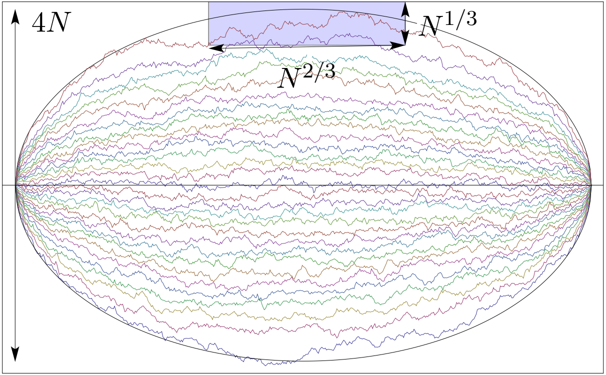


- Distribution law

$$\lim_{N\to\infty} \frac{X_1(\alpha N)}{N^{1/2}} = \text{Largest eigenvalue of a } m \times m \text{ GUE matrix}$$

- Local process: Dyson's Brownian Motion

- Consider now $m = N$ non-intersecting Brownian Bridges with $X_k(-N) = X_k(N) = 0$.



- Macroscopic shape: $h(\alpha) := \lim_{N\to\infty} \frac{\mathbb{E}(X_1(\alpha N))}{N} = 2\sqrt{1-\alpha^2}$.

- Consider now $m = N$ non-intersecting Brownian Bridges with $X_k(-N) = X_k(N) = 0$.
- Fluctuations scale

$$\sqrt{\mathrm{Var}(X_1(\alpha N))} = \mathcal{O}(N^{1/3}).$$

- Distribution law: Tracy-Widom F_2

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{X_1(\alpha N) - Nh(\alpha)}{\sigma(\alpha)N^{1/3}} \leq s\right) = F_2(s)$$

F_2 discovered by Tracy, Widom '94

- Local process: Airy$_2$

$$\lim_{N\to\infty} \frac{X_1(\alpha N + \tau c(\alpha)N^{2/3}) - Nh(\alpha + \tau c(\alpha)N^{-1/3})}{\sigma(\alpha)N^{1/3}} = \mathcal{A}_2(\tau)$$

- Airy$_2$ process discovered in a stochastic growth model in the Kardar-Parisi-Zhang universality class Prähofer, Spohn '02
  Measured experimentally (liquid crytals) Takeuchi, Sano '10

- Consider the point process $\xi(t,x) = \sum_{k=1}^{N} \delta_{X_k(t)}(x)$.
- Scaling at the edge: $t = \tau N^{2/3}$ and $x = 2N + sN^{1/3}$

$$\lim_{N\to\infty} \rho^{(n)}(s_1,\tau_1;\dots;s_n,\tau_n) = \det[K_{\rm Ai}(s_i,\tau_i;s_j,\tau_j)]_{1\le i,j\le n}$$

where $K_{\rm Ai}$ is the extended Airy kernel

$$K_{\rm Ai}(s_1,\tau_1;s_2,\tau_2) = \frac{-1}{(2\pi{\rm i})^2}\int_{\langle}{\rm d}w\int_{\rangle}{\rm d}z\,\frac{e^{z^3/3+\tau_1 z^2+(\tau_1^2-s_1)z}}{e^{w^3/3+\tau_2 w^2+(\tau_2^2-s_2)w}}\frac{1}{z-w} - \frac{\mathbb{1}[\tau_1>\tau_2]}{\sqrt{4\pi(\tau_1-\tau_2)}}e^{-(s_1-s_2+\tau_2^2-\tau_1^2)^2/4(\tau_1-\tau_2)}$$

- Scaling in the bulk

$$\lim_{N\to\infty} \rho^{(n)}(x_1,t_1;\ldots;x_n,t_n) = \det[K_{\rm Sine}(x_i,t_i;x_j,t_j)]_{1\leq i,j\leq n}$$

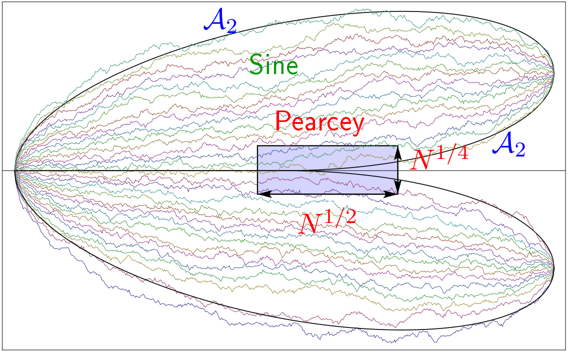
where $K_{\rm Sine}$ is the extended Sine kernel. In particular,

$$K_{\rm Sine}((x,0),(y,0)) = \frac{\sin(\pi(x-y))}{\pi(x-y)}.$$

- Now consider $2N$ non-intersecting Brownian Bridges with

$$x_k(-N) = 0, \quad 1 \leq k \leq 2N$$

$$x_k(N) = bN, \quad 1 \leq k \leq N, \quad x_k(N) = -bN, \quad N+1 \leq k \leq 2N.$$



- Macroscopic shape of X_N and X_{N+1} come together, say at $t = a(b)N$: a cusp!

- Now consider $2N$ non-intersecting Brownian Bridges with

$$x_k(-N) = 0, \quad 1 \leq k \leq 2N$$

$$x_k(N) = bN, \; 1 \leq k \leq N, \quad x_k(N) = -bN, \; N+1 \leq k \leq 2N.$$

- Scaling at the cusp: Density of BB is $\mathcal{O}(N^{-1/4})$ and locally like Brownian motions:

$$t = a(b)N + \tau c(b)N^{1/2}, \quad x = s\sigma(b)N^{1/4}$$

- Rescaled point process is determinantal with (as $N \to \infty$) n-point correlation given by the Pearcey kernel.

Tracy, Widom '05

- Consider the point process $\xi(t,x) = \sum_{k=1}^{N} \delta_{X_k(t)}(x)$.
- Scaling at the edge: $t = a(b)N + \tau c(b)N^{1/2}$ and $x = s\sigma(b)N^{1/4}$

$$\lim_{N\to\infty} \rho^{(n)}(s_1,\tau_1;\dots;s_n,\tau_n) = \det[K_{\mathcal{P}}(s_i,\tau_i;s_j,\tau_j)]_{1\le i,j\le n}$$

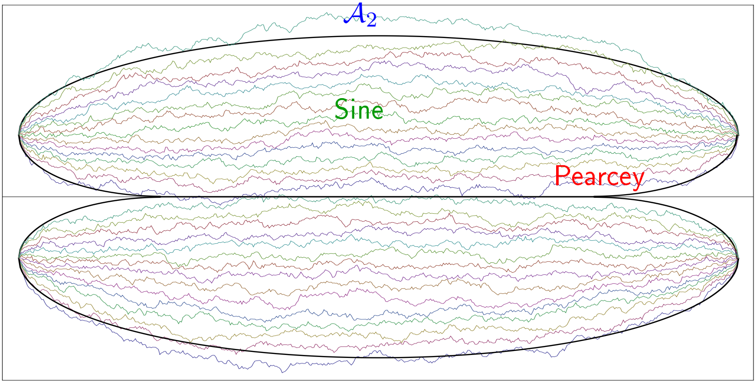
where $K_{\mathcal{P}}$ is the extended Pearcey kernel

$$\begin{aligned} &K_{\mathcal{P}}(s_1,\tau_1;s_2,\tau_2) \\ &\quad= \frac{-1}{(2\pi \mathrm{i})^2}\int_{\mathrm{i}\mathbb{R}} \mathrm{d}w \int_{\times} \mathrm{d}z \frac{e^{z^4/4-\tau_1 z^2/2+s_1 z}}{e^{w^4/4-\tau_2 w^2/2+s_2 w}} \frac{1}{z-w} \\ &\qquad - \frac{\mathbb{1}[\tau_2>\tau_1]}{\sqrt{2\pi(\tau_2-\tau_1)}} e^{-(s_1-s_2)^2/2(\tau_2-\tau_1)} \end{aligned}$$

- Now consider $2N$ non-intersecting Brownian Bridges with

$$x_k(\pm N) = bN, \quad 1 \le k \le N, \qquad x_k(\pm N) = -bN, \quad N+1 \le k \le 2N.$$

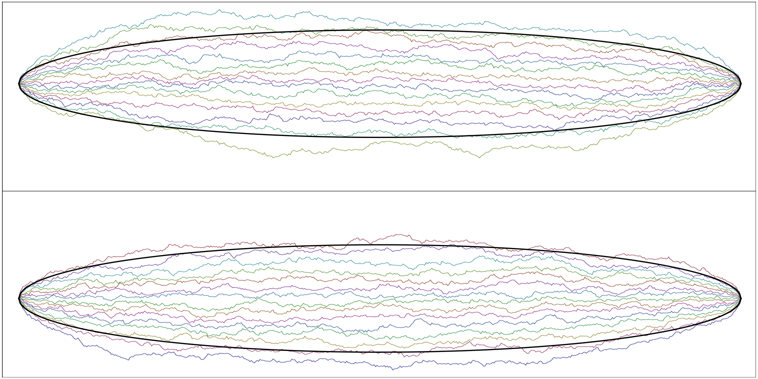
- Macroscopic shape: $0 < b < 2$, there are two cusps around $\pm a(b)N \Rightarrow$ Pearcey process in scale $(N^{1/2}, N^{1/4})$.



- Now consider $2N$ non-intersecting Brownian Bridges with

$$x_k(\pm N) = bN, \quad 1 \leq k \leq N, \qquad x_k(\pm N) = -bN, \quad N+1 \leq k \leq 2N.$$

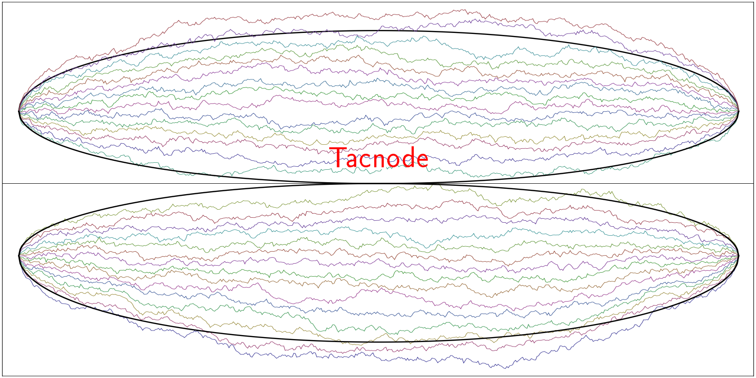
- Macroscopic shape: $b > 2$, there are two (asymptotically) independent blocks of N non-intersecting Brownian Bridges: Airy$_2$ processes in the scale $(N^{2/3}, N^{1/3})$.



- Now consider $2N$ non-intersecting Brownian Bridges with

$$x_k(\pm N) = bN, \quad 1 \le k \le N, \qquad x_k(\pm N) = -bN, \quad N+1 \le k \le 2N.$$

- Macroscopic shape: $b = 2 + o(1)$, the two cusps joints into a tacnode.



Q: Is the tacnode process like two Pearcey joining together or like two Airy_2 processe colliding?

Q: Is the tacnode process like two Pearcey joining together or like two Airy$_2$ processe colliding?

A: Like two Airy$_2$ processes touching. Take $bN = 2N + \sigma N^{1/3}$.

- σ modulate the "strength of interaction" between the two set of Brownian Bridges.
- Under Airy "microscope": $t = \tau N^{2/3}$ and $x = sN^{1/3}$ we obtain that the limit process has determinantal correlations governed by the symmetric tacnode kernel $\mathcal{K}_\sigma$

Simplified expressions in the recent work by

`Adler, Johansson, van Moerbeke: arXiv:1112.5532`

One-time kernel at $\tau = 0$:

$$
\begin{aligned}
\mathcal{K}_\sigma(s_1, s_2) &= \mathcal{C}(s_1 - s_2) \\
&+ \int_{\delta+\mathrm{i}\mathbb{R}} \frac{du}{2\pi\mathrm{i}} \int_{-\delta+\mathrm{i}\mathbb{R}} \frac{dv}{2\pi\mathrm{i}} \quad \frac{e^{\frac{u^3}{3}-\sigma u}}{e^{\frac{v^3}{3}-\sigma v}} \left(\frac{e^{s_1 u}}{e^{s_2 v}} + \frac{e^{s_1 v}}{e^{s_2 u}} \right) \frac{(1-\hat{\mathcal{P}}(u))(1-\hat{\mathcal{P}}(-v))}{u-v} \\
&+ \int_{2\delta+\mathrm{i}\mathbb{R}} \frac{du}{2\pi\mathrm{i}} \int_{\delta+\mathrm{i}\mathbb{R}} \frac{dv}{2\pi\mathrm{i}} \quad \frac{e^{\frac{u^3}{3}-\sigma u}}{e^{-\frac{v^3}{3}-\sigma v}} \left(\frac{e^{s_1 u}}{e^{s_2 v}} + \frac{e^{s_1 v}}{e^{s_2 u}} \right) \frac{(1-\hat{\mathcal{P}}(u))\hat{\mathcal{Q}}(-v)}{u-v} \\
&+ \int_{-\delta+\mathrm{i}\mathbb{R}} \frac{du}{2\pi\mathrm{i}} \int_{-2\delta+\mathrm{i}\mathbb{R}} \frac{dv}{2\pi\mathrm{i}} \quad \frac{e^{-\frac{u^3}{3}-\sigma u}}{e^{\frac{v^3}{3}-\sigma v}} \left(\frac{e^{s_1 u}}{e^{s_2 v}} + \frac{e^{s_1 v}}{e^{s_2 u}} \right) \frac{(1-\hat{\mathcal{P}}(-v))\hat{\mathcal{Q}}(u)}{u-v} \\
&+ \int_{-\delta+\mathrm{i}\mathbb{R}} \frac{du}{2\pi\mathrm{i}} \int_{\delta+\mathrm{i}\mathbb{R}} \frac{dv}{2\pi\mathrm{i}} \quad \frac{e^{-\frac{u^3}{3}-\sigma u}}{e^{-\frac{v^3}{3}-\sigma v}} \left(\frac{e^{s_1 u}}{e^{s_2 v}} + \frac{e^{s_1 v}}{e^{s_2 u}} \right) \frac{\hat{\mathcal{Q}}(-v)\hat{\mathcal{Q}}(u)}{u-v}.
\end{aligned}
$$

where

$$\hat{\mathcal{Q}}(u) := \int_{\tilde{\sigma}}^{\infty} d\kappa\, \mathcal{Q}(\kappa) e^{\kappa u 2^{1/3}}, \quad \text{with} \quad \mathcal{Q}(\kappa) := [(\mathbb{1} - K_{\mathrm{Ai}})^{-1}_{(\tilde{\sigma},\infty)} \mathrm{Ai}](\kappa)$$

$$\hat{\mathcal{P}}(u) := -\int_{0}^{\infty} d\kappa\, e^{-\kappa u 2^{1/3}} \int_{\tilde{\sigma}}^{\infty} d\mu\, \mathcal{Q}(\mu) \mathrm{Ai}(\mu + \kappa), \quad \text{with} \quad \tilde{\sigma} := 2^{2/3}\sigma$$

$$\mathcal{C}(s_1) := 2^{-1/3} \int_{\tilde{\sigma}}^{\infty} d\kappa\ \mathcal{Q}(\kappa) \Big(\mathrm{Ai}(\kappa - 2^{-1/3} s_1) + \mathcal{Q}(\kappa + 2^{-1/3}|s_1|) - \int_{\tilde{\sigma}}^{\infty} d\lambda\ \mathcal{Q}(\lambda) K_{\mathrm{Ai}}(\kappa, \lambda - 2^{-1/3}|s_1|) \Big)$$

Alternative formulation:

$$\mathcal{K}_\sigma(s_1;s_2) = \mathcal{C}(s_1-s_2)+\int_0^\infty d\gamma \begin{pmatrix} \mathcal{A}(s_1-\gamma)\mathcal{A}(s_2-\gamma)+\mathcal{A}(-s_1-\gamma)\mathcal{A}(-s_2-\gamma) \\ -\mathcal{A}(s_1-\gamma)\mathcal{B}(s_2-\gamma)-\mathcal{B}(-s_1-\gamma)\mathcal{A}(-s_2-\gamma) \\ -\mathcal{B}(s_1-\gamma)\mathcal{A}(s_2-\gamma)-\mathcal{A}(-s_1-\gamma)\mathcal{B}(-s_2-\gamma) \\ -\mathcal{B}(s_1+\gamma)\mathcal{B}(s_2+\gamma)-\mathcal{B}(-s_1+\gamma)\mathcal{B}(-s_2+\gamma) \end{pmatrix}$$

where

$$\mathcal{A}(s_1) := \mathrm{Ai}(\sigma-s_1)+\int_{\tilde\sigma}^\infty d\kappa \int_0^\infty d\mu\ \mathcal{Q}(\kappa)\mathrm{Ai}(\kappa+\mu)\mathrm{Ai}(2^{1/3}\mu+\sigma-s_1)$$

$$\mathcal{B}(s_1) := \int_{\tilde\sigma}^\infty d\kappa\ \mathcal{Q}(\kappa)\mathrm{Ai}(2^{1/3}\kappa+s_1-\sigma)$$

(1) Discrete (space) continuous time random walks

Adler, Ferrari, van Moerbeke: arXiv:1007.1163

(2) Brownian Bridges, not only symmetric, use Riemann-Hilbert

Delvaux, Kuijlaars, Zhang: arXiv:1009.2457

(3) Brownian Bridges, symmetric Johansson: arXiv:1105.4027

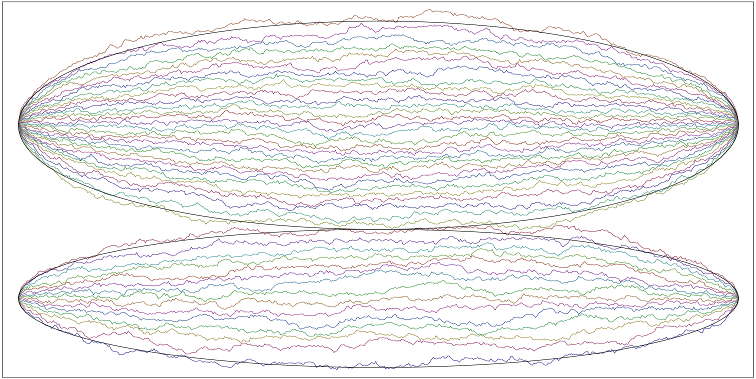
(4) Brownian Bridges, asymptotics for asymmetric (= generic) case, based on (3)

Ferrari, Vető: arXiv:1112.5002

(5) Space and time discrete: double Aztec diamond, symmetric, based on the approaches of (1) and (3)

Adler, Johansson, van Moerbeke: arXiv:1112.5532

- For the equivalence of formulations of (1) and (3), see (4).
- (3), (5): see next two talks

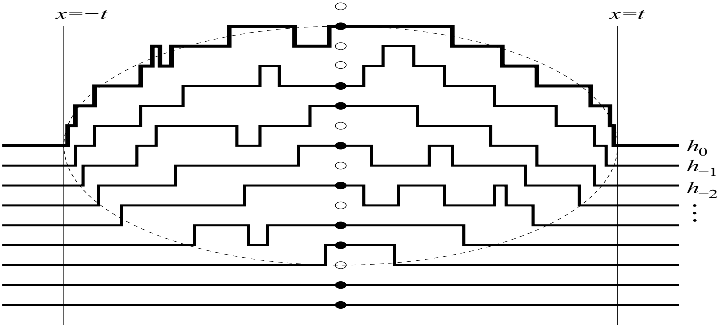


Asymmetric tacnode: two-parameter (ratio of curvatures λ and "strength of interaction" σ) family process with asymptotic kernel obtained in Ferrari, Vető: arXiv:1112.5002

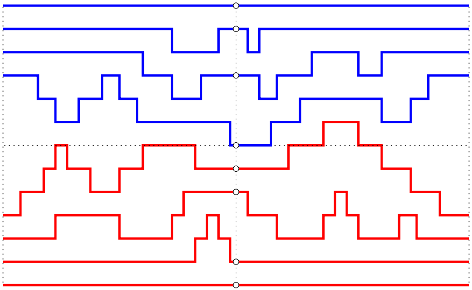
Remark: the Airy_2 and Pearcey Process have no free parameters.

- Discrete model: consider ∞ many non-intersecting continuous time random walks from $-1, -2, \ldots$.
- This is the multilayer polynuclear growth (PNG) model where the Airy$_2$ process was discovered (top layer).

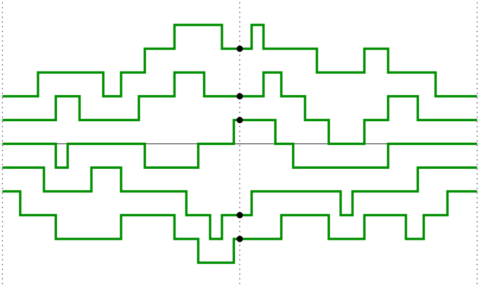
Prähofer,Spohn'02



- Discrete model: consider ∞ many non-intersecting continuous time random walks from $-1, -2, \ldots$.
- Idea: Take two multilayer PNG models, one with top layer starting at $-m-1$, the other up-side down with lower layer starting at $m+1$.



- Step 1: Let us start with only $2m+1$ walkers $\tilde{x}_k(t)$, $k \in I_m = \{-m, \ldots, m\}$, starting and leaving at $-m, -m+1, \ldots, m-1, m$.



- This point process $\tilde{\eta}(x) := \sum_{k \in I_m} \delta_{x_k(0),x}$ is determinantal with kernel $\widetilde{K}_m$. How to get the kernel?

- In Step 1, to get the kernel of the $2m+1$ random walks we use:

(a) Karlin-Mc Gregor formula

(b) Orthogonalize in the "Fourier representation"

(c) Transform a Toeplitz determinant into a Fredholm determinant by the Borodin-Okounkov Formula

One-time kernel ($t_1 = t_2 = 0$).

$$\widetilde{K}_m(x,y) = \frac{V_m}{(2\pi\mathrm{i})^2} \oint_{\Gamma_0} dz \oint_{\Gamma_{0,z}} dw \frac{e^{T(z-z^{-1})}}{e^{T(w-w^{-1})}} \frac{w^{y-m-1}}{z^{x-m}} \frac{H_{2m+1}(w)H_{2m+1}(z^{-1})}{z-w}$$
$$+ \frac{V_m}{(2\pi\mathrm{i})^2} \oint_{\Gamma_0} dw \oint_{\Gamma_{0,w}} dz \frac{e^{T(w-w^{-1})}}{e^{T(z-z^{-1})}} \frac{w^{y+m}}{z^{x+m+1}} \frac{H_{2m+1}(z)H_{2m+1}(w^{-1})}{w-z}$$
$$+ \frac{V_m}{2\pi\mathrm{i}} \oint_{\Gamma_0} dz \frac{1}{z^{x-y+1}} H_{2m+1}(z^{-1})H_{2m+1}(z),$$

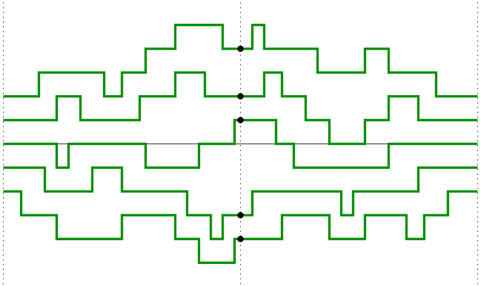
with $V_m := 1/(H_{2m+1}(0)H_{2m+2}(0))$. The function H_n is itself the Fredholm determinant

$$H_n(z^{-1}) := \det(\mathbb{1} - \mathcal{K}(z^{-1}))_{\ell^2(\{2m+1,2m+2,\dots\})}$$

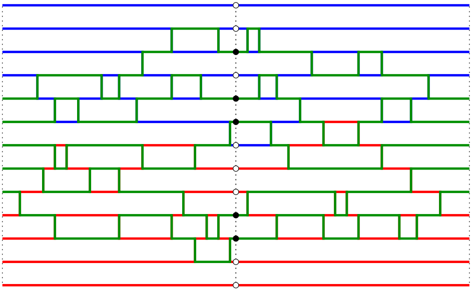
of the kernel

$$\mathcal{K}(z^{-1})_{k,\ell} := \frac{(-1)^{k+\ell}}{(2\pi\mathrm{i})^2} \oint_{\Gamma_0} du \oint_{\Gamma_{0,u}} dv \frac{u^\ell}{v^{k+1}} \frac{1}{v-u} \frac{u-z}{v-z} \frac{e^{2T(u-u^{-1})}}{e^{2T(v-v^{-1})}},$$

- Step 2: Particle-hole transformation



- Step 2: Particle-hole transformation



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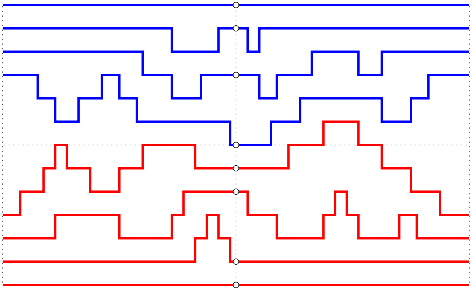


- Step 2: Particle-hole transformation
- The point process $\tilde{\eta}(x) := \sum_{k=-m}^{m} \delta_{\tilde{x}_k(0),x}$ is determinantal with kernel $\widetilde{K}_m$
- We want to characterize the complementary point process,

$$\eta(x) := \sum_{k \notin \{-m,\dots,m\}} \delta_{x_k(0),x}.$$

It is determinantal with kernel $K_m = \mathbb{1} - \widetilde{K}_m$

Borodin, Olshanski, Okounkov '00.

One-time kernel ($t_1 = t_2 = 0$).

$$K_m(x,y) = -\frac{V_m}{(2\pi \mathrm{i})^2}\oint_{\Gamma_0} dz \oint_{\Gamma_{0,z}} dw \frac{e^{T(z-z^{-1})}}{e^{T(w-w^{-1})}} \frac{w^{y-m-1}}{z^{x-m}} \frac{H_{2m+1}(w)H_{2m+1}(z^{-1})}{z-w}$$
$$-\frac{V_m}{(2\pi \mathrm{i})^2}\oint_{\Gamma_0} dw \oint_{\Gamma_{0,w}} dz \frac{e^{T(w-w^{-1})}}{e^{T(z-z^{-1})}} \frac{w^{y+m}}{z^{x+m+1}} \frac{H_{2m+1}(z)H_{2m+1}(w^{-1})}{w-z}$$
$$-\mathbb{1}_{[x\neq y]}\frac{V_m}{2\pi \mathrm{i}}\oint_{\Gamma_0} dz \frac{1}{z^{x-y+1}} H_{2m+1}(z^{-1})H_{2m+1}(z),$$

with $V_m := 1/(H_{2m+1}(0)H_{2m+2}(0))$. The function H_n is itself the Fredholm determinant

$$H_n(z^{-1}) := \det(\mathbb{1} - \mathcal{K}(z^{-1}))_{\ell^2(\{2m+1,2m+2,\dots\})}$$

of the kernel

$$\mathcal{K}(z^{-1})_{k,\ell} := \frac{(-1)^{k+\ell}}{(2\pi \mathrm{i})^2}\oint_{\Gamma_0} du \oint_{\Gamma_{0,u}} dv \frac{u^\ell}{v^{k+1}} \frac{1}{v-u} \frac{u-z}{v-z} \frac{e^{2T(u-u^{-1})}}{e^{2T(v-v^{-1})}},$$

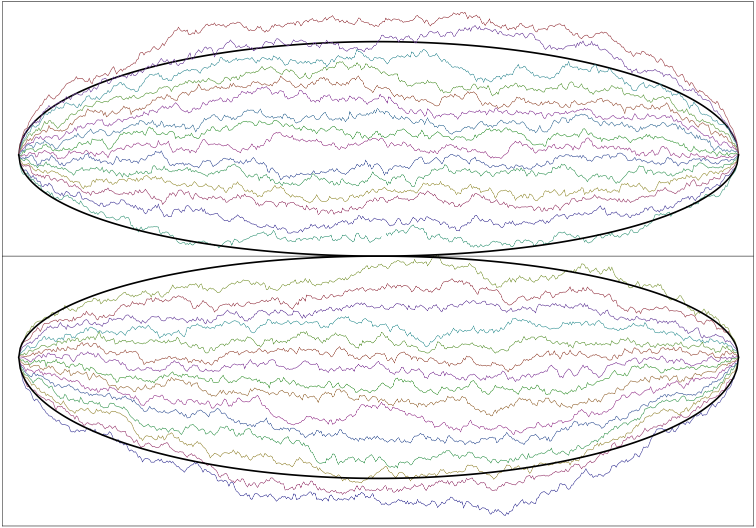
- Step 3: Asymptotic analysis

(a) First we "reshaped" the kernel K_m so that all the terms would have a limit as $T \to \infty$ from the naïve steepest descent analysis, though not easy to carry out rigorously on the complex integral representations.

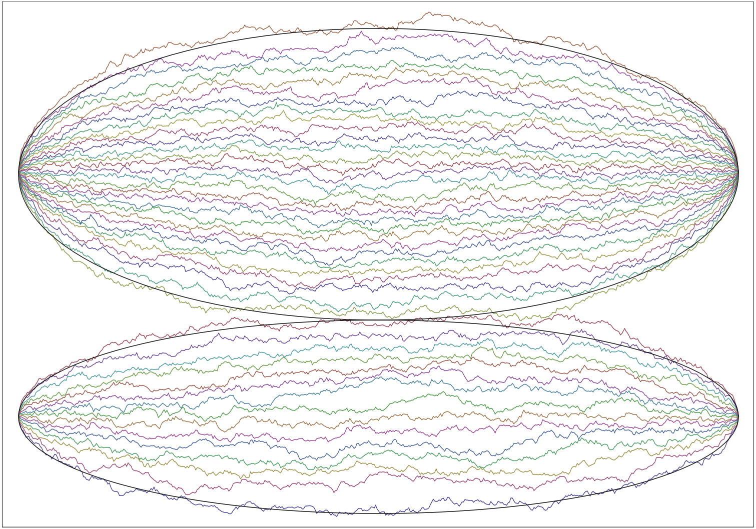
(b) Rewrite all complex integrals in the kernel K_m in terms of Bessel function.

(c) Limit $T \to \infty$ and get Airy functions instead

- Rewriting of the Airy functions as complex integrals, leads to the result of the naïve steepest descent analysis.



The asymmetric tacnode



Ferrari, Vető: arXiv:1112.5002