Vicious Brownian motion, O'Connell's process, and equilibrium Toda lattice

Makoto KATORI (Chuo Univ., Tokyo)

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1. Introduction

The present talk is based on the papers

[**O'Connell 12a**] N. O'Connell: Directed polymers and the quantum Toda lattice. *Ann. Probab.* **40**, 437-458 (2012)

[**Baudoin-O'Connell 11**] F. Baudoin, N. O'Connell: Exponential functionals of Brownian motion and class-one Whittaker functions. *Ann. Inst. H. Poincaré, Probab. Statist.*, **47**, 1096-1120 (2011)

[**O'Connell 12b**] O'Connell, N.: Whittaker functions and related stochastic processes. arXiv:math.PR/1201.4849

and on my work

[**K11**] M. Katori: O'Connell's process as a vicious Brownian motion. *Phys. Rev.* **E84**, 061144/1-11 (2011); arXiv:math-ph/1110.1845

[**K12**] M. Katori: Survival probability of mutually killing Brownian motions and the O'Connell process. *J. Stat. Phys.* (in press) DOI 10.1007/s10955-012-0472-3 ; arXiv:math.PR/1112.4009

The O'Connell process

N-particle diffusion process in one dimension
with parameters (drifts) $\boldsymbol{\nu} = (\nu_1, \dots, \nu_N) \in \mathbb{R}^N$.

- the infinitesimal generator

$$\begin{aligned} \mathcal{L}_{\boldsymbol{\nu}} &= -(\psi_{\boldsymbol{\nu}})^{-1}\Big(\mathcal{H} + |\boldsymbol{\nu}|^2\Big)\psi_{\boldsymbol{\nu}} \\ &= \frac{1}{2}\Delta + \nabla \log \psi_{\boldsymbol{\nu}}(\boldsymbol{x}) \cdot \nabla, \end{aligned}$$

where

$\psi_{\boldsymbol{\nu}}(\boldsymbol{x})$ = the class-one Whittaker function,
$\mathcal{H}$ = the Hamiltonian of the $\mathrm{GL}(N, \mathbb{R})$-quantum Toda lattice.

- multi-dimensional (many-particle) extension of the Matsumoto-Yor process
- A geometric lifting ('inverse tropical analogue')
  of Dyson's Brownian motion model with $\beta = 2$
  (eigenvalue process of GUE = noncolliding BM)

2. Whittaker function and survival probability

$N \in \{2, 3, \dots\}, \boldsymbol{x} = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N, t \in [0, \infty)$.
Schrödinger equation for identical N particles (mass m, charge q) of open Toda lattice in an external vector potential $\boldsymbol{A}$

$$i\hbar \frac{\partial}{\partial t}\Phi(t, \boldsymbol{x}) = \left(-\frac{\hbar^2}{2m}\Delta - i\hbar \frac{q}{2mc}\boldsymbol{A} \cdot \nabla + \frac{1}{\xi^2}\sum_{j=1}^{N-1} e^{-(x_{j+1}-x_j)/\xi} \right) \Phi(t, \boldsymbol{x}),$$

where ξ = the range of interaction.

- natural units c = light velocity $= 1, \hbar$ = Planck const.$/(2\pi) = 1$.
  mass $m = 1$, charge $q = 1$.

- We set

$$t \quad \Longrightarrow \quad -it \quad \text{(Wick rotation)}$$
$$\frac{i}{2}\boldsymbol{A} \quad \Longrightarrow \quad \frac{1}{\xi}\boldsymbol{\nu}, \quad \Phi(t, \boldsymbol{x}) \quad \Longrightarrow \quad u(t, \boldsymbol{x}).$$

  Then we have a diffusion equation

$$\frac{\partial}{\partial t}u(t, \boldsymbol{x}) = \left(\frac{1}{2}\Delta + \frac{1}{\xi}\boldsymbol{\nu} \cdot \nabla - \frac{1}{\xi^2}\sum_{j=1}^{N-1} e^{-(x_{j+1}-x_j)/\xi} \right) u(t, \boldsymbol{x}).$$

By drift transformation $u \mapsto \widehat{u}$:

$$u(t,\boldsymbol{x}) = \exp\left(-t\frac{|\boldsymbol{\nu}|^2}{2\xi^2} - \frac{\boldsymbol{\nu}}{\xi}\cdot\boldsymbol{x}\right)\widehat{u}(t,\boldsymbol{x}),$$

the equation is transformed to

$$\frac{\partial}{\partial t}\widehat{u}(t,\boldsymbol{x}) = -\mathcal{H}\widehat{u}(t,\boldsymbol{x})$$

with the Toda lattice 'Hamiltonian'

$$\mathcal{H} = -\frac{1}{2}\Delta + \frac{1}{\xi^2}\sum_{j=1}^{N-1} e^{-(x_{j+1}-x_j)/\xi}.$$

- generalized eigenvalue/eigenfunction problem

$$\mathcal{H}\Psi_\lambda(\boldsymbol{x}) = \lambda\Psi_\lambda(\boldsymbol{x})$$

is solved as

$$\lambda = -\frac{1}{2}\sum_{j=1}^{N}\mu_j^2, \quad \boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_N) \in \mathbb{C}^N$$

and

$$\Psi_\lambda(\boldsymbol{x}) = m(\boldsymbol{x}/\xi, \boldsymbol{\mu})$$

fundamental Whittaker function, which behaves as

$$m(\boldsymbol{x}/\xi, \boldsymbol{\mu}) \simeq e^{\boldsymbol{\mu}\cdot\boldsymbol{x}/\xi} \quad \text{as} \quad \xi \to 0.$$

- class-one Whittaker function is defined by

$$\psi_{\boldsymbol{\mu}}(\boldsymbol{x}/\xi) = \prod_{1 \leq j < \ell \leq N} \frac{\pi}{\sin\pi(\mu_\ell - \mu_j)} \sum_{\sigma \in \mathfrak{S}_N} \mathrm{sgn}(\sigma) m(\boldsymbol{x}/\xi, \sigma(\boldsymbol{\mu}))$$

for $\boldsymbol{\mu} \in \mathbb{W}_N \equiv \{\boldsymbol{\mu} \in \mathbb{R}^N : \mu_1 < \mu_2 < \cdots < \mu_N\}$
(Weyl chamber of type A_{N-1}).

Then

$$\psi_{\sigma(\boldsymbol{\mu})}(\boldsymbol{x}/\xi) = \psi_{\boldsymbol{\mu}}(\boldsymbol{x}/\xi), \quad \forall\sigma \in \mathfrak{S}_N,$$

where $\mathfrak{S}_N$ is the symmetry group, $\sigma(\boldsymbol{\mu}) = (\mu_{\sigma(1)}, \dots, \mu_{\sigma(N)})$.
(the alternating sum formula of [**Baudoin-O'Connell 11**]).

$$\frac{\partial}{\partial t}\widehat{u}(t,\boldsymbol{x}) = -\mathcal{H}\widehat{u}(t,\boldsymbol{x})$$

$$\Longrightarrow \quad \text{class-one Whittaker function solutions}$$

$$\left\{\exp\left(\frac{t}{2\xi^2}\sum_{j=1}^{N}\mu_j^2\right)\psi_{\boldsymbol{\mu}}(\boldsymbol{x}/\xi)\ :\ \boldsymbol{\mu}\in\mathbb{C}^N\right\}$$

$$\frac{\partial}{\partial t}u(t,\boldsymbol{x}) = -\left(\mathcal{H}+\frac{1}{\xi}\boldsymbol{\nu}\cdot\nabla\right)u(t,\boldsymbol{x})$$

$$\Longrightarrow \quad u(t,\boldsymbol{x}) = \exp\left\{-t\frac{|\boldsymbol{\nu}|^2}{2\xi^2}-\boldsymbol{\nu}\cdot\frac{\boldsymbol{x}}{\xi}\right\}u(t,\boldsymbol{x})$$

$$= \exp\left\{\frac{t}{2\xi^2}\left(\sum_{j=1}^{N}\mu_j^2-|\boldsymbol{\nu}|^2\right)\right\}e^{-\boldsymbol{\nu}\cdot\boldsymbol{x}/\xi}\psi_{\boldsymbol{\mu}}(\boldsymbol{x}/\xi).$$

Stationary solution is obtained by setting $\boldsymbol{\mu}=\boldsymbol{\nu}$,

$$u(\boldsymbol{x}) = e^{-\boldsymbol{\nu}\cdot\boldsymbol{x}/\xi}\psi_{\boldsymbol{\nu}}(\boldsymbol{x}/\xi).$$

On the other hand,

killing term

$$\frac{\partial}{\partial t}u(t,\boldsymbol{x}) = -\left(\mathcal{H} - \frac{1}{\xi}\boldsymbol{\nu}\cdot\nabla\right)u(t,\boldsymbol{x})$$

$$\Longleftrightarrow \quad \frac{\partial}{\partial t}u(t,\boldsymbol{x}) = \frac{1}{2}\Delta u(t,\boldsymbol{x}) + \frac{1}{\xi}\boldsymbol{\nu}\cdot\nabla u(t,\boldsymbol{x}) - \underline{\frac{1}{\xi^2}\sum_{j=1}^{N-1} e^{-(x_{j+1}-x_j)/\xi}u(t,\boldsymbol{x})}$$

stationary solution given by Feynman-Kac formula
for $\boldsymbol{x}, \boldsymbol{\nu} \in \mathbb{W}_N$

$$u(t,\boldsymbol{x}) = \mathbb{E}^{\boldsymbol{x}}\left[\exp\left(-\frac{1}{\xi^2}\sum_{j=1}^{N-1}\int_0^\infty e^{-\{B_{j+1}^{\nu_{j+1}}(s)-B_j^{\nu_j}(s)\}/\xi}ds\right)\right],$$

where

$$\boldsymbol{B}^{\boldsymbol{\nu}}(t) = (B_1^{\nu_1}(t), B_2^{\nu_2}, \dots, B_N^{\nu_N}), \quad t \in [0,\infty)$$

N-dim. Brownian motion (BM) with drift $\boldsymbol{\nu}$

$\mathbb{E}^{\boldsymbol{x}}[\,\cdot\,]$ = expectation w.r.t. the BM starting from $\boldsymbol{x}$

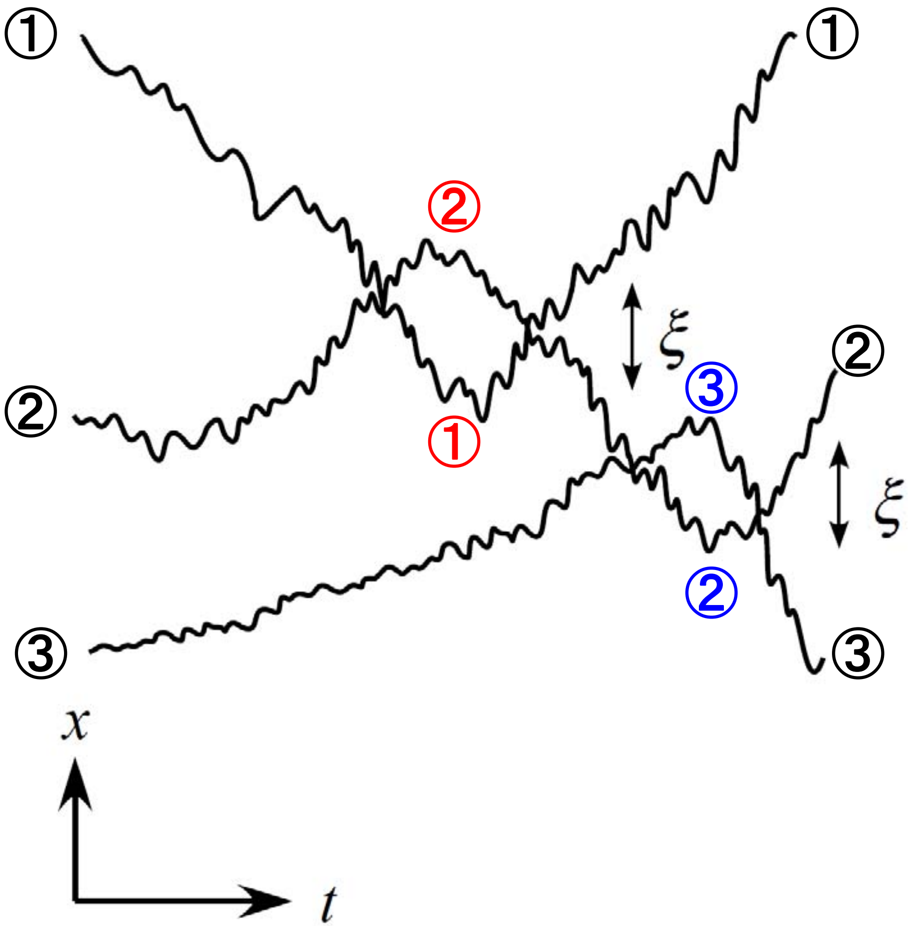
This solution is interpreted as the survival probability

$$\mathrm{P}_{\boldsymbol{\nu}}\left(\text{survival}\middle|\boldsymbol{x}/\xi\right) = \mathbb{E}^{\boldsymbol{x}}\left[\exp\left(-\frac{1}{\xi^2}\sum_{j=1}^{N-1}\int_0^\infty e^{-(B_{j+1}^{\nu_{j+1}}(s)-B_j^{\nu_j}(s))/\xi}ds\right)\right],$$

$\boldsymbol{x}, \boldsymbol{\nu} \in \mathbb{W}_N$.

The probability that all N particles survive forever
in the N-particle system of one-dim. BMs with drift
with the mutually killing term (with a range ξ)

$$-\frac{1}{\xi^2}\sum_{j=1}^{N-1} e^{-(x_{j+1}-x_j)/\xi}.$$



The probability that all N particles survive forever in the N-particle system of one-dim. BMs with drift with the mutually killing term (with a range ξ)

$$-\frac{1}{\xi^2}\sum_{j=1}^{N-1} e^{-(x_{j+1}-x_j)/\xi}.$$

Compare the two solution by looking at the asymptotics

$$\begin{cases} \text{the 'Whittaker solution'} \\ \qquad e^{-\boldsymbol{\nu}\cdot\boldsymbol{x}/\xi}\psi_{\boldsymbol{\nu}}(\boldsymbol{x}/\xi) \quad \underset{\boldsymbol{x},\boldsymbol{\nu}\in\mathbb{W}_N,\ \xi\to 0}{\Longrightarrow} \quad \prod_{1\le j<\ell\le N} \frac{\pi}{\sin\pi(\nu_\ell-\nu_j)} \\ \text{the 'Feynman-Kac solution'} \\ \qquad \mathrm{P}_{\boldsymbol{\nu}}\left(\text{survival}\,\middle|\,\boldsymbol{x}/\xi\right) \quad \underset{\boldsymbol{x},\boldsymbol{\nu}\in\mathbb{W}_N,\ \xi\to 0}{\Longrightarrow} \quad 1 \end{cases}$$

Equality 1

$$\psi_{\boldsymbol{\nu}}(\boldsymbol{x}/\xi) = e^{\boldsymbol{\nu}\cdot\boldsymbol{x}/\xi} \prod_{1\le j<\ell\le N} \frac{\pi}{\sin\pi(\nu_\ell-\nu_j)} \times \mathrm{P}_{\boldsymbol{\nu}}\left(\text{survival}\,\middle|\,\boldsymbol{x}/\xi\right), \quad \boldsymbol{x},\boldsymbol{\nu}\in\mathbb{W}_N$$

[O'Connell 12b]

For $\boldsymbol{\nu} = 0$

$$\mathrm{P}_{\boldsymbol{\nu}}\left(\text{survival up to time } t \,\middle|\, \boldsymbol{x}/\xi\right) \underset{t \to \infty}{\simeq} c(N) t^{-N(N-1)/4} \psi_0(\boldsymbol{x}/\xi),$$

with a constant $c(N) > 0$. **[K12]**

3. Givental's integral

- Consider a lower triangular array with size N

$$\boldsymbol{T} = \Big(T_{j,k}, 1 \le k \le j \le N\Big) = \begin{pmatrix} T_{1,1} & & & \\ T_{2,1} & T_{2,2} & & \\ & \cdots & \cdots & \\ T_{N,1} & T_{N,2} & \cdots & T_{N,N} \end{pmatrix} \in \mathbb{R}^{N(N+1)/2}$$

We write the rows of $\boldsymbol{T}$ as

$$\begin{aligned} \boldsymbol{T}^{(1)} &= (T_{1,1}) \in \mathbb{R} \\ \boldsymbol{T}^{(2)} &= (T_{2,1}, T_{2,2}) \in \mathbb{R}^2 \\ &\cdots \\ \boldsymbol{T}^{(N)} &= (T_{N,1}, T_{N,2}, \cdots, T_{N,N}) \in \mathbb{R}^N. \end{aligned}$$

and define the type of $\boldsymbol{T}$ as

$$\operatorname{type} \boldsymbol{T} = \left(T_{1,1}, T_{2,1} + T_{2,2} - T_{1,1}, \ldots, \sum_{j=1}^{N} T_{N,j} - \sum_{j=1}^{N-1} T_{N-1,j}\right) \in \mathbb{R}^N.$$

Let

$$\mathcal{F}(\boldsymbol{T}/\xi) = -\sum_{j=1}^{N}\sum_{k=1}^{j}\left\{e^{-(T_{j,k}-T_{j+1,k})/\xi} + e^{-(T_{j+1,k+1}-T_{j,k})/\xi}\right\}.$$

Then Givental (1997) gives the integral expression

$$\psi_{\boldsymbol{\nu}}(\boldsymbol{x}/\xi) = \xi^{-N(N-1)/2}\int_{\boldsymbol{T}^{(N)}=\boldsymbol{x}} e^{\boldsymbol{\nu}\cdot\mathrm{type}\,\boldsymbol{T}/\xi} e^{\mathcal{F}(\boldsymbol{T}/\xi)} d\boldsymbol{T},$$

where

$$\int_{\boldsymbol{T}^{(N)}=\boldsymbol{x}}(\cdots)d\boldsymbol{T} \equiv \prod_{j=1}^{N-1}\prod_{k=1}^{j}\int_{-\infty}^{\infty} dT_{j,k}\prod_{\ell=1}^{N}\delta(T_{N,\ell}-x_\ell)(\cdots).$$

4. Vicious BM (Random matrix theory) limit

By the alternating sum formula [**Baudwin-O'Connell 11**], for $\boldsymbol{x}, \boldsymbol{\nu} \in \mathbb{W}_N$

$$\begin{aligned}
\psi_{\xi\boldsymbol{\nu}}(\boldsymbol{x}/\xi) &= \prod_{1 \le j < \ell \le N} \frac{\pi}{\sin \pi\xi(\nu_\ell - \nu_j)} \sum_{\sigma \in \mathfrak{S}_N} \mathrm{sgn}(\sigma) m(\boldsymbol{x}/\xi, \xi\sigma(\boldsymbol{\nu})) \\
&\simeq \xi^{-N(N-1)/2} \frac{1}{h(\boldsymbol{\nu})} \sum_{\sigma \in \mathfrak{S}_N} \mathrm{sgn}(\sigma) e^{\sigma(\boldsymbol{\nu}) \cdot \boldsymbol{x}} \quad \text{as} \quad \xi \to 0 \\
&= \xi^{-N(N-1)/2} \frac{1}{h(\boldsymbol{\nu})} \det_{1 \le j, \ell \le N} \left[e^{\nu_j x_\ell} \right].
\end{aligned}$$

where we have used the fact $\Gamma(z) \simeq 1/z$ as $z \to 0$ and

$$h(\boldsymbol{\nu}) \equiv \prod_{1 \le j < \ell \le N} (\nu_\ell - \nu_j) \quad \text{(Vandermonde determinant)}$$

Equality 1

$$\psi_{\boldsymbol{\nu}}(\boldsymbol{x}/\xi) = e^{\boldsymbol{\nu}\cdot\boldsymbol{x}/\xi} \prod_{1\leq j<\ell\leq N} \frac{\pi}{\sin\pi(\nu_\ell - \nu_j)} \times \mathrm{P}_{\boldsymbol{\nu}}\Big(\text{survival}\,\Big|\,\boldsymbol{x}/\xi\Big), \quad \boldsymbol{x},\boldsymbol{\nu}\in\mathbb{W}_N$$

[O'Connell 12b]

$$\lim_{\xi\to 0} \mathrm{P}_{\xi\boldsymbol{\nu}}\Big(\text{survival}\,\Big|\,\boldsymbol{x}/\xi\Big) = e^{-\boldsymbol{\nu}\cdot\boldsymbol{x}} \det_{1\leq j,\ell\leq N}\Big[e^{\nu_j x_\ell}\Big],$$

$$\text{for} \quad \boldsymbol{\nu},\boldsymbol{x}\in\mathbb{W}_N.$$

Let $T_{\mathbb{W}}$ = the first exit time from $\mathbb{W}_N$ of the BM with drift $\boldsymbol{\nu}$.
Then the above is equal to $\mathbb{P}_{\boldsymbol{\nu}}^{\boldsymbol{x}}\Big[T_{\mathbb{W}}=\infty\Big]$, $\boldsymbol{\nu},\boldsymbol{x}\in\mathbb{W}_N$.

Equality 2

$$\mathbb{P}_{\boldsymbol{\nu}}^{\boldsymbol{x}}\Big[T_{\mathbb{W}}=\infty\Big] = e^{-\boldsymbol{\nu}\cdot\boldsymbol{x}} \det_{1\leq j,\ell\leq N}\Big[e^{\nu_j x_\ell}\Big], \quad \boldsymbol{\nu},\boldsymbol{x}\in\mathbb{W}_N.$$

[**Biane, Bougerol, O'Connell 05**] : Duke Math J. **130** (2005) 127-167
[**O'Connell 12b**]

By Givental's integral representation of $\psi_{\boldsymbol{\nu}}(\boldsymbol{x}/\xi)$

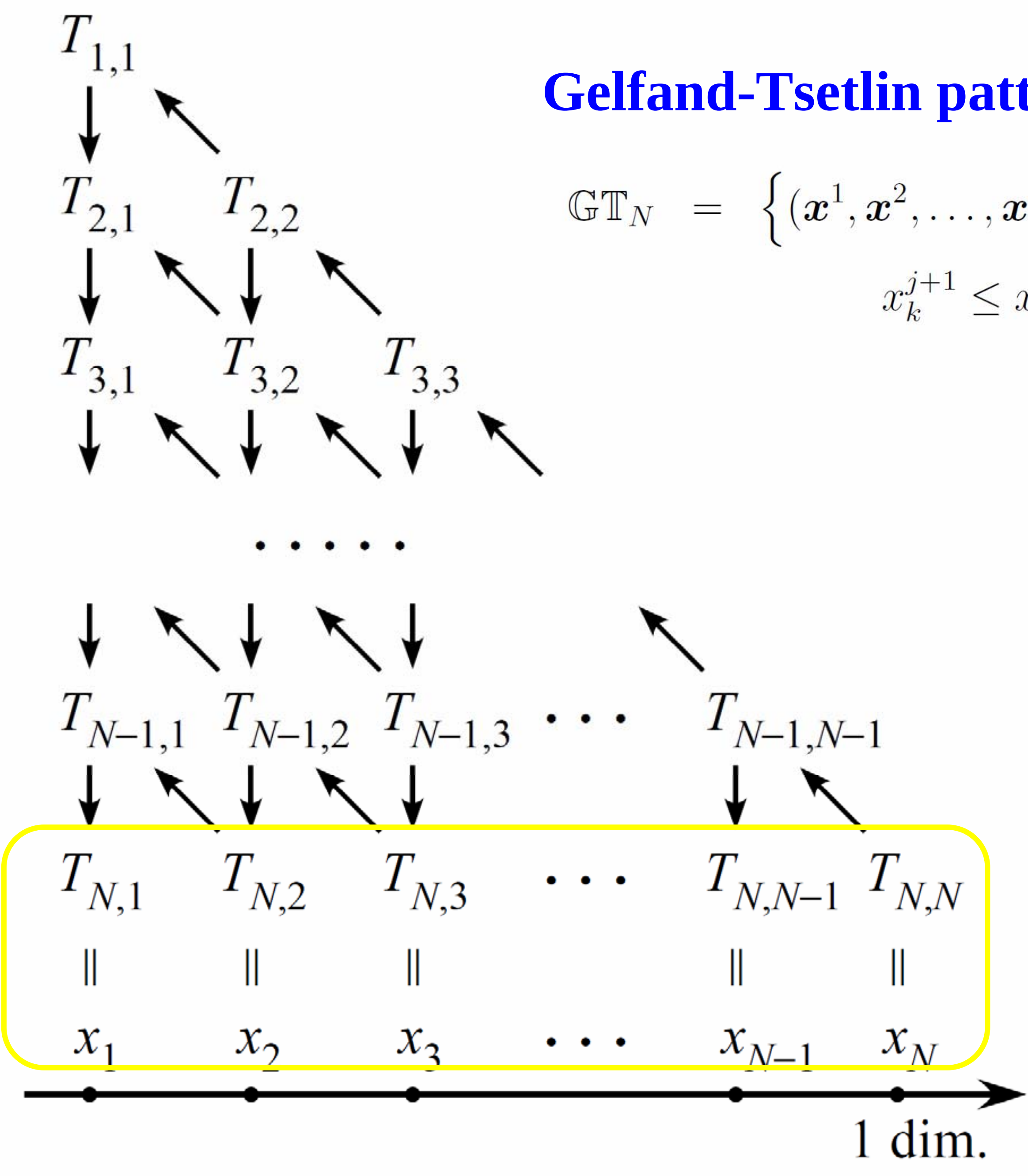
$$\begin{aligned}\lim_{\xi\to 0}\xi^{N(N-1)/2}\psi_{\xi\boldsymbol{\nu}}(\boldsymbol{x}/\xi) &= \lim_{\xi\to 0}\int_{\boldsymbol{T}^{(N)}=\boldsymbol{x}} e^{\boldsymbol{\nu}\cdot\mathrm{type}\,\boldsymbol{T}} e^{\mathcal{F}(\boldsymbol{T}/\xi)}\, d\boldsymbol{T} \\ &= \int_{\mathbb{GT}_N:\boldsymbol{T}^{(N)}=\boldsymbol{x}} e^{\boldsymbol{\nu}\cdot\mathrm{type}\,\boldsymbol{T}}\, d\boldsymbol{T},\end{aligned}$$

where

$$\begin{aligned}\mathbb{GT}_N &= \Big\{(\boldsymbol{x}^1,\boldsymbol{x}^2,\ldots,\boldsymbol{x}^N)\in\overline{\mathbb{W}}_1\times\overline{\mathbb{W}}_2\times\cdots\times\overline{\mathbb{W}}_N : \\ &\qquad x_k^{j+1}\le x_k^j\le x_{k+1}^{j+1}, 1\le k\le j\le N-1\Big\},\end{aligned}$$

with $\overline{\mathbb{W}}_j=\{\boldsymbol{x}=(x_1,\ldots,x_j)\in\mathbb{R}^j : x_1\le\cdots\le x_j\}, 1\le j\le N.$

Gelfand-Tsetlin pattern



$$\mathbb{GT}_N = \Big\{(\boldsymbol{x}^1, \boldsymbol{x}^2, \dots, \boldsymbol{x}^N) \in \overline{\mathbb{W}}_1 \times \overline{\mathbb{W}}_2 \times \cdots \times \overline{\mathbb{W}}_N : x_k^{j+1} \le x_k^j \le x_{k+1}^{j+1}, 1 \le k \le j \le N-1\Big\}$$

conditioned on the bottom values

$$\boldsymbol{T}^{(N)} = \boldsymbol{x}$$

By Givental's integral representation of $\psi_{\boldsymbol{\nu}}(\boldsymbol{x}/\xi)$

$$\begin{aligned}
\lim_{\xi\to 0}\xi^{N(N-1)/2}\psi_{\xi\boldsymbol{\nu}}(\boldsymbol{x}/\xi) &= \lim_{\xi\to 0}\int_{\boldsymbol{T}^{(N)}=\boldsymbol{x}} e^{\boldsymbol{\nu}\cdot\mathrm{type}\,\boldsymbol{T}} e^{\mathcal{F}(\boldsymbol{T}/\xi)}\, d\boldsymbol{T} \\
&= \int_{\mathbb{GT}_N:\boldsymbol{T}^{(N)}=\boldsymbol{x}} e^{\boldsymbol{\nu}\cdot\mathrm{type}\,\boldsymbol{T}}\, d\boldsymbol{T},
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{GT}_N &= \Big\{(\boldsymbol{x}^1,\boldsymbol{x}^2,\ldots,\boldsymbol{x}^N)\in\overline{\mathbb{W}}_1\times\overline{\mathbb{W}}_2\times\cdots\times\overline{\mathbb{W}}_N : \\
&\qquad x_k^{j+1}\le x_k^j\le x_{k+1}^{j+1}, 1\le k\le j\le N-1\Big\},
\end{aligned}$$

with $\overline{\mathbb{W}}_j=\{\boldsymbol{x}=(x_1,\ldots,x_j)\in\mathbb{R}^j : x_1\le\cdots\le x_j\}, 1\le j\le N.$

Equality 3

$$\int_{\mathbb{GT}_N:\boldsymbol{T}^{(N)}=\boldsymbol{x}} e^{\boldsymbol{\nu}\cdot\mathrm{type}\,\boldsymbol{T}}\, d\boldsymbol{T} = \frac{1}{h(\boldsymbol{\nu})}\det_{1\le j,\ell\le N}\left[e^{\nu_j x_\ell}\right]$$

Harish-Chandra-Itzykson-Zuber (HCIZ) integral

$$\frac{1}{h(\boldsymbol{\nu})h(\boldsymbol{x})} \det_{1 \le j,\ell \le N} \left[e^{\nu_j x_\ell} \right] = \int_{\mathbb{U}(N)} \exp \left[\mathrm{tr}(\Lambda_{\boldsymbol{\nu}} U^* \Lambda_{\boldsymbol{x}} U) \right] dU,$$

where

$$\Lambda_{\boldsymbol{\nu}} = \mathrm{diag}(\nu_1, \cdots, \nu_N), \quad \Lambda_{\boldsymbol{x}} = \mathrm{diag}(x_1, \dots, x_N),$$
$$\int_{\mathbb{U}(N)} (\cdots) dU = \text{integral over } \mathbb{U}(N) \text{ w.r.t. the Haar measure.}$$

Since

$$H \equiv U^* \Lambda_{\boldsymbol{x}} U = N \times N \text{ Hermitian matrix} \in \mathbb{H}(N) \text{ with eigenvalues } \boldsymbol{x} = (x_1, \dots, x_N) \in \mathbb{W}_N,$$

if we set

$$a_j = H_{jj}, \quad 1 \le j \le N : \text{diagonal elements of } H,$$

we see

$$\mathrm{tr}(\Lambda_{\boldsymbol{\nu}} U^* \Lambda_{\boldsymbol{x}} U) = \boldsymbol{\nu} \cdot \boldsymbol{a}.$$

Then

HCIZ integral $\Longleftrightarrow$

$$\frac{1}{h(\boldsymbol{\nu})} \det_{1 \le j,\ell \le N} \left[e^{\nu_j x_\ell} \right] = h(\boldsymbol{x}) \int_{\mathbb{R}^N} d\boldsymbol{a}\, e^{\boldsymbol{\nu} \cdot \boldsymbol{a}} \int_{\substack{\mathbb{H}(N):\text{eigenvalues}=\boldsymbol{x} \\ \text{diagonals}=\boldsymbol{a}}} dH.$$

Equality 3

$$\int_{\mathbb{GT}_N : \boldsymbol{T}^{(N)} = \boldsymbol{x}} e^{\boldsymbol{\nu} \cdot \mathrm{type}\, \boldsymbol{T}} \, d\boldsymbol{T} = \frac{1}{h(\boldsymbol{\nu})} \det_{1 \le j, \ell \le N} \left[e^{\nu_j x_\ell} \right]$$

$$h(\boldsymbol{x}) \int_{\mathbb{R}^N} d\boldsymbol{a}\, e^{\boldsymbol{\nu} \cdot \boldsymbol{a}} \int_{\substack{\mathbb{H}(N):\text{eigenvalues}=\boldsymbol{x} \\ \text{diagonals}=\boldsymbol{a}}} dH$$

$$= \int_{\mathbb{GT}_N : \boldsymbol{T}^{(N)} = \boldsymbol{x}} e^{\boldsymbol{\nu} \cdot \mathrm{type}\, \boldsymbol{T}} \, d\boldsymbol{T}$$

$$= \int_{\mathbb{R}^N} d\boldsymbol{a}\, e^{\boldsymbol{\nu} \cdot \boldsymbol{a}} \int_{\substack{\mathbb{GT}_N : \boldsymbol{T}^{(N)} = \boldsymbol{x} \\ \mathrm{type}\, \boldsymbol{T} = \boldsymbol{a}}} d\boldsymbol{T}.$$

Hermite matrix H	$\Longleftrightarrow$	lower triangular array $\boldsymbol{T}$
eigenvalues $\boldsymbol{x} = (x_1, \ldots, x_N)$	$\Longleftrightarrow$	$\boldsymbol{T}^{(N)} = (T_{N,1}, \ldots, T_{N,N})$
diagonal elements $\boldsymbol{a} = (H_{1,1}, \ldots, H_{N,N})$	$\Longleftrightarrow$	type $\boldsymbol{T}$

[O'Connell 12a, O'Connell 12b]

5. Geometric lifting of Dyson model

Dyson's Brownian motion model (GUE, $\beta = 2$)
transition probability density from $\boldsymbol{x} \in \mathbb{W}_N$ to $\boldsymbol{y} \in \mathbb{W}_N$ with time duration $t \geq 0$

$$p(t, \boldsymbol{y}|\boldsymbol{x}) = \frac{h(\boldsymbol{y})}{h(\boldsymbol{x})} \det_{1 \leq j, \ell \leq N} \left[\frac{1}{\sqrt{2\pi t}} e^{-(x_j - y_\ell)^2/2t} \right]$$

h-transform by the Vandermonde determinant
of the Karlin-McGregor-Lindström-Gessel-Viennot determinant

$\boldsymbol{\nu}, \boldsymbol{x}, \boldsymbol{y} \in \mathbb{W}_N$

$$P_{\boldsymbol{\nu}}(t, \boldsymbol{y}|\boldsymbol{x}) = \frac{\mathrm{P}_{\boldsymbol{\nu}}\left(\text{survival}\middle|\boldsymbol{y}/\xi\right)}{\mathrm{P}_{\boldsymbol{\nu}}\left(\text{survival}\middle|\boldsymbol{x}/\xi\right)} e^{-t|\boldsymbol{\nu}|^2/(2\xi^2) - \boldsymbol{\nu}\cdot(\boldsymbol{x}-\boldsymbol{y})/\xi} Q(t, \boldsymbol{y}|\boldsymbol{x})$$

Equality 1

$$= e^{-t|\boldsymbol{\nu}|^2/(2\xi^2)} \frac{\psi_{\boldsymbol{\nu}}(\boldsymbol{y}/\xi)}{\psi_{\boldsymbol{\nu}}(\boldsymbol{x}/\xi)} Q(t, \boldsymbol{y}|\boldsymbol{x}),$$

$\boldsymbol{\nu}, \boldsymbol{x}, \boldsymbol{y} \in \mathbb{W}_N$

$$\begin{aligned} P_{\boldsymbol{\nu}}(t, \boldsymbol{y}|\boldsymbol{x}) &= \frac{\mathrm{P}_{\boldsymbol{\nu}}\left(\text{survival}\middle|\boldsymbol{y}/\xi\right)}{\mathrm{P}_{\boldsymbol{\nu}}\left(\text{survival}\middle|\boldsymbol{x}/\xi\right)} e^{-t|\boldsymbol{\nu}|^2/(2\xi^2)-\boldsymbol{\nu}\cdot(\boldsymbol{x}-\boldsymbol{y})/\xi} Q(t, \boldsymbol{y}|\boldsymbol{x}) \\ &= e^{-t|\boldsymbol{\nu}|^2/(2\xi^2)} \frac{\psi_{\boldsymbol{\nu}}(\boldsymbol{y}/\xi)}{\psi_{\boldsymbol{\nu}}(\boldsymbol{x}/\xi)} Q(t, \boldsymbol{y}|\boldsymbol{x}), \end{aligned}$$

where

$$Q(t, \boldsymbol{y}|\boldsymbol{x}) = \int_{\mathbb{R}^N} e^{-t|\boldsymbol{k}|^2/2} \psi_{i\xi\boldsymbol{k}}(\boldsymbol{x}/\xi)\psi_{-i\xi\boldsymbol{k}}(\boldsymbol{y}/\xi) s(\xi\boldsymbol{k}) d\boldsymbol{k}$$

with the density of Sklyanin measure

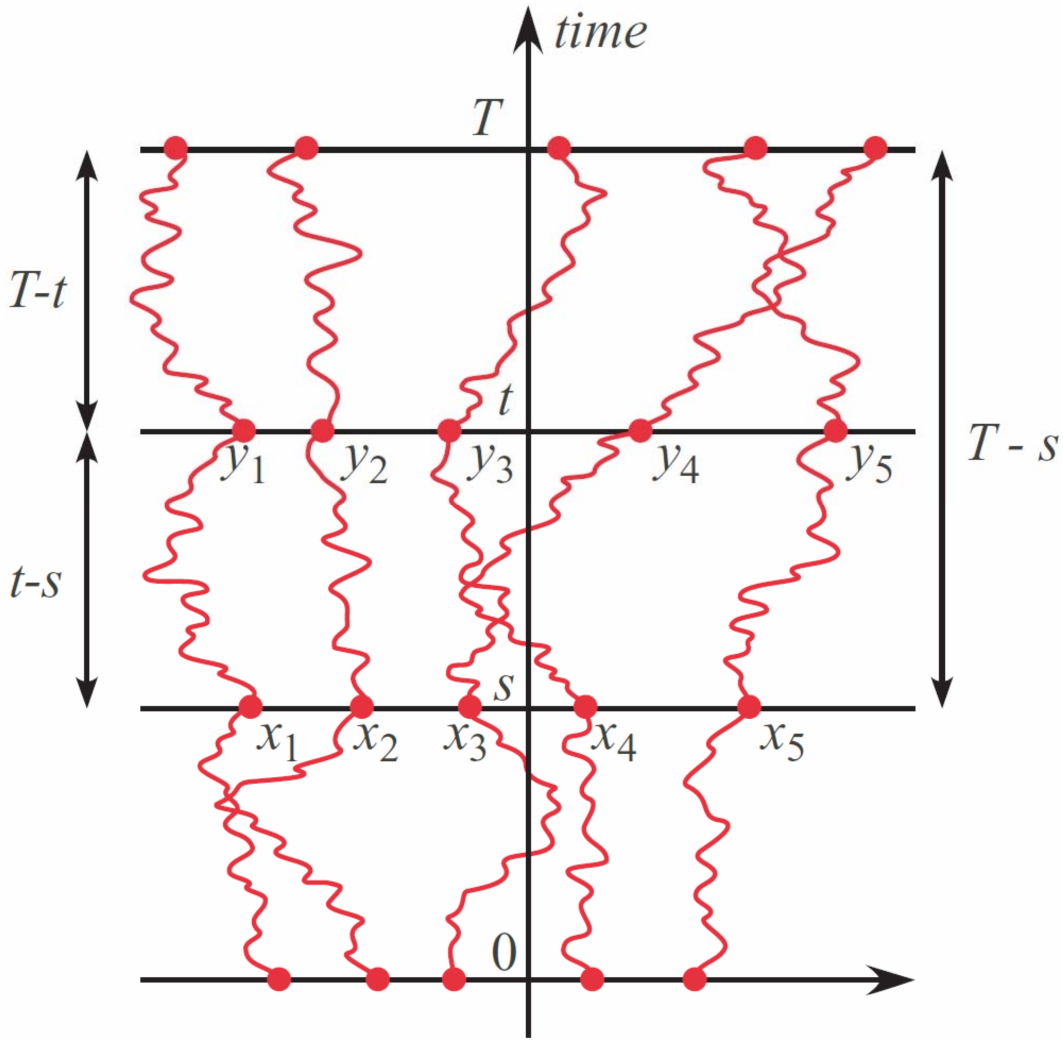
$$\begin{aligned} s(\xi\boldsymbol{k}) &= \frac{1}{(2\pi)^N N!} \prod_{1 \le j < \ell \le N} |\Gamma(i\xi(k_\ell - k_j))|^{-2} \\ &= \frac{1}{(2\pi)^N N!} h(\xi\boldsymbol{k}) \prod_{1 \le j < \ell \le N} \frac{\sinh \pi\xi(k_\ell - k_j)}{\pi}, \quad \boldsymbol{k} \in \mathbb{R}^N. \end{aligned}$$

For $\boldsymbol{\nu} = 0$

$$P_0(t, \boldsymbol{y}|\boldsymbol{x}) = \frac{\psi_0(\boldsymbol{y}/\xi)}{\psi_0(\boldsymbol{x}/\xi)} Q(t, \boldsymbol{y}|\boldsymbol{x}).$$

We can show that

$$\lim_{\xi \to 0} P_0(t, \boldsymbol{y}|\boldsymbol{x}) = p(t, \boldsymbol{y}|\boldsymbol{x}) \qquad \textbf{[K11], [K12]}$$



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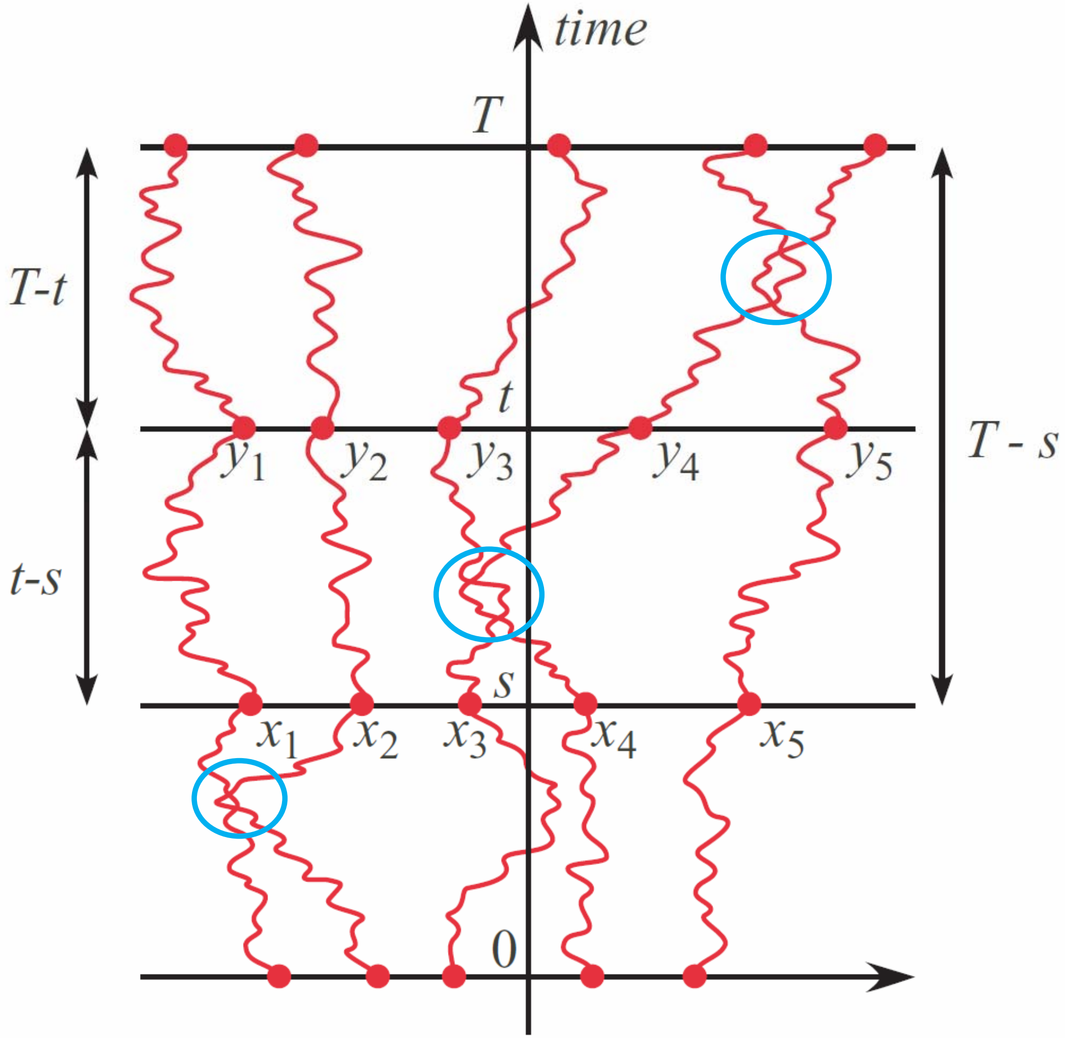


vicious walkers

softened

locally friendly walkers

- the transition probability density

$$P_{\boldsymbol{\nu}}(t,\boldsymbol{y}|\boldsymbol{x}) = e^{-t|\boldsymbol{\nu}|^2/(2\xi^2)} \frac{\psi_{\boldsymbol{\nu}}(\boldsymbol{y}/\xi)}{\psi_{\boldsymbol{\nu}}(\boldsymbol{x}/\xi)} Q(t,\boldsymbol{y}|\boldsymbol{x})$$

solves the diffusion equation associated with O'Connell's infinitesimal generator

$$\mathcal{L}_{\boldsymbol{\nu}} = -(\psi_{\boldsymbol{\nu}})^{-1}\left(\mathcal{H} + |\boldsymbol{\nu}|^2\right)\psi_{\boldsymbol{\nu}} = \frac{1}{2}\Delta_{\boldsymbol{x}} + \nabla_{\boldsymbol{x}} \log \psi_{\boldsymbol{\nu}}(\boldsymbol{x}) \cdot \nabla_{\boldsymbol{x}},$$

which is regarded as the Kolmogorov backward equation

$$\frac{\partial}{\partial t} P_{\boldsymbol{\nu}}(t,\boldsymbol{y}|\boldsymbol{x}) = \mathcal{L}_{\boldsymbol{\nu}} P_{\boldsymbol{\nu}}(t,\boldsymbol{y}|\boldsymbol{x}), \quad t \in [0,\infty), \quad P_{\boldsymbol{\nu}}(0,\boldsymbol{y}|\boldsymbol{x}) = \delta(\boldsymbol{y}-\boldsymbol{x}).$$

for the O'Connell process.

- multi-time joint distribution for $0 < t_1 < t_2 < \cdots < t_M < \infty$

$$\begin{aligned}
&\mathbb{P}^{\boldsymbol{x}}_{\boldsymbol{\nu}}(t_1, \boldsymbol{x}^{(1)}; t_2, \boldsymbol{x}^{(2)}; \cdots; t_M, \boldsymbol{x}^{(M)}) \\
&\quad = e^{-t_M|\boldsymbol{\nu}|^2/(2\xi^2)} \frac{\psi_{\boldsymbol{\nu}}(\boldsymbol{x}^{(M)}/\xi)}{\psi_{\boldsymbol{\nu}}(\boldsymbol{x}/\xi)} \prod_{m=0}^{M-1} Q(t_{m+1}-t_m, \boldsymbol{x}^{(m+1)}|\boldsymbol{x}^{(m)}), \quad \boldsymbol{x}^{(0)} \equiv \boldsymbol{x}, t_0 = 0.
\end{aligned}$$

The O'Connell process

$$\boldsymbol{X}(t) = (X_1(t), X_2(t), \ldots, X_N(t)), \quad t \in [0,\infty).$$

6. Equilibrium state of Toda lattice on T (Borodin-Corwin method)

If we think

$$
\begin{aligned}
P_0(t, \boldsymbol{y}|\boldsymbol{x}) &= \frac{\psi_0(\boldsymbol{y}/\xi)}{\psi_0(\boldsymbol{x}/\xi)} Q(t, \boldsymbol{y}|\boldsymbol{x}) \\
&= \frac{1}{\psi_0(\boldsymbol{x}/\xi)} \lim_{\boldsymbol{\nu} \to 0} \psi_{i\boldsymbol{\nu}}(\boldsymbol{y}/\xi) Q(t, \boldsymbol{y}|\boldsymbol{x}),
\end{aligned}
$$

the following formula is useful to calculate expectations.

For $1 \le r \le N-1, \boldsymbol{\nu} \in \mathbb{R}^N$,

$$
\sum_{I \subset \{1,\ldots,N\}, |I|=r} \prod_{j \in I, k \in \{1,\ldots,N\} \setminus I} \frac{1}{i(\nu_k - \nu_j)} \psi_{i(\boldsymbol{\nu} + i\boldsymbol{e}_I)}(\boldsymbol{x}) = \exp\left(-\sum_{j=1}^{r} x_j \right) \psi_{i\boldsymbol{\nu}}(\boldsymbol{x}),
$$

where $\boldsymbol{e}_I$ is the vector with ones in the slots of label I and zeros otherwise;

$$
(\boldsymbol{e}_I)_j = \begin{cases} 1, & j \in I, \\ 0, & j \in \{1, \ldots, N\} \setminus I. \end{cases}
$$

[**Borodin-Corwin 11**] A. Borodin, I. Corwin : Macdonald processes. arXiv:math.PR/1111.4408

Assume that analytic continuation is performed
for the ortogonal relation of Whittaker functions w.r.t. $\boldsymbol{k}, \boldsymbol{k}'$
(see [**Borodin-Corwin 11**])

$$\int_{\mathbb{R}^N} \psi_{i\boldsymbol{k}}(\boldsymbol{x})\psi_{-i\boldsymbol{k}'}(\boldsymbol{x})d\boldsymbol{x} = \frac{1}{s(\boldsymbol{k})N!}\sum_{\sigma\in\mathfrak{S}_N}\delta(\boldsymbol{k}-\sigma(\boldsymbol{k}'))."$$

Then for example, $0 \le t_1 < t_2 < \infty$ (a two-time observable)

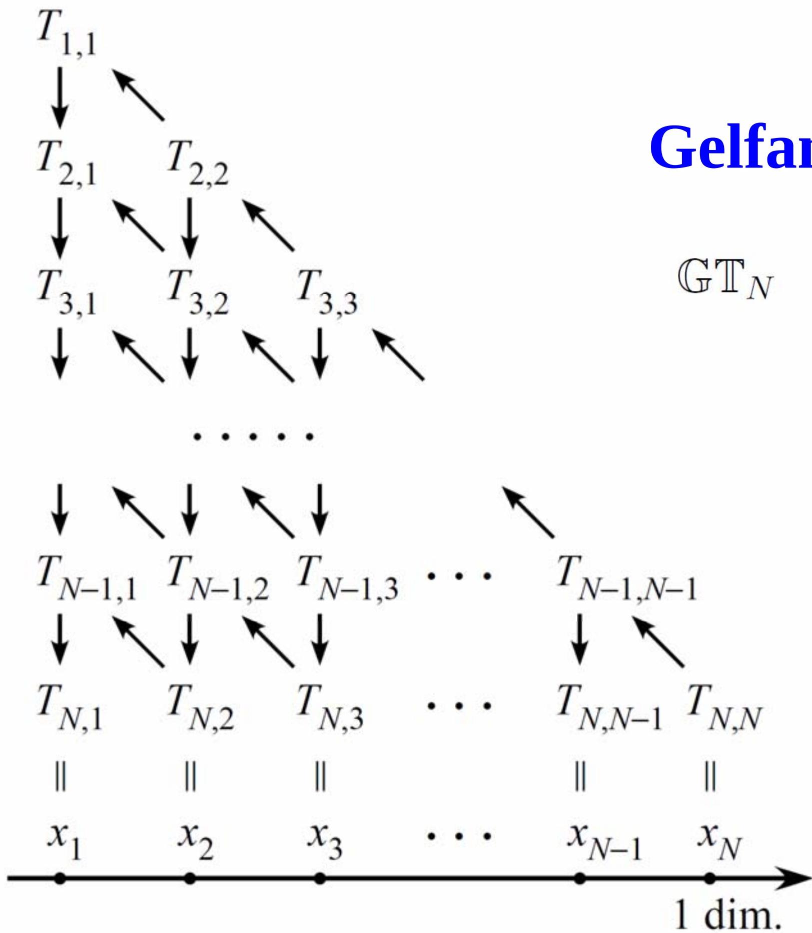
$$\begin{aligned}
&\mathbb{E}_0^{\boldsymbol{x}}\left[e^{-X_1(t_1)/\xi}e^{-X_1(t_2)/\xi}\right] \\
&= \lim_{\boldsymbol{\nu}\to 0}\Bigg[\sum_{j=1}^{N}\prod_{\ell\neq j}\frac{e^{-(t_2-t_1)(2i\nu_j-1)/(2\xi^2)}e^{-t_1(4i\nu_j-4)/(2\xi^2)}}{i(\nu_\ell-\nu_j)(i\nu_\ell-i\nu_j+1)}\mathbf{E}^{\boldsymbol{x}}\left[e^{(i\boldsymbol{\nu}-2\boldsymbol{e}_{\{j\}})\cdot\mathrm{type}\,\boldsymbol{T}}\right] \\
&\quad+\sum_{1\le j_1\neq j_2\le N}\prod_{\ell_2\neq j_2}\frac{e^{-(t_2-t_1)(2i\upsilon_{j_2}-1)/(2\xi^2)}}{i(\nu_{\ell_2}-\nu_{j_2})}\times\frac{1}{i(\nu_{j_2}-\nu_{j_1})} \\
&\qquad\times\prod_{\ell_1\neq j_1,\ell_1\neq j_2}\frac{e^{-t_1\{2i(\nu_{j_1}+\nu_{j_2})-2\}/(2\xi^2)}}{i\nu_{\ell_1}-i\nu_{j_1}+1}\mathbf{E}^{\boldsymbol{x}}\left[e^{(i\boldsymbol{\nu}-\boldsymbol{e}_{\{j_1,j_2\}})\cdot\mathrm{type}\,\boldsymbol{T}}\right]\Bigg],
\end{aligned}$$

where

$$\mathbf{E}^{\boldsymbol{x}}\left[f(\boldsymbol{T})\right]=\int_{\boldsymbol{T}^{(N)}=\boldsymbol{x}}f(\boldsymbol{T})\frac{e^{\mathcal{F}(\boldsymbol{T}/\xi)}}{Z^{\boldsymbol{x}}}d\boldsymbol{T}\quad\text{with}\quad Z^{\boldsymbol{x}}=\int_{\boldsymbol{T}^{(N)}=\boldsymbol{x}}e^{\mathcal{F}(\boldsymbol{T}/\xi)}d\boldsymbol{T}.$$

Gibbs state with the Boltzman weight $e^{-\beta E(\boldsymbol{T})}=e^{\mathcal{F}(\boldsymbol{T}/\xi)}$.
on the space of real triangular arrays $\{\boldsymbol{T}:\boldsymbol{T}^{(N)}=\boldsymbol{x}\}\simeq\mathbb{R}^{N(N-1)/2}$

Gelfand-Tsetlin pattern



$$\mathbb{GT}_N = \Big\{(\boldsymbol{x}^1, \boldsymbol{x}^2, \dots, \boldsymbol{x}^N) \in \overline{\mathbb{W}}_1 \times \overline{\mathbb{W}}_2 \times \cdots \times \overline{\mathbb{W}}_N : x_k^{j+1} \le x_k^j \le x_{k+1}^{j+1}, 1 \le k \le j \le N-1\Big\}$$

conditioned on the bottom values $\boldsymbol{T}^{(N)} = \boldsymbol{x}$

$$\mathbf{E}^{\boldsymbol{x}}\Big[f(\boldsymbol{T})\Big] = \int_{\boldsymbol{T}^{(N)}=\boldsymbol{x}} f(\boldsymbol{T}) \frac{e^{\mathcal{F}(\boldsymbol{T}/\xi)}}{Z^{\boldsymbol{x}}} d\boldsymbol{T} \quad \text{with} \quad Z^{\boldsymbol{x}} = \int_{\boldsymbol{T}^{(N)}=\boldsymbol{x}} e^{\mathcal{F}(\boldsymbol{T}/\xi)} d\boldsymbol{T}.$$

Gibbs state with the Boltzman weight $e^{-\beta E(\boldsymbol{T})} = e^{\mathcal{F}(\boldsymbol{T}/\xi)}$.
on the space of real triangular arrays $\{\boldsymbol{T} : \boldsymbol{T}^{(N)} = \boldsymbol{x}\} \simeq \mathbb{R}^{N(N-1)/2}$

7. Concluding remarks

multi-variate extensions

1 dim. BM cond. to stay positive

radial part of 3 dim. BM

BES(3)

Pitman's thm

$2 \max_{0 \le s \le t} B(s) - B(t)$

vicious BMs cond. to survive

eigenvalue process of matrix-v. BM

noncollid BM (Dyson model)

Interlacing process construction

Determinantal Process

geometric lifting

1 dim. killing BM cond. to survive forever

hyperbolic space

Matsumoto-Yor Process

generalized Pitman's thm

$\xi \log\left\{\frac{1}{\xi^2}\int_0^t e^{2B(s)/\xi}\,ds\right\} - B(t)$

mutually killing BM cond. to survive

Gelfand-Tsetlin pattern

O'Connell Process

Q. Toda lattice

Whittaker functions

Macdonald Process

KPZ universality

multi-variate extensions