Stationary two-point correlation for the KPZ equation

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(Based on collaborations with T. Imamura)

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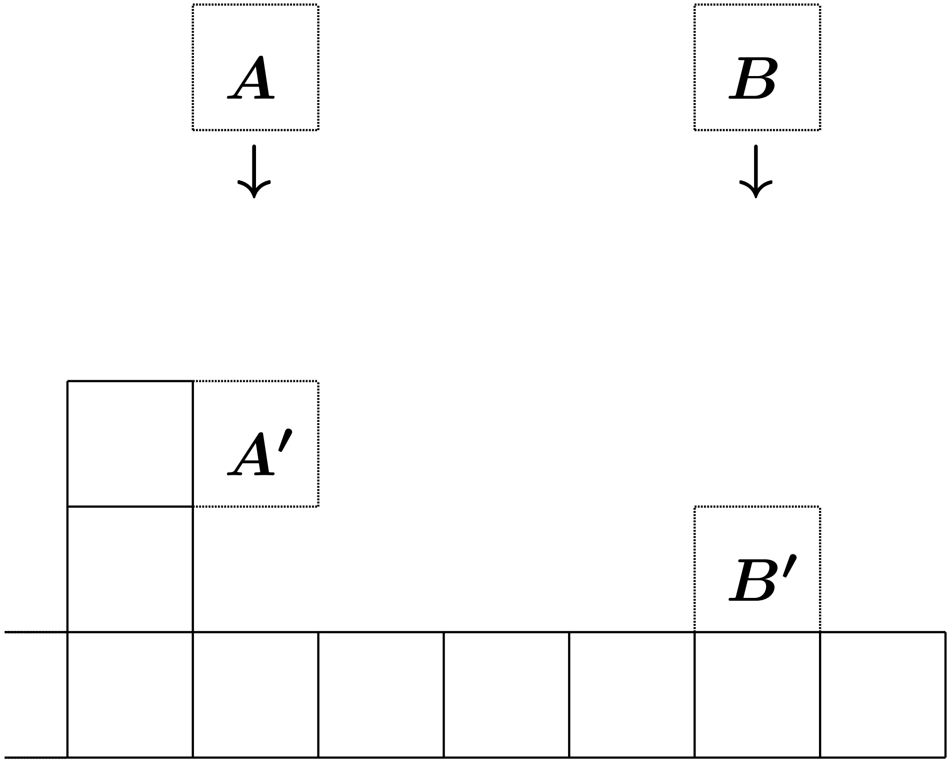
References: arxiv:1111.4634

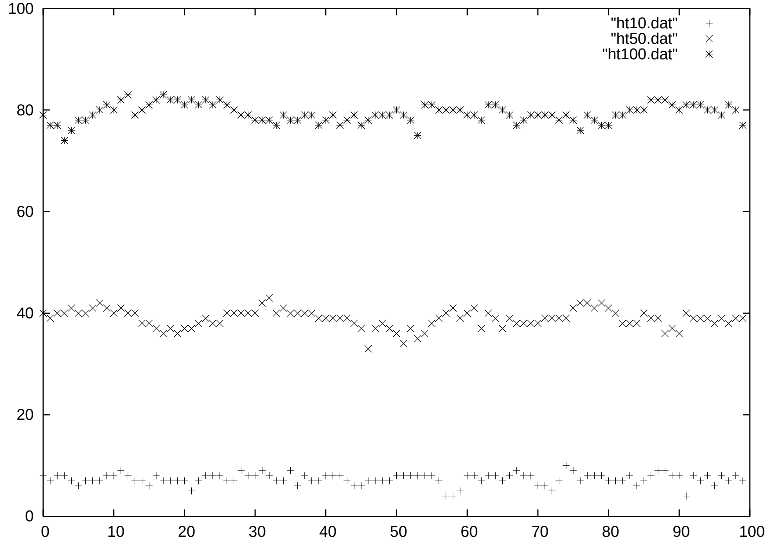
1. Introduction: 1D surface growth

An example: ballistic deposition model



3 snapshots for flat substrate

Kinetic roughening



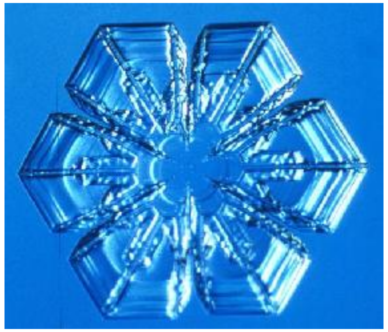
Motivations

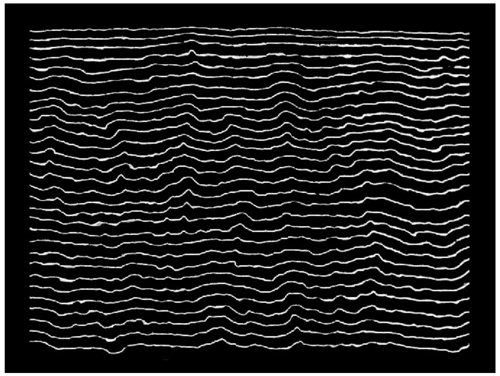
- Ubiquitous and interesting physical phenomenon in itself
- Beautiful hidden mathematical structure (e.g. Macdonald)
- Two aspects from non-eq stat. mech: dynamic and stationary

  Kinetic roughening (dynamical)

  Nonequilibrium steady state (NESS)

  Nonlinearity + Noise

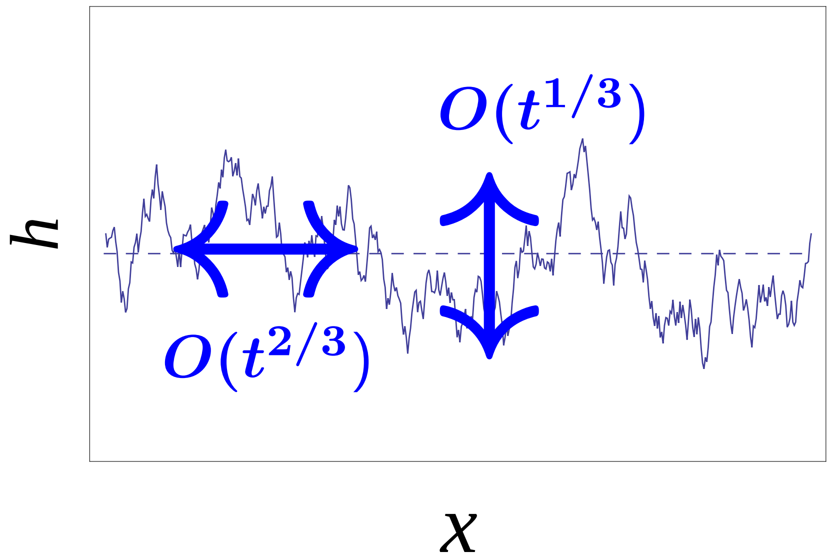




Scaling

$h(x,t)$: surface height at position x and at time t

- In stationary state, height looks like a random walk. $h(x,t) - h(0,t) \sim O(x^{1/2})$ for large x
- $h(x,t) \sim vt + O(t^{1/3})$ for large t
- $h(at^{2/3},t), h(bt^{2/3},t)$ has nontrivial correlation.



Kardar-Parisi-Zhang(KPZ) equation

1986 Kardar Parisi Zhang

$$\partial_t h(x,t) = \frac{1}{2}\lambda(\partial_x h(x,t))^2 + \nu\partial_x^2 h(x,t) + \sqrt{D}\eta(x,t)$$

where η is the Gaussian noise with mean 0 and covariance $\langle\eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$

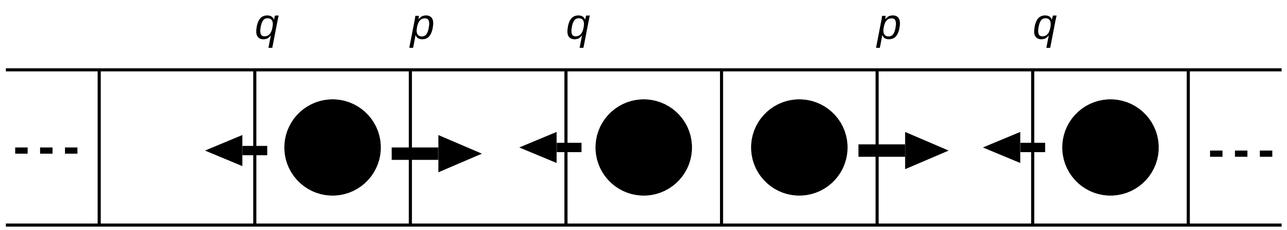
- The Brownian motion is stationary.
- By a dynamical RG analysis, one can see the KPZ equation exhibit the correct scaling. $\rightarrow$ KPZ universality class

- By $x \to \alpha^2 x,\ t \to 2\nu\alpha^4 t,\ h \to \frac{\lambda}{2\nu} h, \quad \alpha = \frac{\lambda^{1/2}}{(2\nu)^{3/2}}$, we can and will do set $\nu = \frac{1}{2}, \lambda = D = 1$.
- Noisy Burgers equation: For $u(x,t) = \partial_x h(x,t)$,
$$\partial_t u = \frac{1}{2}\partial_x^2 u + \frac{1}{2}\partial_x u^2 + \partial_x \eta(x,t)$$
- KPZ equation is not really wel-defined.

  We consider the Cole-Hopf solution,
$$h(x,t) = \log\left(Z(x,t)\right)$$
  where $Z(x,t)$ is the solution of the stochastic heat equation,
$$dZ(x,t) = \frac{1}{2}\frac{\partial^2 Z(x,t)}{\partial x^2}dt + Z(x,t)dB(x,t).$$
  where $B(x,t)$ is the cylindrical Brownian motion.

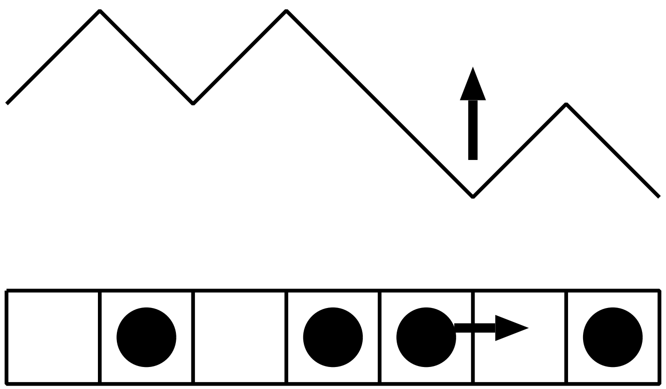
2. Scaling limit results from discrete models

An example: ASEP(asymmetric simple exclusion process)

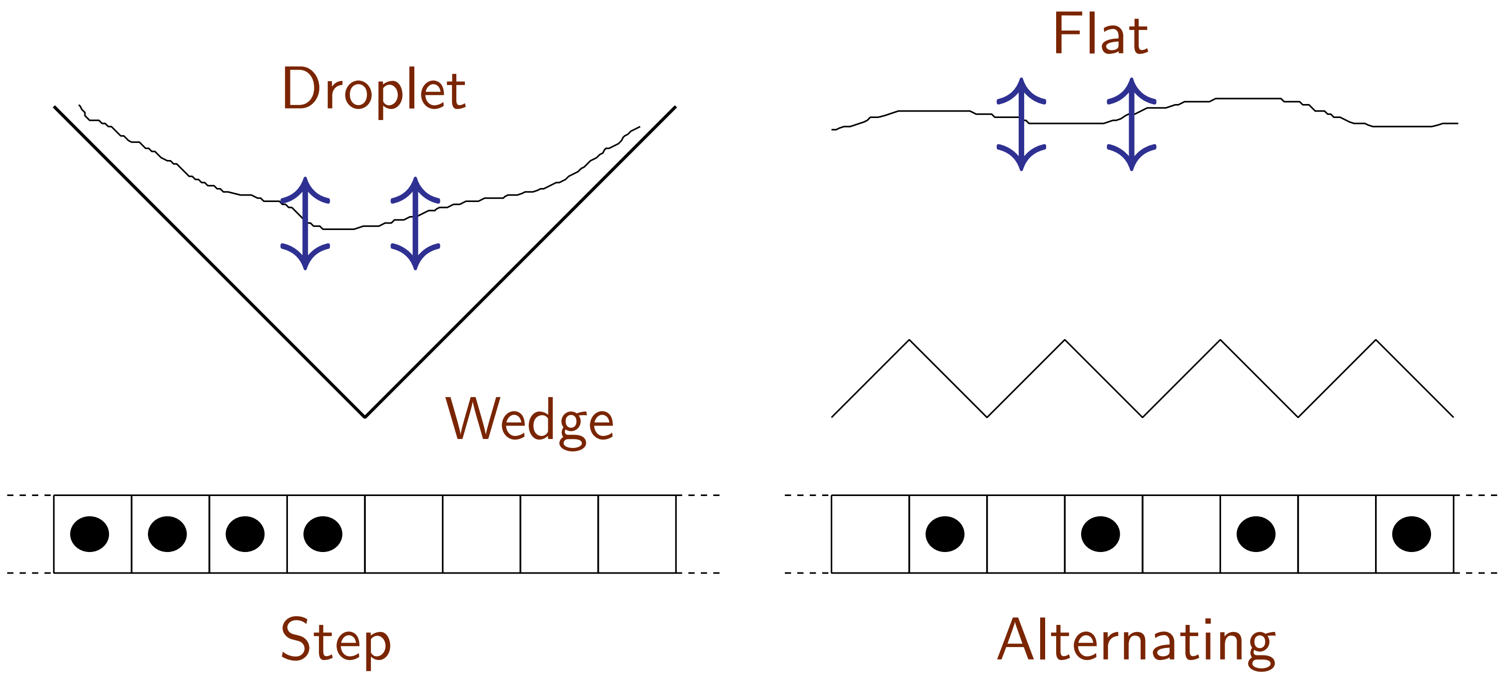


Bernoulli measure is stationary.

Mapping to surface growth



Surface growth and 2 initial conditions besides stationary



Integrated current $N(x,t)$ in ASEP $\Leftrightarrow$ Height $h(x,t)$ in surface growth

Current distributions for ASEP with wedge initial conditions

2000 Johansson (TASEP) 2008 Tracy-Widom (ASEP)

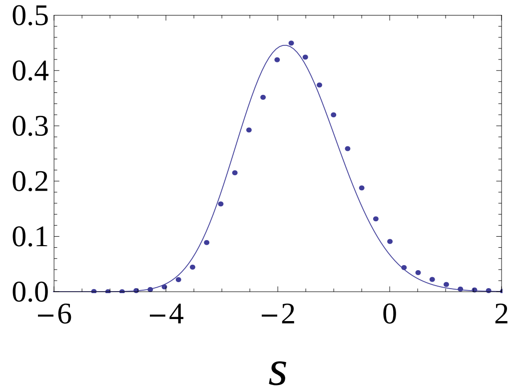
$$N(0, t/(q-p)) \simeq \frac{1}{4}t - 2^{-4/3}t^{1/3}\xi_{\mathrm{TW}}$$

Here $N(x=0,t)$ is the integrated current of ASEP at the origin and ξ_{TW} obeys the GUE Tracy-Widom distributions;

$$F_{\mathrm{TW}}(s) = \mathbb{P}[\xi_{\mathrm{TW}} \le s] = \det(1 - P_s K_{\mathrm{Ai}} P_s)$$

wher P_s: projection onto the interval $[s, \infty)$ and K_{Ai} is the Airy kernel

$$K_{\mathrm{Ai}}(x,y) = \int_0^\infty d\lambda \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)$$



Other cases

	wedge	flat	stationary
1pt	GUE TW	GOE TW	$\boldsymbol{F_0}$
multi	Airy$_{\mathbf{2}}$	Airy$_{\mathbf{1}}$	Airy$_{\mathbf{0}}$(?)

2000 Baik Rains GOE TW

2000 Baik Rains, Prähofer Ferrari Spohn $\boldsymbol{F_0}$

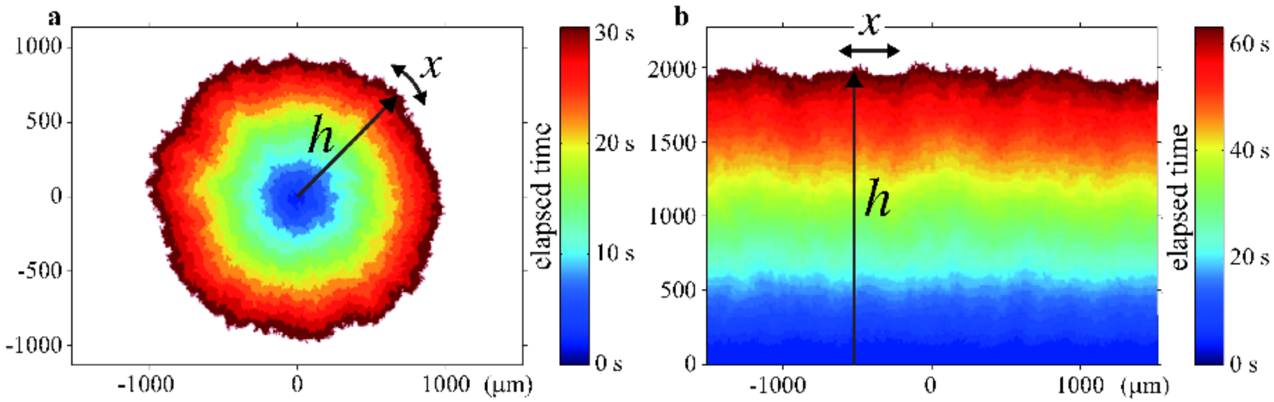
2001 Prähofer Spohn, Johansson Airy$_{\mathbf{2}}$

2005 Borodin Ferrari Prähofer S Airy$_{\mathbf{1}}$

2009 Baik Ferrari Péché Airy$_{\mathbf{0}}$(?)

3. Experiments by liquid crystal turbulence

2010-2012 Takeuchi Sano (see arXiv:1203.2530)



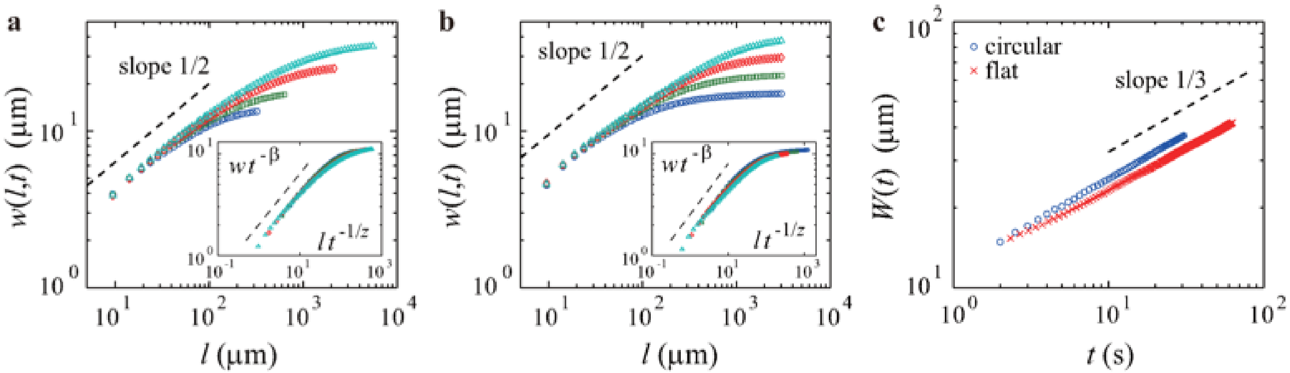
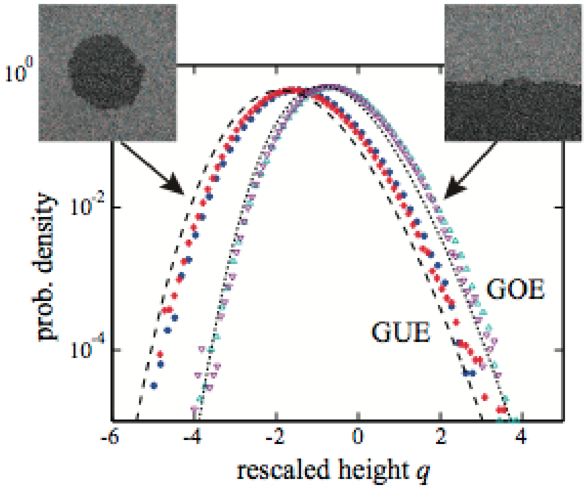
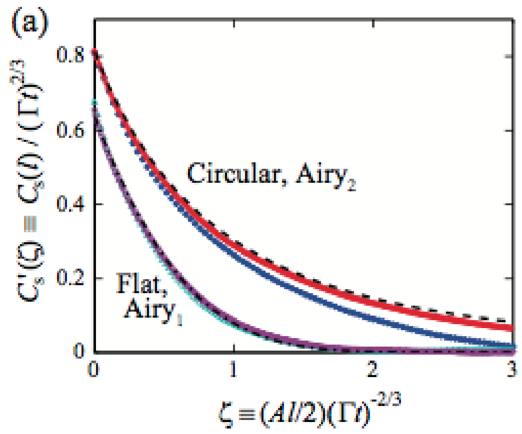


Figure 2 | Family-Vicsek scaling. a,b, Interface width $w(l, t)$ against the length scale l at different times t for the circular (a) and flat (b) interfaces. The four data correspond, from bottom to top, to $t = 2.0$ s, 4.0 s, 12.0 s and 30.0 s for the panel a and to $t = 4.0$ s, 10.0 s, 25.0 s and 60.0 s for the panel b. The insets show the same data with the rescaled axes. c, Growth of the overall width $W(t) \equiv \sqrt{\langle [h(x,t) - \langle h \rangle]^2 \rangle}$. The dashed lines are guides for the eyes showing the exponent values of the KPZ class.

Fluctuations in experiments

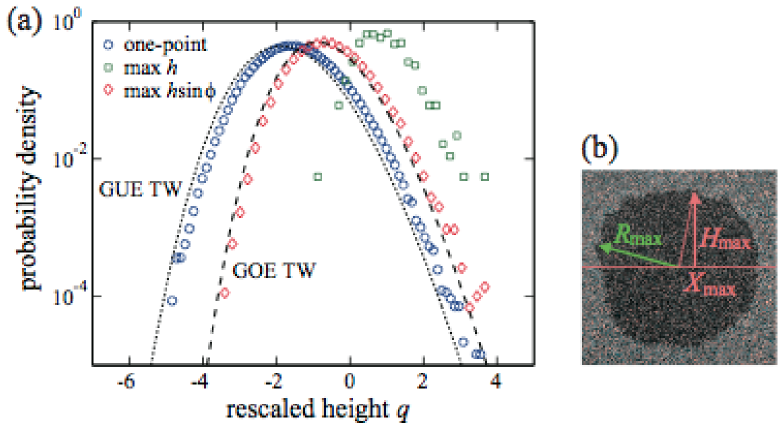
1pt





Airy$_{1,2}$

Max[Airy$_2 - x^2$]



Challenges for us: time correlation, persistence...

4. The narrow wedge KPZ equation

2010 Sasamoto Spohn, Amir Corwin Quastel

- Narrow wedge initial condition
- Based on (i) the fact that the weakly ASEP is KPZ equation (1997 Bertini Giacomin) and (ii) a formula for step ASEP by 2009 Tracy Widom
- In the book by Barabási Stanley [1995], they write "the KPZ equation cannot be solved in closed form"

Before this

2009 Balaźs, Quastel, and Seppäläinen

The 1/3 exponent for the stationary case

Narrow wedge initial condition

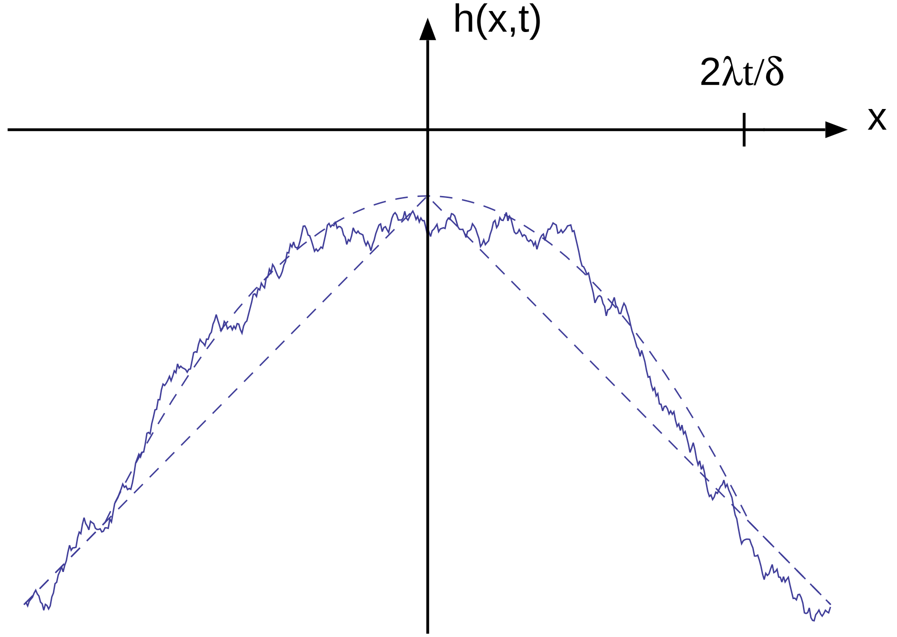
We consider the initial condition, $Z(x, 0) = \delta(x)$.

This corresponds to the droplet growth with the following narrow wedge initial conditions:

$$h(x, 0) = -|x|/\delta\,, \quad \delta \ll 1$$

For finite t, the macroscopic shape is

$$h(x, t) = \begin{cases} -x^2/2t & \text{for } |x| \le t/\delta\,, \\ (1/2\delta^2)t - |x|/\delta & \text{for } |x| > t/\delta \end{cases}$$


h(x,t)
2λt/δ
x

Distribution

$$h(x,t) = -x^2/2t - \tfrac{1}{12}\gamma_t^3 + \gamma_t \xi_t$$

where $\gamma_t = (t/2)^{1/3}$.

The distribution function of ξ_t

$$F_t(s) = \mathbb{P}[\xi_t \le s] = 1 - \int_{-\infty}^{\infty} \exp\big[- \mathrm{e}^{\gamma_t(s-u)}\big]$$

$$\times\big(\det(1 - P_u(B_t - P_{\mathrm{Ai}})P_u) - \det(1 - P_u B_t P_u)\big)\mathrm{d}u$$

where $P_{\mathrm{Ai}}(x,y) = \mathrm{Ai}(x)\mathrm{Ai}(y)$.

P_u is the projection onto $[u, \infty)$ and the kernel B_t is

$$B_t(x,y) = K_{\mathrm{Ai}}(x,y) + \int_0^\infty \mathrm{d}\lambda (\mathrm{e}^{\gamma_t \lambda} - 1)^{-1}$$
$$\times \big(\mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda) - \mathrm{Ai}(x-\lambda)\mathrm{Ai}(y-\lambda)\big) \, .$$

Developments(Not all!)

- Structural

  2010 O'Connell A directed polymer model related to q-Toda

  2011 COSZ Tropical RSK for inverse gamma polymer

  2011 Borodin Corwin Macdonald process

- Probabilistic

- Generalizations by replica method

  2010 Calabrese Le Doussal Rosso, Dotsenko Narrow wedge

  2010 Prolhac Spohn Multi-point distributions

  2011 Calabrese Le Dossal Flat

  2011 Imamura Sasamoto Half-BM and stationary

5. Stationary case

- Narrow wedge is technically the simplest (transient).
- Flat case is a well-studied case in surface growth (transient)
- Stationary case is important for stochastic process and nonequilibrium statistical mechanics
  - Two-point correlation function
  - Experiments: Scattering, direct observation
  - A lot of approximate methods (renormalization, mode-coupling, etc.) have been applied to this case.
  - Nonequilibrium steady state(NESS): No principle. Dynamics is even harder.

Modification of initial condition

Original: two sided BM

$$h(x,0) = \begin{cases} B_-(-x), & x < 0, \\ B_+(x), & x > 0, \end{cases}$$

where $B_\pm(x)$ are two independent standard BMs is stationary.

Modification: we consider a generalized initial condition

$$h(x,0) = \begin{cases} \tilde{B}(-x) + v_- x, & x < 0, \\ B(x) - v_+ x, & x > 0, \end{cases}$$

where $B(x), \tilde{B}(x)$ are independent standard BMs and $v_\pm$ are the strength of the drifts.

Result

For the generalized initial condition with $v_\pm$

$$F_{v_\pm,t}(s) := \text{Prob}\left[h(x,t) + \gamma_t^3/12 \le \gamma_t s\right]$$
$$= \frac{\Gamma(v_+ + v_-)}{\Gamma(v_+ + v_- + \gamma_t^{-1} d/ds)}\left[1 - \int_{-\infty}^{\infty} du e^{-e^{\gamma_t(s-u)}} \nu_{v_\pm,t}(u)\right]$$

Here $\nu_{v_\pm,t}(u)$ is expressed as a difference of two Fredholm determinants,

$$\nu_{v_\pm,t}(u) = \det\left(1 - P_u(B_t^\Gamma - P_{\mathrm{Ai}}^\Gamma)P_u\right) - \det\left(1 - P_u B_t^\Gamma P_u\right),$$

where P_s represents the projection onto (s, ∞),

$$P_{\mathrm{Ai}}^\Gamma(\xi_1, \xi_2) = \mathrm{Ai}_\Gamma^\Gamma\left(\xi_1, \frac{1}{\gamma_t}, v_-, v_+\right)\mathrm{Ai}_\Gamma^\Gamma\left(\xi_2, \frac{1}{\gamma_t}, v_+, v_-\right)$$

$$B_t^{\Gamma}(\xi_1,\xi_2) = \int_{-\infty}^{\infty} dy \frac{1}{1-e^{-\gamma_t y}} \mathrm{Ai}_{\Gamma}^{\Gamma}\left(\xi_1 + y, \frac{1}{\gamma_t}, v_-, v_+\right)$$
$$\times \mathrm{Ai}_{\Gamma}^{\Gamma}\left(\xi_2 + y, \frac{1}{\gamma_t}, v_+, v_-\right),$$

and

$$\mathrm{Ai}_{\Gamma}^{\Gamma}(a,b,c,d) = \frac{1}{2\pi} \int_{\Gamma_{i\frac{d}{b}}} dz e^{iza + i\frac{z^3}{3}} \frac{\Gamma\left(ibz + d\right)}{\Gamma\left(-ibz + c\right)},$$

where Γ_{z_p} represents the contour from $-\infty$ to ∞ and, along the way, passing below the pole at $z = id/b$.

Height distribution for the stationary KPZ equation

$$F_{0,t}(s) = \frac{1}{\Gamma(1+\gamma_t^{-1}d/ds)} \int_{-\infty}^{\infty} du \gamma_t e^{\gamma_t(s-u)+e^{-\gamma_t(s-u)}} \nu_{0,t}(u)$$

where $\nu_{0,t}(u)$ is obtained from $\nu_{v_\pm,t}(u)$ by taking $v_\pm \to 0$ limit.

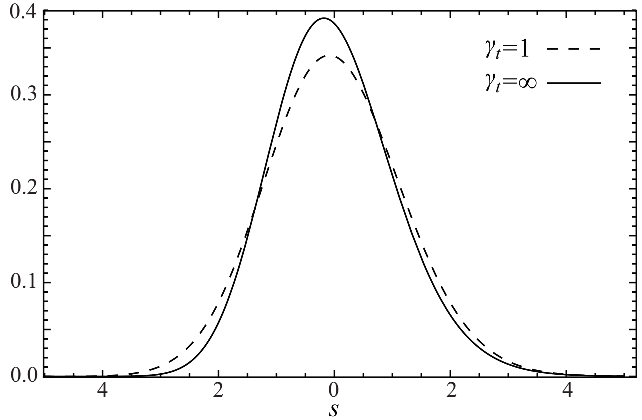


Figure 1: Stationary height distributions for the KPZ equation for $\gamma_t = 1$ case. The solid curve is F_0.

Stationary 2pt correlation function

$$C(x,t) = \langle (h(x,t) - \langle h(x,t) \rangle)^2 \rangle$$

$$g_t(y) = (2t)^{-2/3} C\left((2t)^{2/3} y, t\right)$$

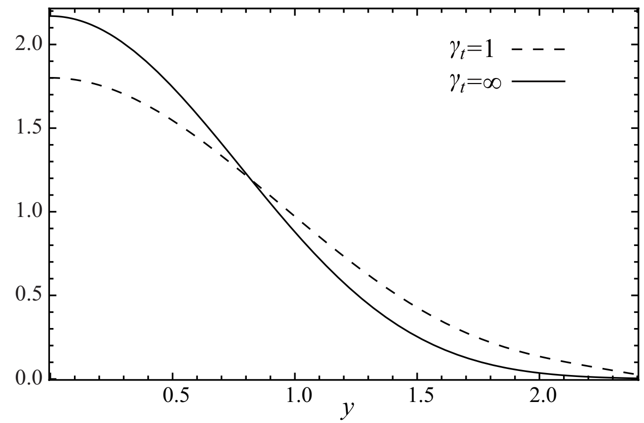


Figure 2: Stationary 2pt correlation function $g_t''(y)$ for $\gamma_t = 1$. The solid curve is the corresponding quantity in the scaling limit $g''(y)$.

Derivation

Cole-Hopf transformation

1997 Bertini and Giacomin

$$h(x,t) = \log\left(Z(x,t)\right)$$

$Z(x,t)$ is the solution of the stochastic heat equation,

$$\frac{\partial Z(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 Z(x,t)}{\partial x^2} + \eta(x,t)Z(x,t).$$

and can be considered as directed polymer in random potential η.

Feynmann-Kac and Generating function

Feynmann-Kac expression for the partition function,

$$Z(x,t) = \mathbb{E}_x \left(\exp\left[\int_0^t \eta\left(b(s), t-s\right) ds \right] Z(b(t), 0) \right)$$

We consider the Nth replica partition function $\langle Z^N(x,t) \rangle$ and compute their generating function $G_t(s)$ defined as

$$G_t(s) = \sum_{N=0}^{\infty} \frac{\left(-e^{-\gamma_t s}\right)^N}{N!} \left\langle Z^N(0,t) \right\rangle e^{N \frac{\gamma_t^3}{12}}$$

with $\gamma_t = (t/2)^{1/3}$.

Apparently the series is divergent but should be a "shadow" of a rigorous version at a higher level.

Replica method

For a system with randomness, e.g. for random Ising model,

$$H = \sum_{\langle ij \rangle} J_{ij} s_i s_j$$

where i is site, $s_i = \pm 1$ is Ising spin, J_{ij} is iid random variable(e.g. Bernoulli), we are interested in the averaged free energy $\langle \log Z \rangle$, $Z = \sum_{s_i = \pm 1} e^{-H}$.

In replica method, one often resorts to the following identity,

$$\langle \log Z \rangle = \lim_{n \to 0} \frac{\langle Z^n \rangle - 1}{n},$$

which needs an analytic continuation wrt n.

δ-Bose gas

Taking the Gaussian average over the noise η, one finds that the replica partition function can be written as

$$\langle Z^N(x,t)\rangle$$

$$= \prod_{j=1}^{N} \int_{-\infty}^{\infty} dy_j \int_{x_j(0)=y_j}^{x_j(t)=x} D[x_j(\tau)] \exp\left[-\int_0^t d\tau \left(\sum_{j=1}^{N} \frac{1}{2} \left(\frac{dx}{d\tau} \right)^2 \right.\right.$$

$$\left.\left. - \sum_{j\neq k=1}^{N} \delta\left(x_j(\tau) - x_k(\tau)\right) \right) \right] \times \left\langle \exp\left(\sum_{k=1}^{N} h(y_k, 0) \right) \right\rangle$$

$$= \langle x | e^{-H_N t} | \Phi \rangle.$$

H_N is the Hamiltonian of the δ-Bose gas,

$$H_N = -\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} - \frac{1}{2}\sum_{j\neq k}^{N}\delta(x_j - x_k),$$

$|\Phi\rangle$ represents the state corresponding to the initial condition. We compute $\langle Z^N(x,t)\rangle$ by expanding in terms of the eigenstates of H_N,

$$\langle Z(x,t)^N\rangle = \sum_z \langle x|\Psi_z\rangle\langle\Psi_z|\Phi\rangle e^{-E_z t}$$

where E_z and $|\Psi_z\rangle$ are the eigenvalue and the eigenfunction of H_N: $H_N|\Psi_z\rangle = E_z|\Psi_z\rangle$.

[Old fashoned...probably possible to do like BC.]

The state $|\Phi\rangle$ can be calculated because the initial condition is Gaussian. For the region where $x_1 < \ldots < x_l < 0 < x_{l+1} < \ldots < x_N, 1 \le l \le N$ it is given by

$$\langle x_1, \cdots, x_N|\Phi\rangle = e^{v_- \sum_{j=1}^{l} x_j - v_+ \sum_{j=l+1}^{N} x_j}$$
$$\times \prod_{j=1}^{l} e^{\frac{1}{2}(2l-2j+1)x_j} \prod_{j=1}^{N-l} e^{\frac{1}{2}(N-l-2j+1)x_{l+j}}$$

We symmetrize wrt $x_1, \ldots, x_N$.

Bethe states

By the Bethe ansatz, the eigenfunction is given as

$$\langle x_1, \cdots, x_N | \Psi_z \rangle = C_z \sum_{P \in S_N} \mathrm{sgn} P$$

$$\times \prod_{1 \le j < k \le N} \left(z_{P(j)} - z_{P(k)} + i \,\mathrm{sgn}(x_j - x_k) \right) \exp \left(i \sum_{l=1}^{N} z_{P(l)} x_l \right)$$

N momenta z_j $(1 \le j \le N)$ are parametrized as

$$z_j = q_\alpha - \frac{i}{2} \left(n_\alpha + 1 - 2 r_\alpha \right), \quad \text{for } j = \sum_{\beta=1}^{\alpha-1} n_\beta + r_\alpha .$$

$(1 \le \alpha \le M$ and $1 \le r_\alpha \le n_\alpha)$. They are divided into M groups where $1 \le M \le N$ and the αth group consists of n_α quasimomenta $z_j^\prime s$ which shares the common real part q_α.

$$C_z = \left(\frac{\prod_{\alpha=1}^{M} n_\alpha}{N!} \prod_{1 \le j < k \le N} \frac{1}{|z_j - z_k - i|^2} \right)^{1/2}$$

$$E_z = \frac{1}{2} \sum_{j=1}^{N} z_j^2 = \frac{1}{2} \sum_{\alpha=1}^{M} n_\alpha q_\alpha^2 - \frac{1}{24} \sum_{\alpha=1}^{M} \left(n_\alpha^3 - n_\alpha \right).$$

Expanding the moment in terms of the Bethe states, we have

$$\langle Z^N(x,t) \rangle$$

$$= \sum_{M=1}^{N} \frac{N!}{M!} \prod_{j=1}^{N} \int_{-\infty}^{\infty} dy_j \left(\int_{-\infty}^{\infty} \prod_{\alpha=1}^{M} \frac{dq_\alpha}{2\pi} \sum_{n_\alpha=1}^{\infty} \right) \delta_{\sum_{\beta=1}^{M} n_\beta, N}$$

$$\times e^{-E_z t} \langle x | \Psi_z \rangle \langle \Psi_z | y_1, \cdots, y_N \rangle \langle y_1, \cdots, y_N | \Phi \rangle.$$

The completeness of Bethe states is known (e.g. Prolhac Spohn).

We see

$$\langle \Psi_z | \Phi \rangle = N! C_z \sum_{P \in S_N} \mathrm{sgn} P \prod_{1 \le j < k \le N} \left(z^*_{P(j)} - z^*_{P(k)} + i \right)$$

$$\times \sum_{l=0}^{N} (-1)^l \prod_{m=1}^{l} \frac{1}{\sum_{j=1}^{m} (-i z^*_{P_j} + v_-) - m^2/2}$$

$$\times \prod_{m=1}^{N-l} \frac{1}{\sum_{j=N-m+1}^{N} (-i z^*_{P_j} - v_+) + m^2/2}.$$

Combinatorial identities

(1)

$$\sum_{P \in S_N} \mathrm{sgn} P \prod_{1 \le j < k \le N} \big(w_{P(j)} - w_{P(k)} + i f(j,k) \big)$$

$$= N! \prod_{1 \le j < k \le N} (w_j - w_k)$$

(2)For any complex numbers w_j $(1 \leq j \leq N)$ and a,

$$\sum_{P \in S_N} \mathrm{sgn} P \prod_{1 \leq j < k \leq N} \left(w_{P(j)} - w_{P(k)} + a \right)$$

$$\times \sum_{l=0}^{N} (-1)^l \prod_{m=1}^{l} \frac{1}{\sum_{j=1}^{m} (w_{P(j)} + v_-) - m^2 a/2}$$

$$\times \prod_{m=1}^{N-l} \frac{1}{\sum_{j=N-m+1}^{N} (w_{Pj} - v_+) + m^2 a/2}$$

$$= \frac{\prod_{m=1}^{N} (v_+ + v_- - am) \prod_{1 \leq j < k \leq N} (w_j - w_k)}{\prod_{m=1}^{N} (w_m + v_- - a/2)(w_m - v_+ + a/2)}.$$

A similar identity in the context of ASEP has not been found.

Generating function

$$G_t(s) = \sum_{N=0}^{\infty} \prod_{l=1}^{N} (v_+ + v_- - l) \sum_{M=1}^{N} \frac{(-e^{-\gamma_t s})^N}{M!}$$

$$\prod_{\alpha=1}^{M} \left(\int_0^\infty d\omega_\alpha \sum_{n_\alpha=1}^{\infty} \right) \delta_{\sum_{\beta=1}^{M} n_\beta, N}$$

$$\det \left(\int_C \frac{dq}{\pi} \frac{e^{-\gamma_t^3 n_j q^2 + \frac{\gamma_t^3}{12} n_j^3 - n_j(\omega_j + \omega_k) - 2iq(\omega_j - \omega_k)}}{\prod_{r=1}^{n_j} (-iq + v_- + \frac{1}{2}(n_j - 2r))(iq + v_+ + \frac{1}{2}(n_j - 2r))} \right)$$

where the contour is $C = \mathbb{R} - ic$ with c taken large enough.

This generating function itself is not a Fredholm determinant due to the novel factor $\prod_{l=1}^{N}(v_+ + v_- - l)$.

We consider a further generalized initial condition in which the initial overall height χ obeys a certain probability distribution.

$$\tilde{h} = h + \chi$$

where h is the original height for which $h(0,0) = 0$. The random variable χ is taken to be independent of h.

Moments $$\langle e^{N\tilde{h}} \rangle = \langle e^{Nh} \rangle \langle e^{N\chi} \rangle.$$

We postulate that χ is distributed as the inverse gamma distribution with parameter $v_+ + v_-$, i.e., if $1/\chi$ obeys the gamma distribution with the same parameter. Its Nth moment is $1/\prod_{l=1}^{N}(v_+ + v_- - l)$ which compensates the extra factor.

Distributions

$$F(s) = \frac{1}{\kappa(\gamma_t^{-1} \frac{d}{ds})} \tilde{F}(s),$$

where $\tilde{F}(s) = \text{Prob}[\tilde{h}(0,t) \le \gamma_t s]$, $F(s) = \text{Prob}[h(0,t) \le \gamma_t s]$ and κ is the Laplace transform of the pdf of χ. For the inverse gamma distribution, $\kappa(\xi) = \Gamma(v + \xi)/\Gamma(v)$, by which we get the formula for the generating function.

Summary

- Explicit formulas for the stationary situation of the KPZ equation by replica method.

  Height distribution and two point correlation function.

- Questions:

  A rigorous version.

  Other initial and boundary conditions?

- See also the poster by Imamura.

Random matrix theory

GUE (Gaussian Unitary Ensemble) hermitian matrices

$$A = \begin{bmatrix} u_{11} & u_{12} + iv_{12} & \cdots & u_{1N} + iv_{1N} \\ u_{12} - iv_{12} & u_{22} & \cdots & u_{2N} + iv_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ u_{1N} - iv_{1N} & u_{2N} - iv_{2N} & \cdots & u_{NN} \end{bmatrix}$$

$$u_{jj} \sim N(0, 1/2) \quad u_{jk}, v_{jk} \sim N(0, 1/4)$$

The largest eigenvalue $x_{\max}$ $\cdots$ GUE TW distribution

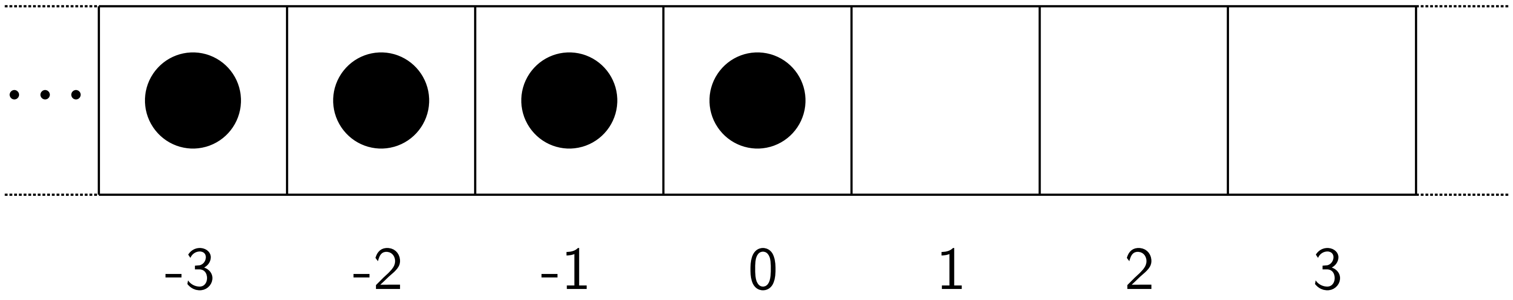
GOE (Gaussian Orthogonal Ensemble) real symmetric matrices

$\cdots$ GOE TW distribution

Connection to random matrix: Johansson

TASEP(Totally ASEP, hop only in one direction)

Step initial condition ($t = 0$)



$N(t)$: Number of particles which crossed (0,1) up to time t

LUE formula

$$\mathbb{P}[N(t) \geq N] = \frac{1}{Z_N} \int_{[0,t]^N} \prod_{i<j} (x_i - x_j)^2 \prod_i e^{-x_i} dx_1 \cdots dx_N$$