Extremes of vicious walkers: application to the directed polymer and KPZ interfaces

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G. S., preprint arXiv:1203.1658

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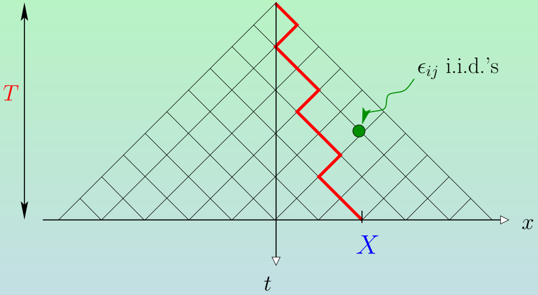
Further references:

- G. S., S. N. Majumdar, A. Comtet, J. Randon-Furling, Phys. Rev. Lett. **101**, 150601 (2008)
- J. Rambeau, G. S., Europhys. Lett. **91**, 60006 (2010)
- P. J. Forrester, S. N. Majumdar, G. S., Nucl. Phys. B **844**, 500 (2011)
- J. Rambeau, G. S., Phys. Rev. E **83**, 061146 (2011)

Motivations: Directed Polymer in a Random Medium

- DPRM with one free end ("point to line")



$$E(\mathcal{C}) = \sum_{\langle i,j \rangle \in \mathcal{C}} \epsilon_{ij}$$

- $E_{\rm opt}$ $\equiv$ Energy of the optimal polymer
- X $\equiv$ Transverse coordinate of the optimal polymer

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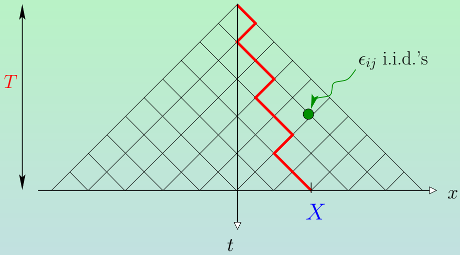

$\mathbb{E}(E_{\text{opt}}) \sim aT$, $\mathbb{E}(X) = 0$

$E_{\text{opt}} - \mathbb{E}(E_{\text{opt}}) \sim \mathcal{O}(T^{1/3})$

$X \sim \mathcal{O}(T^{2/3})$

Q: what is the joint pdf of E_{opt}, X ?

- $E_{\text{opt}} \equiv$ Energy of the optimal polymer
- $X \equiv$ Transverse coordinate of the optimal polymer

Fluctuations in DPRM and the Airy$_2$ process minus a parabola

- The "Airy$_2$ process minus a parabola"

$$Y(u) = \mathcal{A}_2(u) - u^2$$

where $\mathcal{A}_2(u)$ is the Airy$_2$ process   Prähofer, Spohn, 02

- Fluctuations in the DPRM

$$\lim_{T\to\infty} \frac{T^{-\frac{1}{3}}}{e_0}\left(E_{\rm opt}(T) - \mathbb{E}\left(E_{\rm opt}(T)\right)\right) = \max_{u\in\mathbb{R}} Y(u) = \mathcal{M}$$

and

$$\lim_{T\to\infty} \frac{T^{-\frac{2}{3}}}{\xi} X = \arg\max_{u\in\mathbb{R}} Y(u) = \mathcal{T}$$

Johansson, 03

$$\mathcal{M} = \max_{u\in\mathbb{R}} \; \mathcal{A}_2(u) - u^2 \; , \; \mathcal{T} = \arg\max_{u\in\mathbb{R}} \; \mathcal{A}_2(u) - u^2$$

- Joint pdf $\hat{P}(m,t)$ of $\mathcal{M},\mathcal{T}$ Moreno Flores, Quastel, Remenik, arXiv:1106.2716

$$\hat{P}(m,t) = 2^{1/3}\mathcal{F}_1(2^{2/3}m)\int_0^\infty dx \int_0^\infty dy \, \Phi_{-t,m}(2^{1/3}x)\rho_{2^{2/3}m}(x,y)\,\Phi_{t,m}(2^{1/3}y)$$

where

$$\Phi_{t,m}(x) = 2e^{x\,t}[t\mathrm{Ai}(t^2+m+x) + \mathrm{Ai}'(t^2+m+x)]$$

and

$$\rho_m(x,y) = (I - \Pi_0\mathcal{B}_m\Pi_0)^{-1}(x,y) \; , \; \mathcal{B}_m(x,y) = \mathrm{Ai}(x+y+m)$$

$$\mathcal{M} = \max_{u\in\mathbb{R}} \mathcal{A}_2(u) - u^2 \;,\; \mathcal{T} = \arg\max_{u\in\mathbb{R}} \mathcal{A}_2(u) - u^2$$

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- Marginal distribution $\hat{P}(t)$ of $\mathcal{T}$ Quastel, Remenik, arXiv:1203.2907

$$\log \hat{P}(t) = -ct^3 + o(t^3) \;,\; t\to\infty \text{ with } \frac{4}{3} \le c \le \frac{32}{3}$$

Extreme statistics of Airy$_2$ process minus a parabola

- This work: joint pdf $\hat{P}(m,t)$ of $\mathcal{M},\mathcal{T}$ G. S., arXiv:1203.1658

$$\hat{P}(m,t) = \frac{\pi^2}{2^{\frac{14}{3}}} \mathcal{F}_1(2^{2/3}m) \int_{2^{2/3}m}^{\infty} f(x, 2^{4/3}t) f(x, -2^{4/3}t)\, dx$$

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$$\underbrace{\frac{\partial}{\partial\zeta}\Psi = A\Psi\,,\;\frac{\partial}{\partial x}\Psi = B\Psi}_{\text{Lax Pair}}\,,\;\Psi = \begin{pmatrix}\Phi_1(\zeta,x)\\ \Phi_2(\zeta,x)\end{pmatrix}$$

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Lax Pair

$$A(\zeta,x) = \begin{pmatrix} 4\zeta q & 4\zeta^2 + x + 2q^2 + 2q' \\ -4\zeta^2 - x - 2q^2 + 2q' & -4\zeta q \end{pmatrix}\;,\; B(\zeta,x) = \begin{pmatrix} q & \zeta \\ -\zeta & -q \end{pmatrix}$$

Extreme statistics of Airy$_2$ process minus a parabola

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$$\log \hat{P}(t) = -ct^3 + o(t^3)\ ,\ t\to\infty \text{ with } c = \frac{4}{3}$$

Outline

1. Non-intersecting Brownian motions
2. Discrete orthogonal polynomials
3. Asymptotic analysis for large N: double scaling limit
4. Conclusion

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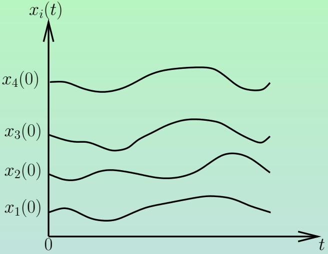
Non-intersecting Brownian motions and Airy$_2$

- N non intersecting Brownian motions in one-dimension

$$x_1(t) < x_2(t) < \ldots < x_N(t) \; , \quad \forall t \geq 0$$



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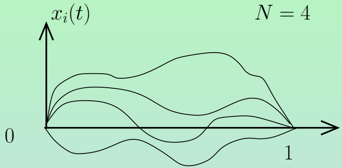

watermelons

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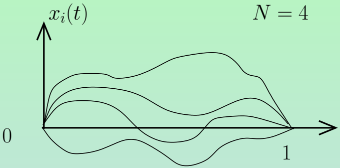

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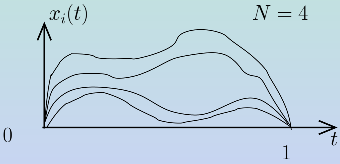
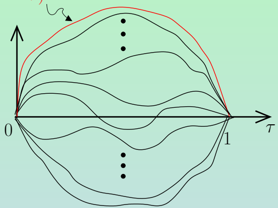


watermelons "with a wall"

Non-intersecting Brownian motions and Airy_2

- Large N limit

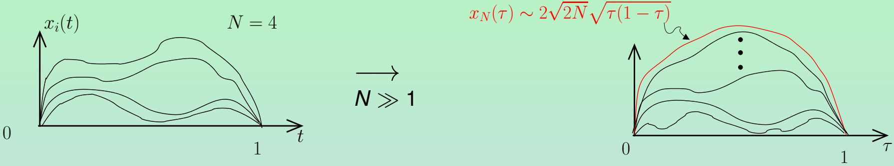


$$\lim_{N\to\infty} \frac{2\left[x_N\left(\frac{1}{2} + \frac{u}{2} N^{-\frac{1}{3}}\right) - \sqrt{N}\right]}{N^{-\frac{1}{6}}} \stackrel{d}{=} \mathcal{A}_2(u) - u^2$$

Prähofer & Spohn '02

$\mathcal{A}_2(u) \equiv \mathrm{Airy}_2$ process

- Focus on non-intersecting excursions



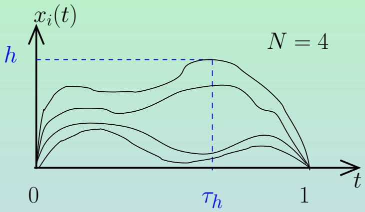
- Expected: the fluctuations of the top path are insensitive to the boundary, for $N \gg 1$

$$\lim_{N\to\infty} \frac{\alpha\left[x_N(\frac{1}{2} + \beta u N^{-\frac{1}{3}}) - \sqrt{2N}\right]}{N^{-\frac{1}{6}}} \overset{d}{=} \mathcal{A}_2(u) - u^2$$

Tracy & Widom '07

$\mathcal{A}_2(u) \equiv$ Airy$_2$ process

Extreme of N non-intersecting excursions



$$0 < x_1(t) < x_2(t) < \ldots < x_N(t)$$

$$\mathcal{H}_N = \max_{t \in [0,1]} x_N(t)$$

$$\mathcal{T}_{\mathcal{H}} = \arg\max_{t \in [0,1]} x_N(t)$$

$P_N(h, \tau_h) \equiv$ joint probability distribution function of $\mathcal{H}_N, \mathcal{T}_{\mathcal{H}}$

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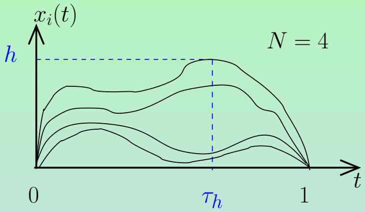

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HERE :

- Compute exactly $P_N(h, \tau_h)$ for any finite N
- Perform the large N limit to describe the extremes of Airy_2 minus a parabola

An exact expression for $P_N(h, \tau_h)$

$$P_N(h,\tau_h) = \frac{A_N}{h^{N(2N+1)+3}} \sum_{(n_1,\cdots,n_N,n'_N)\in\mathbb{Z}^{N+1}} \Bigg[(-1)^{n_N+n'_N}\, n_N^2 {n'_N}^2 \prod_{i=1}^{N-1} n_i^2\, \Delta_N(n_1^2,\ldots,n_{N-1}^2,n_N^2)$$
$$\times \Delta_N(n_1^2,\ldots,n_{N-1}^2,{n'_N}^2)\, e^{-\frac{\pi^2}{2h^2}\sum_{i=1}^{N-1} n_i^2 - \frac{\pi^2}{2h^2}\left[(1-\tau_h){n'_N}^2+\tau_h n_N^2\right]}\Bigg]$$

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$$A_N = \frac{N\,\pi^{2N^2+N+2}}{2^{N^2+N/2+1}\prod_{j=0}^{N-1}\Gamma(2+j)\Gamma\left(\frac{3}{2}+j\right)}$$

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Introducing (discrete) orthogonal polynomials

$$P_N(h, \tau_h) \quad \propto \sum_{(n_1,\cdots,n_N,n'_N)\in\mathbb{Z}^{N+1}} \Bigg[(-1)^{n_N+n'_N}\, n_N^2 {n'_N}^2 \prod_{i=1}^{N-1} n_i^2 \, \Delta_N(n_1^2,\ldots,n_{N-1}^2,n_N^2)$$

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- Discrete orthogonal polynomials

$$\sum_{n=-\infty}^{\infty} p_k(n)p_{k'}(n)e^{-\frac{\pi^2}{2h^2}n^2} = \delta_{k,k'}\, h_k \;,\; p_k(n) = n^k + \cdots$$

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- After some manipulations...

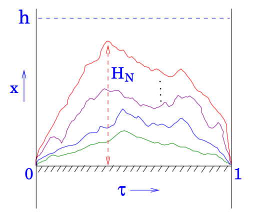
$$P_N(h,\tau_h) \propto \underbrace{\prod_{j=1}^{N} h_{2j-1}} \sum_{n,m}(-1)^{n+m}\, n\, m \sum_{k=1}^{N} \frac{p_{2k-1}(n)p_{2k-1}(m)}{h_{2k-1}}\, e^{-\frac{\pi^2}{2h^2}\left[(1-\tau_h)n^2+\tau_h m^2\right]}$$

$\mathbb{P}[\mathcal{H}_\mathcal{N} \leq h] = F_N(h)$ P. J. Forrester, S. N. Majumdar, G. S. '11

Intermezzo on the marginal distribution of $\mathcal{H}_N$

- Cumulative distribution of the maximum $\mathcal{H}_N$

$$\begin{aligned} F_N(h) &= \mathbb{P}\left[x_N(\tau) \leq h \,,\, \forall\, 0 \leq \tau \leq 1\right] \\ &= \int_0^1 d\tau_h \int_0^h dx\, P_N(x, \tau_h) \end{aligned}$$



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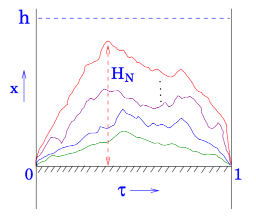

- Exact result for finite N G. S, S. N. Majumdar, A. Comtet, J. Randon-Furling '08

$$F_N(h) = \frac{B_N}{h^{2N^2+N}} \sum_{n_1,\cdots,n_N=0}^{+\infty} \prod_{i=1}^{N} n_i^2 \prod_{1\leq j<k\leq N} (n_j^2 - n_k^2)^2 e^{-\frac{\pi^2}{2h^2}\sum_{i=1}^N n_i^2}$$

$$= \frac{\tilde{B}_N}{h^{2N^2+N}} \prod_{j=1}^{N} h_{2j-1}$$

see also T. Feierl '08, M. Katori *et al.* '08

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$$F_N(h) = \frac{\tilde{B}_N}{h^{2N^2+N}} \prod_{j=1}^{N} h_{2j-1}$$

- Large N analysis

$$F_N(M) \to \mathcal{F}_1\left(2^{11/6} N^{1/6} \left| M - \sqrt{2N} \right|\right)$$

$$\mathcal{F}_1(t) = \exp\left(-\frac{1}{2} \int_t^\infty \left((s-t)\, q^2(s) - q(s) \right) ds \right)$$

$$\equiv \text{Tracy-Widom distribution for } \beta = 1$$

P. J. Forrester, S. N. Majumdar, G.S. '11

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P. J. Forrester, S. N. Majumdar, G.S. '11

- A recent rigorous proof of this result: K. Liechty arXiv:1111.4239

Back to discrete orthogonal polynomials

- Discrete orthogonal polynomials

$$\sum_{n=-\infty}^{\infty} p_k(n) p_{k'}(n) e^{-\frac{\pi^2}{2h^2} n^2} = \delta_{k,k'} h_k \; , \; p_k(n) = n^k + \cdots$$

- After some manipulations...

$$P_N(h, \tau_h) \propto \underbrace{\prod_{j=1}^{N} h_{2j-1}}\sum_{n,m} (-1)^{n+m} n\, m \sum_{k=1}^{N} \frac{p_{2k-1}(n) p_{2k-1}(m)}{h_{2k-1}} e^{-\frac{\pi^2}{2h^2}\left[(1-\tau_h)n^2 + \tau_h m^2\right]}$$

$\mathbb{P}[\mathcal{H}_N \leq h] = F_N(h)$ P. J. Forrester, S. N. Majumdar, G. S. '11

$$P_N(h, \frac{1}{2} + u) = \frac{\tilde{B}_N}{h^3} F_N(h) \sum_{k=1}^{N} G_{2k-1}(h, u) G_{2k-1}(h, -u)$$

$$G_{2k-1}(h, u) = \sum_{n=-\infty}^{\infty} (-1)^n \, n \, \psi_{2k-1}(n) e^{-\frac{u\pi^2}{2h^2} n^2} \; , \psi_k(n) = \frac{p_k(n)}{\sqrt{h_k}} e^{-\frac{\pi^2}{4h^2} n^2}$$

Differential recursion relations (I)

$$\sum_{n=-\infty}^{\infty} p_k(n) p_{k'}(n) e^{-\frac{\pi^2}{2h^2} n^2} = \delta_{k,k'} h_k \; , \; \psi_k(n) = \frac{p_k(n)}{\sqrt{h_k}} e^{-\frac{\pi^2}{4h^2} n^2}$$

- Three terms recursion relation

$$x\, p_k(x) = p_{k+1}(x) + R_k\, p_{k-1}(x) \; , \; R_k = \frac{h_k}{h_{k-1}} \; ,$$

$$x\, \psi_k(x) = \gamma_{k+1} \psi_{k+1}(x) + \gamma_k \psi_{k-1}(x) \; , \; \gamma_k = \sqrt{R_k}$$

- A first differential recursion relation for $G_{2k-1}(h, u)$

$$G_{2k-1}(h,u) = \sum_{n=-\infty}^{\infty} (-1)^n\, n\, \psi_{2k-1}(n) e^{-\frac{u\pi^2}{2h^2} n^2}$$

$$\frac{\partial}{\partial u} G_{2k-1} = -\frac{\pi^2}{2h^2} \left[\gamma_{2k}\gamma_{2k+1} G_{2k+1} + (\gamma_{2k}^2 + \gamma_{2k-1}^2) G_{2k-1} + \gamma_{2k-2}\gamma_{2k-1} G_{2k-3} \right]$$

Differential recursion relations (II)

$$\sum_{n=-\infty}^{\infty} p_k(n) p_{k'}(n) e^{-\frac{\pi^2}{2h^2} n^2} = \delta_{k,k'} h_k \ , \ \psi_k(n) = \frac{p_k(n)}{\sqrt{h_k}} e^{-\frac{\pi^2}{4h^2} n^2}$$

- A differential recursion relation for $\psi_k(n)$

$$-\frac{h^3}{2\pi^2} \frac{\partial}{\partial h} \psi_k = \frac{1}{4} \gamma_k \gamma_{k-1} \psi_{k-2} - \frac{1}{4} \gamma_{k+2} \gamma_{k+1} \psi_{k+2}$$

- A second differential recursion relation for $G_{2k-1}(h,u)$

$$\begin{aligned} -\frac{h^3}{2\pi^2} \frac{\partial}{\partial h} G_{2k-1} &= \left[\left(\frac{1}{4} - \frac{u}{2} \right) \gamma_{2k-1} \gamma_{2k-2} G_{2k-3} - \frac{u}{2} \left(\gamma_{2k}^2 + \gamma_{2k-1}^2 \right) G_{2k-1} \right. \\ &- \left. \left(\frac{1}{4} + \frac{u}{2} \right) \gamma_{2k} \gamma_{2k+1} G_{2k+1} \right] \end{aligned}$$

Outline

1. Non-intersecting Brownian motions
2. Discrete orthogonal polynomials
3. Asymptotic analysis for large N: double scaling limit
4. Conclusion

Double scaling limit

$$h \to \infty \;\&\; N \to \infty \;, \text{ with } N^{\frac{1}{6}}(h - \sqrt{2N}) \text{ fixed}$$

- Analysis of the coefficients $R_k = h_k / h_{k-1}$ for large k

$$R_k = \frac{h^4}{\pi^2} - (-1)^k h^{10/3} f_1(x_k) + h^{8/3} f_2(x_k) + \mathcal{O}(h^2) \;,\; x_k = h^{4/3}\left(1 - \frac{k}{h^2}\right)$$

$$f_1(x) = -\frac{2^{5/3}}{\pi^2} q(2^{2/3} x) \;,\; q''(s) = s q(s) + 2q^3(s) \;,\; q(s) \underset{s \to +\infty}{\sim} \mathrm{Ai}(s)$$

Gross, Matytsin '94

- Analysis of $G_{2k-1}(h,u)$ for large k

$$G_{2k-1}(h,u) = (-1)^k h^{5/3} \left(g_1(x_{2k}, v) + h^{-2/3} g_2(x_{2k}, v) + h^{-4/3} g_3(x_{2k}, v) + \mathcal{O}(h^{-2}) \right)$$

$$x_{2k} = h^{4/3}\left(1 - \frac{2k}{h^2}\right) \;,\; v = u\, h^{2/3}$$

Differential recursion relations in the dble scaling limit

$$\frac{\partial}{\partial u} G_{2k-1} = -\frac{\pi^2}{2h^2}\left[\gamma_{2k}\gamma_{2k+1} G_{2k+1} + (\gamma_{2k}^2 + \gamma_{2k-1}^2) G_{2k-1} + \gamma_{2k-2}\gamma_{2k-1} G_{2k-3}\right]$$

$$G_{2k-1}(h,u) = (-1)^k h^{5/3} g_1(x_{2k}, v) + \mathcal{O}(h)$$
$$g_1(x,v) = f(s = 2^{2/3}x, w = 2^{7/3}v)$$

where $f(s,w)$ satisfies G. S. '12

$$\frac{\partial}{\partial w} f(s,w) = \left[\frac{\partial^2}{\partial s^2} - (q^2 - q')\right] f(s,w)$$

Differential recursion relations in the dble scaling limit

$$-\frac{h^3}{2\pi^2}\frac{\partial}{\partial h}G_{2k-1} = \left[\left(\frac{1}{4}-\frac{u}{2}\right)\gamma_{2k-1}\gamma_{2k-2}G_{2k-3} - \frac{u}{2}\left(\gamma_{2k}^2+\gamma_{2k-1}^2\right)G_{2k-1} - \left(\frac{1}{4}+\frac{u}{2}\right)\gamma_{2k}\gamma_{2k+1}G_{2k+1}\right]$$

$$G_{2k-1}(h,u) = (-1)^k h^{5/3} g_1(x_{2k},v) + \mathcal{O}(h)$$
$$g_1(x,v) = f(s = 2^{2/3}x, w = 2^{7/3}v)$$

where $f(s,w)$ satisfies G. S. '12

$$4\frac{\partial^3 f}{\partial s^3} - 2w\frac{\partial^2 f}{\partial s^2} - \frac{\partial f}{\partial s}\left[6(q^2-q')+s\right] - f\left[3(q^2-q')'+2-2w(q^2-q')\right] = 0$$

with the asymptotic behavior (matching the regime $h \gg \sqrt{2N}$)

$$f(s,w) \sim \frac{-2^{11/3}}{\pi}e^{w^3+\frac{w\,s}{4}}\left(\frac{w}{4}\mathrm{Ai}(w^2/2^{8/3}+s/2^{2/3}) + \frac{1}{2^{2/3}}\mathrm{Ai}'(w^2/2^{8/3}+s/2^{2/3})\right)$$

Solution in terms of a psi-function associated to P II

$$f(s,w) = -\frac{2^{13/2}}{\pi^2}\int_0^\infty \zeta \Phi_2(\zeta, s) e^{-w\zeta^2}\, d\zeta$$

$$\underbrace{\frac{\partial}{\partial \zeta}\Psi = A\Psi \, , \, \frac{\partial}{\partial x}\Psi = B\Psi}_{\text{Lax Pair}} \, , \, \Psi = \begin{pmatrix} \Phi_1(\zeta, x) \\ \Phi_2(\zeta, x) \end{pmatrix} \, , \, \Phi_2(\zeta, x) \sim -\sin\left(\frac{4\zeta^3}{3} + x\zeta\right) \, , \, \zeta \gg 1$$

Lax Pair

$$A(\zeta, x) = \begin{pmatrix} 4\zeta q & 4\zeta^2 + x + 2q^2 + 2q' \\ -4\zeta^2 - x - 2q^2 + 2q' & -4\zeta q \end{pmatrix} \, , \, B(\zeta, x) = \begin{pmatrix} q & \zeta \\ -\zeta & -q \end{pmatrix}$$

Large N limit and back to $\mathcal{A}_2(u) - u^2$

- Large N asymptotics for extremes of excursions

$$\lim_{N\to\infty} 2^{-\frac{9}{2}} N^{-\frac{1}{2}} P_N(\sqrt{2N} + 2^{-\frac{11}{6}} s N^{-\frac{1}{6}}, \frac{1}{2} + 2^{-\frac{8}{3}} w N^{-\frac{1}{3}}) = P(s,w)$$

$$P(s,w) = \frac{\pi^2}{2^{\frac{20}{3}}} \mathcal{F}_1(s) \int_s^\infty f(x,w) f(x,-w)\, dx$$

$$f(x,w) = -\frac{2^{\frac{13}{2}}}{\pi^2} \int_0^\infty \zeta \Phi_2(\zeta,x) e^{-w\zeta^2}\, d\zeta$$

- Results for Airy$_2$ process minus a parabola

$$\hat{P}(m,t) = 4\, P(2^{2/3} m,\, 2^{4/3} t)$$

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Conclusion

- Exact results for extreme statistics of N vicious walkers
- Large N analysis using dble scaling limit of discrete orthogonal polynomials
- Connection between extreme statistics of $\mathcal{A}_2(u) - u^2$ and Painlevé
- Marginal distribution $\hat{P}(t)$ of $\mathcal{T}$ G. S., arXiv:1203.1658

$$\log \hat{P}(t) = -c\, t^3 + o(t^3)\ ,\ t \to \infty \text{ with } c = \frac{4}{3}$$