Stochastic Differential Equations Related to Soft-Edge Scaling Limit

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Introduction

The Gaussian ensembles are introduced as Hermitian matrices with independent elements of Gaussian random variables, and joint distribution invariant under conjugation by appropriate unitary matrices. The ensemble are divided into classes according to the element being real, complex or real quaternion, and their invariance under conjugation by

orthogonal matices (GOE),

unitary matrices (GUE),

unitary symplectic matrices (GSE),

respectively.

Introduction

The distribution of eigenvalues of the ensembles with size $n \times n$ are given by

$$m_\beta^n(d\mathbf{x}_n) = \frac{1}{Z} \prod_{i<j} |x_i - x_j|^\beta \exp\left\{ -\frac{\beta}{4} \sum_{i=1}^n |x_i|^2 \right\} d\mathbf{x}_n,$$

on the configuration space of n particles:

$$\mathbb{W}_n = \{\mathbf{x}_n \in \mathbb{R}^n : x_1 < x_2 < \cdots < x_n\}$$

where the GOE, the GUE, and the GSE correspond with $\beta = 1, 2$ and 4, respectively.

Introduction

(Bulk scaling limit) For the eigenvalues $\{\lambda_1^n, \dots, \lambda_n^n\}$

$$\{\sqrt{n}\lambda_1^n, \dots, \sqrt{n}\lambda_n^n\} \to \mu_{\sin,\beta}, \quad \text{weakly as } n \to \infty.$$

$\mu_{\sin,\beta}$ is a probability measure on the configuration space of unlabelled particles:

$$\mathfrak{M} = \Big\{ \xi : \xi \text{ is a nonnegative integer valued Radon measures in } \mathbb{R} \Big\}$$

Any element ξ of $\mathfrak{M}$ can be represented as: $\xi(\cdot) = \sum_{j \in \mathbb{I}} \delta_{x_j}(\cdot)$ with some sequence $(x_j)_{j \in \mathbb{I}}$ of $\mathbb{R}$ satisfying $\sharp\{j \in \mathbb{I} : x_j \in K\} < \infty$, for any compact set K. $\mathfrak{M}$ is a Polish space with the vague topology.

Introduction

For $\beta = 2$ (GUE) $\mu_{\sin,2}$ is the determinantal point process(DPP) , in which any spatial correlation function ρ_m is given by a determinant with the *sine kernel*

$$K_{\sin,2}(x,y) = K_{\sin}(x,y) \equiv \frac{\sin\{\pi(y-x)\}}{\pi(y-x)}, \quad x,y \in \mathbb{R}.$$

The moment generating function is given by a Fredholm determinant

$$\int_{\mathfrak{M}} \exp\left\{ \int_{\mathbb{R}} f(x)\xi(dx)\right\}\mu_{\sin,2}(d\xi) = \mathop{\mathrm{Det}}_{(x,y)\in\mathbb{R}^2}\Big[\delta_x(y) + K_{\sin}(x,y)\chi(y)\Big],$$

for $f \in \mathrm{C}_c(\mathbb{R})$, where $\chi(\cdot) = e^{f(\cdot)} - 1$.

Introduction

For $\beta = 1$ (GOE) $\mu_{\sin,1}$ is the quaternion DPP with kernels

$$K_{\sin,1}(x,y) = \begin{bmatrix} K_{\sin}(x,y) & -\frac{\partial}{\partial y} K_{\sin}(x,y) \\ -\frac{1}{2}\mathrm{sign}(x-y) + \int_y^x K_{\sin,2}(u,y)du & K_{\sin}(x,y) \end{bmatrix}.$$

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Here we use a natural identification between the 2×2 complex matrices and the quaternions:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

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For $\beta = 4$ (GSE) $\mu_{\sin,4}$ is the quaternion DPP with kernels

$$K_{\sin,4}(x,y) = \begin{bmatrix} K_{\sin}(2x,2y) & -\frac{\partial}{\partial y} K_{\sin}(2x,2y) \\ \int_y^x K_{\sin}(2u,2y)du & K_{\sin}(2x,2y) \end{bmatrix}.$$

Introduction

The moment generating function is given by

$$\int_{\mathfrak{M}} \exp\left\{\int_{\mathbb{R}} f(x)\xi(dx)\right\}\mu_{\sin,\beta}(d\xi)$$

$$= \underset{(x,y)\in\mathbb{R}^2}{\mathrm{Det}}\left[\delta_x(y)\otimes I_2 + K_{\sin,\beta}(x,y)\chi_2(y)\right]^{1/2}$$

$$= \underset{(x,y)\in\mathbb{R}^2}{\mathrm{Pf}}\left[\delta_x(y)\otimes I_2 + K_{\sin,\beta}(x,y)\chi_2(y)\right],$$

for $f \in C_c(\mathbb{R})$, where

$$\chi_2(\cdot) = \begin{pmatrix} e^{f(\cdot)} - 1 \\ e^{f(\cdot)} - 1 \end{pmatrix},$$

and Pf denotes the Fredholm pfaffian.

Introduction

$n \times n$ Hermitian matrix valued process $(n \in \mathbb{N})$

$$M(t) = \begin{pmatrix} M_{11}(t) & M_{12}(t) & \cdots & M_{1n}(t) \\ M_{21}(t) & M_{22}(t) & \cdots & M_{2n}(t) \\ & & \cdots & \\ M_{1}(t) & M_{n2}(t) & \cdots & M_{nn}(t) \end{pmatrix}, \quad M_{\ell k}(t) = M_{k\ell}(t)^{\dagger}.$$

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(GOE) $B^{\mathrm{R}}_{k\ell}(t)$, $1 \le k \le \ell \le n$: indep. BMs

$$M_{k\ell}(t) = \frac{1}{\sqrt{2}} B^{\mathrm{R}}_{k\ell}(t), \quad 1 \le k < \ell \le n,$$
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(GUE) $B^{\mathrm{R}}_{k\ell}(t)$, $B^{\mathrm{I}}_{k\ell}(t)$,$1 \le k \le \ell \le n$: indep. BMs

$$M_{k\ell}(t) = \frac{1}{\sqrt{2}} B^{\mathrm{R}}_{k\ell}(t) + \frac{\sqrt{-1}}{\sqrt{2}} B^{\mathrm{I}}_{k\ell}(t), \quad 1 \le k < \ell \le n,$$

$$M_{kk}(t) = B^{\mathrm{R}}_{kk}(t), \quad 1 \le k \le n,$$

Introduction

(GSE) $B^{\alpha}_{k\ell}(t)$, $\alpha = 0, 1, 2, 3$, $1 \le k \le \ell \le n$: indep. BMs

$$M^0_{k\ell}(t) = \frac{1}{\sqrt{2}} B^0_{k\ell}(t), \quad 1 \le k < \ell \le n,$$
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For $\alpha = 1, 2, 3$,

$$M^{\alpha}_{k\ell}(t) = \frac{\sqrt{-1}}{\sqrt{2}} B^{\alpha}_{k\ell}(t), \quad 1 \le k < \ell \le n,$$
$$M^{\alpha}_{kk}(t) = 0, \quad 1 \le k \le n,$$

$2n \times 2n$ self dual Hermitian matrix valued process

$$M(t) = M^0(t) \otimes I + \sum_{\alpha=1}^{3} M^{\alpha}(t) \otimes e_{\alpha}$$

Introduction

The eigen-valued process is Dyson's Brownian motion model [Dyson 62], which is a one parameter family of the systems solving the following stochastic differential equation:

$$dX_j(t) = dB_j(t) + \frac{\beta}{2} \sum_{\substack{k:1\le k\le n \\ k\neq j}} \frac{dt}{X_j(s) - X_k(s)}, \ 1 \le j \le n \qquad (1)$$

where $B_j(t), j = 1, 2, \dots, n$ are independent one dimensional Brownian motions.

Osada[PTRF: online first] constructed the diffusion process which solves the SDE (1) with $n = \infty$ by the Dirichlet form technique associated with the measure $\mu_{\mathrm{sin},\beta}$, $\beta = 1, 2$ and 4.

Introduction

(Soft edge scaling limit) As $n \to \infty$

$$\{n^{1/6}\lambda_1^n - 2n^{2/3}, \dots, n^{1/6}\lambda_n^n - 2n^{2/3}\} \to \mu_{\mathrm{Ai},\beta}, \text{ weakly,}$$

For $\beta = 2$, $\mu_{\mathrm{Ai},2}$ is the determinantal point process with the Airy kernel

$$K_{\mathrm{Ai}}(x,y) = \begin{cases} \dfrac{\mathrm{A}i(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y} & \text{if } x \neq y \\ (\mathrm{Ai}'(x))^2 - x(\mathrm{Ai}(x))^2 & \text{if } x = y, \end{cases}$$

where $\mathrm{Ai}(\cdot)$ is the Airy function defined by

$$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_{\mathbb{R}} dk\, e^{i(zk+k^3/3)}, \quad z \in \mathbb{C},$$

and $\mathrm{Ai}'(x) = d\mathrm{Ai}(x)/dx$.

Introduction

For $\beta = 1$, $\mu_{\mathrm{Ai},1}$ is the quaternion DPP with the kernel

$$K_{\mathrm{Ai},1}(x,y) = \begin{bmatrix} K_{\mathrm{Ai}}(x,y) & -\frac{\partial}{\partial y} K_{\mathrm{Ai}}(x,y) \\ -\frac{1}{2}\mathrm{sign}(x-y) + \int_y^x K_{\mathrm{Ai}}(u,y)du & K_{\mathrm{Ai}}(x,y) \end{bmatrix}$$
$$+ \frac{1}{2} \begin{bmatrix} \mathrm{Ai}(x)\left(1 - \int_y^\infty \mathrm{Ai}(u)du\right) & -\mathrm{Ai}(x)\mathrm{Ai}(y) \\ \int_y^x \mathrm{Ai}(u)du \left(1 - \int_y^\infty \mathrm{Ai}(u)du\right) & \left(1 - \int_x^\infty \mathrm{Ai}(u)du\right)\mathrm{Ai}(y) \end{bmatrix}.$$

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$$+\frac{1}{2}\begin{bmatrix} \mathrm{Ai}(x)\left(1-\int_y^\infty \mathrm{Ai}(u)du\right) & -\mathrm{Ai}(x)\mathrm{Ai}(y) \\ \int_y^x \mathrm{Ai}(u)du\left(1-\int_y^\infty \mathrm{Ai}(u)du\right) & \left(1-\int_x^\infty \mathrm{Ai}(u)du\right)\mathrm{Ai}(y) \end{bmatrix}.$$

For $\beta = 4$, $\mu_{\mathrm{Ai},4}$ is the quaternion DPP with the kernel

$$K_{\mathrm{Ai},4}(x,y) = \frac{1}{2^{1/3}}\begin{bmatrix} K_{\mathrm{Ai}}(2^{2/3}x, 2^{2/3}y) & -\frac{\partial}{\partial y} K_{\mathrm{Ai}}(2^{2/3}x, 2^{2/3}y) \\ \int_y^x K_{\mathrm{Ai}}(2^{2/3}u, 2^{2/3}y)du & K_{\mathrm{Ai}}(2^{2/3}x, 2^{2/3}y) \end{bmatrix}$$
$$+\frac{1}{2^{2/3}}\begin{bmatrix} -\mathrm{Ai}(2^{2/3}x)\int_y^\infty \mathrm{Ai}(2^{2/3}u)du & -\mathrm{Ai}(2^{2/3}x)\mathrm{Ai}(2^{2/3}y) \\ -\int_y^x \mathrm{Ai}(2^{2/3}u)du\int_y^\infty \mathrm{Ai}(2^{2/3}u)du & -\int_x^\infty \mathrm{Ai}(2^{2/3}u)du\mathrm{Ai}(2^{2/3}y) \end{bmatrix}.$$

Results

Let $\beta = 1, 2$ and 4.

(1) There exists a diffusion process describing an infinite particle system, which has $\mu_{\mathrm{Ai},\beta}$ as a reversible measure.

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(2) The system solves the ISDE given by

$$dX_j(t) = dB_j(t) + \frac{\beta}{2} \lim_{L\to\infty} \left\{ \sum_{\substack{k\neq j,\\ |X_k(t)|<L}} \frac{1}{X_j(t) - X_k(t)} - \int_{|x|<L} \frac{\widehat{\rho}(x)dx}{-x} \right\} dt,$$

$$j \in \mathbb{N},$$

where

$$\widehat{\rho}(x) = \frac{\sqrt{-x}}{\pi} \mathbf{1}(x < 0).$$

Dirichlet spaces

A function f defined on the configuration space $\mathfrak{M}$ is local if $f(\xi) = f(\xi_K)$ for some compact set K.
A local function f is smooth if $f(\sum_{j=1}^n \delta_{x_j}) = \tilde{f}(x_1, x_2, \ldots, x_n)$ with some smooth function $\tilde{f}$ on $\mathbb{R}^n$ with compact support. Put

$$\mathcal{D}_0 = \{f : f \text{ is local and smooth}\}.$$

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We put

$$\mathbb{D}[f, g](\xi) = \frac{1}{2} \sum_{j=1}^{\xi(K)} \frac{\partial \tilde{f}(\mathbf{x})}{\partial x_j} \frac{\partial \tilde{g}(\mathbf{x})}{\partial x_j}$$

and for a probability measure μ we introduce the bilinear form

$$\mathcal{E}^{\mu}(f, g) = \int_{\mathfrak{M}} \mathbb{D}[f, g] d\mu, \quad f, g \in \mathcal{D}_0.$$

quasi Gibbs measure

Let Φ be a free potential, Ψ be an interaction potential. For a given sequence $\{b_r\}$ of $\mathbb{N}$ we introduce a Hamiltonian on $I_r = (-b_r, b_r)$:

$$H_r(\xi) = H_r^{\Phi,\Psi}(\xi) = \sum_{x_j \in I_r} \Phi(x_j) + \sum_{x_j, x_k \in I_r, j<k} \Psi(x_j, x_k)$$

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Definition A probability measure μ is said to be a (Φ, Ψ)-quasi Gibbs measure if there exists an increasing sequence $\{b_r\}$ of $\mathbb{N}$ and measures $\{\mu_{r,k}^m\}$ such that for each $r, m \in \mathbb{N}$ satisfying

$$\mu_{r,k}^m \leq \mu_{r,k+1}^m, \quad k \in \mathbb{N}, \quad \lim_{k\to\infty} \mu_{r,k}^m = \mu(\cdot \cap \{\xi(I_r) = m\}), \textit{weekly}$$

and that for all $r, m, k \in \mathbb{N}$ and for $\mu_{r,k}^m$-a.s. $\xi \in \mathfrak{M}$

$$c^{-1} e^{-H_r(\xi)} \mathbf{1}_{\{\xi(I_r)=m\}} \Lambda(d\zeta) \leq \mu_{r,k}^m(\pi_{I_r} \in d\zeta | \xi_{I_r^c}) \leq c e^{-H_r(\xi)} \mathbf{1}_{\{\xi(I_r)=m\}} \Lambda(d\zeta)$$

Here Λ is the Poisson random measure with intensity measure dx.

quasi regular Dirichlet space

Theorem (Bulk) [Osada:to appear in AOP]
Let $\beta = 1, 2, 4$.
(1) The probability measure $\mu_{\sin,\beta}$ is a quasi Gibbs measure with $\Phi(x) = 0$ and $\Psi(x) = -\beta \log |x - y|$.
(2) The closure of $(\mathcal{E}^{\mu_{\sin,\beta}}, \mathcal{D}_0, L^2(\mathfrak{M}, \mu_{\sin,\beta}))$ is a quasi Dirichlet space, and there exists a $\mu_{\sin}$-reversible diffusion process $(\Xi^{\sin,\beta}(t), P)$ associated with the Diriclet space.

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Theorem 1 (Soft-Edge) [Osada-T]
Let $\beta = 1, 2, 4$.
(1) The probability measure $\mu_{\mathrm{Ai},\beta}$ is a quasi Gibbs measure with $\Phi(x) = 0$ and $\Psi(x) = -\beta \log |x - y|$.
(2) The closure of $(\mathcal{E}^{\mu_{\mathrm{Ai},\beta}}, \mathcal{D}_0, L^2(\mathfrak{M}, \mu_{\mathrm{Ai},\beta}))$ is a quasi Dirichlet space, and there exists a μ_{Ai}-reversible diffusion process $(\Xi^{\mathrm{Ai},\beta}(t), P)$ associated with the Diriclet space.

Extended Airy kerenel

Let $(\xi^{\mathrm{Ai}}(t), P)$ be the determinantal process with the extended Airy kernel

$$\mathcal{K}^{\mathrm{Ai}}(s,x;t,y) = \begin{cases} \int_{-\infty}^{0} du\, e^{(t-s)u/2}\mathrm{Ai}(x-u)\mathrm{Ai}(y-u), & \text{if } s \le t, \\ -\int_{0}^{\infty} du\, e^{(t-s)u/2}\mathrm{Ai}(x-u)\mathrm{Ai}(y-u), & \text{if } s > t. \end{cases}$$

The process is reversible process with reversible measure $\mu_{\mathrm{Ai},2}$.
[Forrester-Nagao-Honner (1999)],[Prähofer-Spohn (2002)]

Theorem [Katori-T: MPRF(2011)]
The process $(\xi^{\mathrm{Ai}}(t), P)$ is a continuos reversible Markov process.

Open problem. $(\xi^{\mathrm{Ai}}(t), P) = (\Xi^{\mathrm{Ai},2}(t), P)$?

Proposition [Katori-T: JSP(2007)]
The Dirichlet space of the process $(\xi^{\mathrm{Ai}}(t), P)$ is a closed extension of $(\mathcal{E}^{\mu_{\mathrm{Ai}}}, \mathcal{D}_0, L^2(\mathfrak{M}, \mu_{\mathrm{Ai}}))$.

Infinite dimensional Stochstic Differential Equations

Theorem 2[Osada-T]
Let $\beta = 1, 2, 4$. The process $\Xi^{\mathrm{Ai},\beta}(t) = \sum_{j\in\mathbb{N}} \delta_{X_j(t)}$ satisfies

$$dX_j(t) = dB_j(t) \quad + \quad \frac{\beta}{2} \lim_{L\to\infty} \left\{ \sum_{k\neq j:|X_k(t)|<L} \frac{1}{X_j(t) - X_k(t)} - \int_{|x|<L} \frac{\widehat{\rho}(x)}{-x} dx \right\} dt, \qquad j \in \mathbb{N},$$

where $B_j(t)$, $j \in \mathbb{N}$ are independent Brownian motions and

$$\widehat{\rho}(x) = \frac{\sqrt{-x}}{\pi} \mathbf{1}(x < 0).$$

Log derivative

Let $\mu_{\mathbf{x}}$ be the Palm measure conditioned at $\mathbf{x} = (x_1, \dots, x_k \in \mathbb{R}^k$

$$\mu_{\mathbf{x}} = \mu\left(\cdot - \sum_{j=1}^{k} \delta_{x_j} \middle| \xi(x_j) \geq 1 \text{ for } j = 1, 2, \dots, k \right).$$

Let μ^k be the Campbell measure of μ:

$$\mu^k(A \times B) = \int_A \mu_{\mathbf{x}}(B) \rho^k(\mathbf{x}) d\mathbf{x}, \quad A \in \mathcal{B}(\mathbb{R}^k), B \in \mathcal{B}(\mathfrak{M}).$$

We call $\mathbf{d}^\mu \in L^1_{loc}(\mathbb{R} \times \mathfrak{M}, \mu^1)$ the log derivative of μ if $\mathbf{d}^\mu$ satisfies

$$\int_{\mathbb{R}\times\mathfrak{M}} \mathbf{d}^\mu(x, \eta) f(x, \eta) d\mu^1(x, \eta) = -\int_{\mathbb{R}\times\mathfrak{M}} \nabla_x f(x, \eta) d\mu^1(x, \eta),$$

$$f \in C_c^\infty(\mathbb{R}) \otimes \mathcal{D}_0.$$

ISDE

Theorem [Osada, PTRF (on line first)]
Assume that there exists a log derivative $\mathbf{d}^{\mu}$ (and some conditions). There exists $\mathfrak{M}_0 \subset \mathfrak{M}$ such that $\mu(\mathfrak{M}_0) = 1$, and for any $\xi = \sum_{j \in \mathbb{N}} \delta_{x_j} \in \mathfrak{M}_0$, there exists $\mathbb{R}^{\mathbb{N}}$-valued continuous process $\mathbf{X}(t) = (X_j(t))_{j=1}^{\infty}$ satisfying $\mathbf{X}(0) = \mathbf{x} = (x_j)_{j=1}^{\infty}$ and

$$dX_j(t) = dB_j(t) + \frac{1}{2}\mathbf{d}^{\mu}\left(X_j(t), \sum_{k:k\neq j} \delta_{X_k(t)}\right)dt, \quad j \in \mathbb{N}.$$

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Lemma (Bulk) [Osada, PTRF online first] Let $\beta = 1, 2, 4$. For $x \in \mathbb{R}$ and $\eta = \sum_{j\in\mathbb{N}} \delta_{y_j}$ with $\eta(\{x\}) = 0$,

$$\mathbf{d}^{\mu_{\sin,\beta}}(x,\eta) = \beta \lim_{L\to\infty} \sum_{j:|x-y_j|\leq L} \frac{1}{x-y_j}$$

Key lemma

The key part in the proof of Theorem 2 is to determine the log derivative of $\mu_{\mathrm{Ai},\beta}$.

Key lemma (Soft-Edge) Let $\beta = 1, 2, 4$. For $x \in \mathbb{R}$ and $\eta = \sum_{j\in\mathbb{N}} \delta_{y_j}$ with $\eta(\{x\}) = 0$,

$$\mathsf{d}^{\mu_{\mathrm{Ai},\beta}}(x,\eta) = \beta \lim_{L\to\infty} \left\{ \sum_{j:|x-y_j|\le L} \frac{1}{x-y_j} - \int_{|u|\le L} \frac{\widehat{\rho}(u)}{-u} du \right\}.$$

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Proof

To prove the key lemma, we use n particle sysytem:

$$m_\beta^n(d\mathbf{u}_n) = \frac{1}{Z}\prod_{i<j}|u_i - u_j|^\beta \exp\Big\{-\frac{\beta}{4}\sum_{i=1}^n |u_i|^2\Big\}d\mathbf{u}_n,$$

We put $u_j = 2\sqrt{n} + \frac{x_j}{n^{1/6}}$ and intrduce the measure defined by

$$\mu_{\mathcal{A},\beta}^n(d\mathbf{x}_n) = \frac{1}{Z}\prod_{i<j}|x_i - x_j|^\beta \exp\left\{-\frac{\beta}{4}\sum_{i=1}^n |2\sqrt{n} + n^{-1/6}x_i|^2\right\} d\mathbf{x}_n,$$

The log derivative $\mathbf{d}^n$ of the measure $\mu_{\mathcal{A},\beta}^n$ is given by

$$\mathbf{d}^n(x,\eta) = \mathbf{d}^n\left(x, \sum_{j=1}^{n-1}\delta_{y_j}\right) = \beta\Big\{\sum_{j=1}^{n-1}\frac{1}{x-y_j} - n^{1/3} - \frac{n^{-1/3}}{2}x\Big\}.$$

Proof

Lemma 3 is derived from the fact that

$$\mathbf{d}^{\mu^{\mathrm{Ai}}}(x,\eta) = \lim_{n\to\infty} \mathbf{d}^{n}(x,\eta) = \beta \lim_{L\to\infty} \left\{ \sum_{|x-y_j|<L} \frac{1}{x-y_j} - \int_{|u|\le L} \frac{\widehat{\rho}(y)}{-y} du \right\} \tag{2}$$

Proof

Lemma 3 is derived from the fact that

$$\mathbf{d}^{\mu^{\mathrm{Ai}}}(x,\eta) = \lim_{n\to\infty} \mathbf{d}^n(x,\eta) = \beta \lim_{L\to\infty} \left\{ \sum_{|x-y_j|<L} \frac{1}{x-y_j} - \int_{|u|\le L} \frac{\widehat{\rho}(y)}{-y} du \right\} \tag{2}$$

To check (2) we divide $\mathbf{d}^n/\beta$ into three parts:

$$\mathrm{g}_L^n(x,\eta) = \sum_{|x-y_j|<L} \frac{1}{x-y_j} - \int_{|x-u|<L} \frac{\rho^n_{\mathcal{A},\beta,x}(u)}{x-u} du,$$

$$w_L^n(x,\eta) = \sum_{|x-y_j|\ge L} \frac{1}{x-y_j} - \int_{|x-u|\ge L} \frac{\rho^n_{\mathcal{A},\beta,x}(u)}{x-u} du,$$

$$u^n(x) = \int_{\mathbb{R}} \frac{\rho^n_{\mathcal{A},\beta,x}(u)}{x-u} du - n^{1/3} - \frac{n^{-1/3}}{2} x.$$

Proof

The fact (2) is obtained if the following conditions hold:

$$\lim_{n\to\infty} \mathrm{g}_L^n(x,\eta) = \mathrm{g}_L(x,\eta), \quad \text{in } L^{\hat{p}}(\mu^1_{\mathrm{Ai},\beta}) \text{ for any } L>0, \tag{3}$$

$$\lim_{L\to\infty} \limsup_{n\to\infty} \int_{[-r,r]\times\mathfrak{M}} |w_L^n(x,\mathrm{y})|^{\hat{p}} d\mu_{\mathcal{A}}^{n,1}(dxd\eta) = 0, \tag{4}$$

$$\lim_{n\to\infty} u^n(x) = u(x), \quad \text{in } L^{\hat{p}}_{loc}(\mathbb{R}, dx) \ , \tag{5}$$

with

$$\mathrm{g}_L(x,\eta) = \sum_{|x-y_j|<L} \frac{1}{x-y_j} - \int_{|x-u|<L} \frac{\rho_{\mathrm{Ai},\beta,x}(u)}{x-u} du,$$

and

$$u(x) = \lim_{L\to\infty} \left\{ \int_{|u|\le L} \frac{\rho_{\mathrm{Ai},\beta,x}(u)}{x-u} du - \int_{|u|\le L} \frac{\widehat{\rho}(u)}{-u} du \right\} \in L^{\hat{p}}_{loc}(\mathbb{R}, dx).$$

Proof

In the case $\beta = 2$, $\mu_{\mathcal{A}}^n$ is the DPP with the correlation kernel

$$K_{\mathcal{A}}^n(x,y) = n^{1/3}\frac{\Psi_n(x)\Psi_{n-1}(y) - \Psi_{n-1}(x)\Psi_n(y)}{x-y}$$

where $\Psi_n(x) = n^{1/12}\varphi_n\left(\sqrt{2n} + \frac{x}{\sqrt{2}n^{1/6}}\right)$, and $\varphi_k(x)$ is the normalized orthogonal functions on $\mathbb{R}$ comprising the Hermite polynomials $H_k(x)$.

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where $\Psi_n(x) = n^{1/12}\varphi_n\left(\sqrt{2n} + \frac{x}{\sqrt{2}n^{1/6}}\right)$, and $\varphi_k(x)$ is the normalized orthogonal functions on $\mathbb{R}$ comprising the Hermite polynomials $H_k(x)$. We also use the function

$$\widehat{\rho}^n(u) = \frac{1}{\pi}\sqrt{-u\left(1 + \frac{u}{4n^{2/3}}\right)}, \quad -4n^{2/3} \le u \le 0$$

and the facts

$$\int_{\mathbb{R}} \frac{\widehat{\rho}^n(u)}{-u} du = n^{1/3}, \quad \lim_{n\to\infty} \widehat{\rho}^n(u) \to \widehat{\rho}(u).$$

Thanks

Thank you for your attention!