Domino tilings, non-intersecting Random Motions and the tacnode Process

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JOINT WORK with

MARK ADLER & KURT JOHANSSON

Three models, leading to Tacnode process: (Universality!)
(1) continuous time and discrete space process: (random walks)

Mark Adler, Patrik Ferrari & PvM : *Non-intersecting random walks in the neighborhood of a symmetric tacnode, Annals of Prob 2011 (arXiv:1007.1163)*

(2) continuous time and continuous space process

K. Johansson: *Non-colliding Brownian Motions and the extended tacnode process* (arXiv:1105.4027)

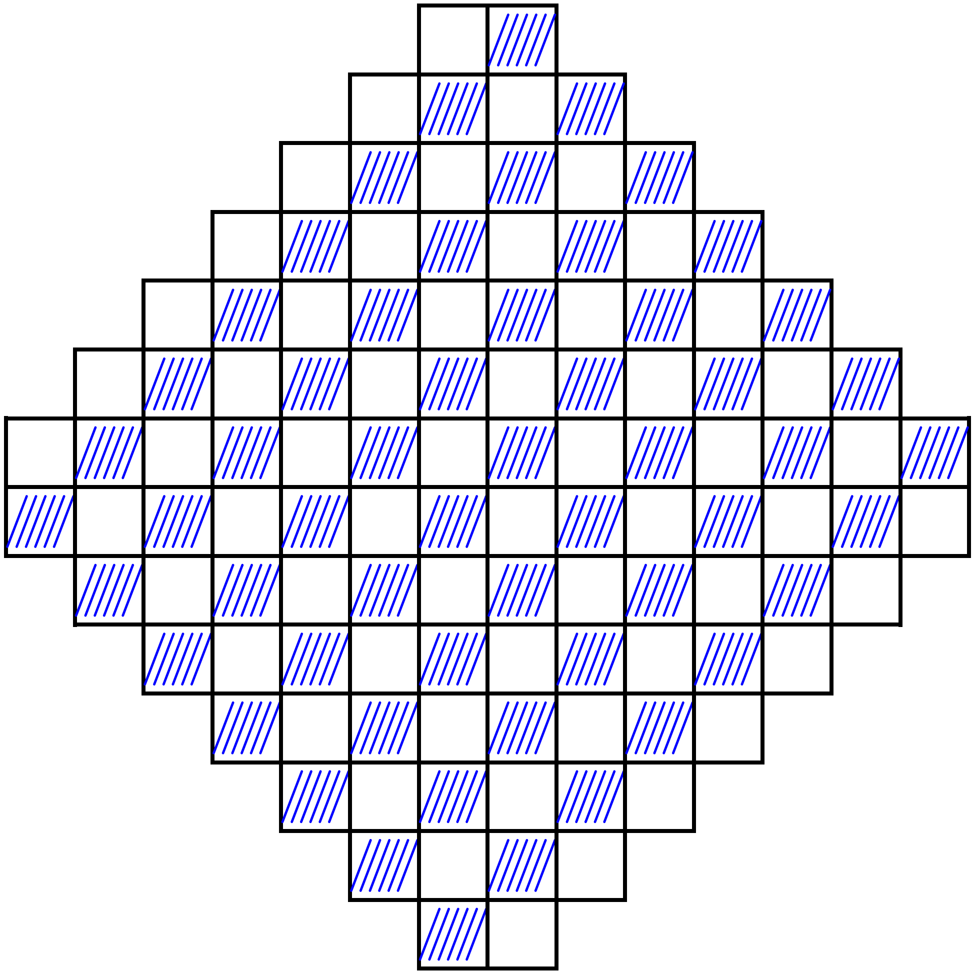
Delvaux-Kuijlaars-Zhang: *Critical behavior of non-intersecting Brownian motions at a Tacnode* (arXiv:1009.2457)

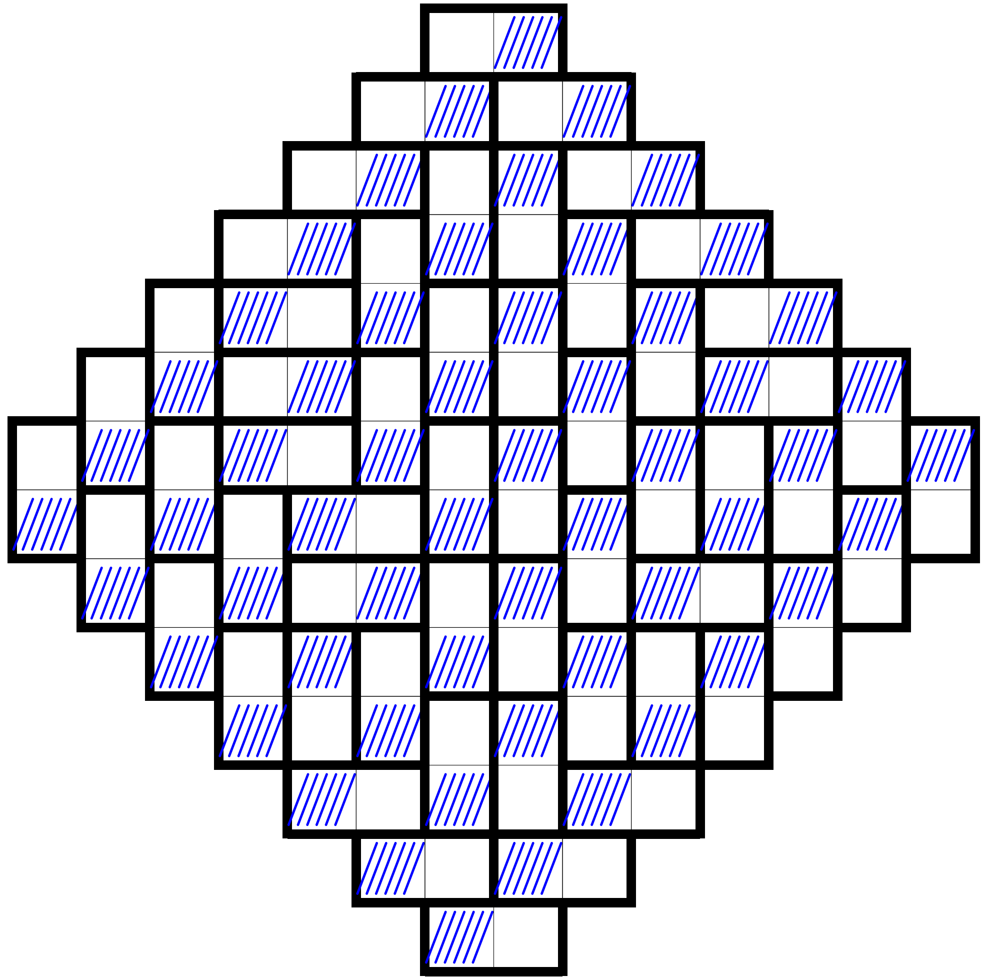
(3) discrete time and discrete space process (Domino tilings of Double Aztec Diamonds):

Mark Adler, Kurt Johansson & PvM: *Double Aztec Diamonds and the Tacnode Process* (arXiv:1112.5532)(to appear 2012)

1. Domino tilings of Double Aztec Diamonds

Domino tilings of an Aztec diamond :





4 types of coverings by domino's!

Kasteleyn 1961, Elkies, Kuperberg, Larsen, Propp '92

Cohn, Elkies, Propp '96, Johansson '00, '03, '05

M. Fulmek, C. Krattenthaler '00, Johansson-Nordenstam '06

Double Aztec diamond (Adler-Johansson-PvM '11)

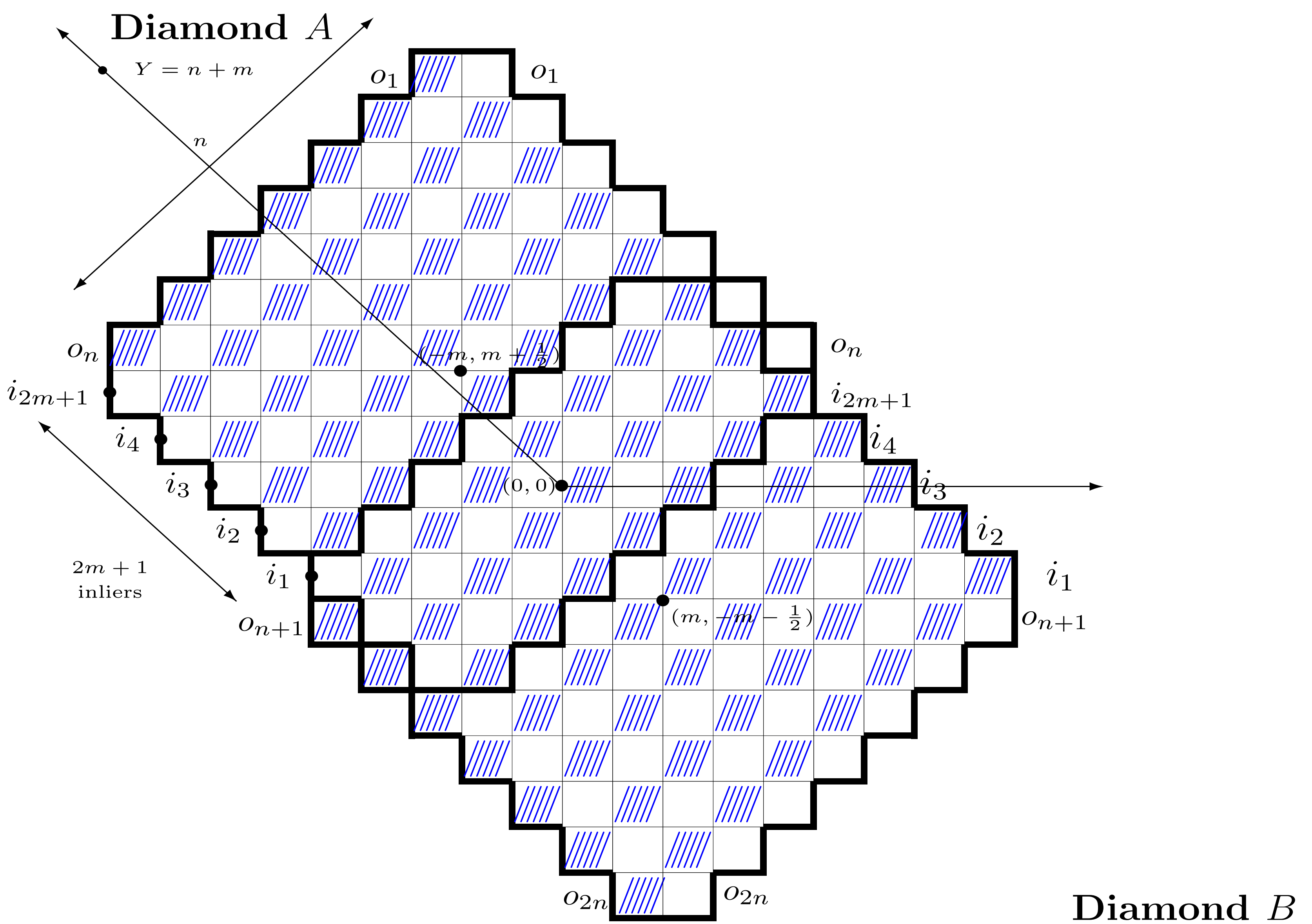
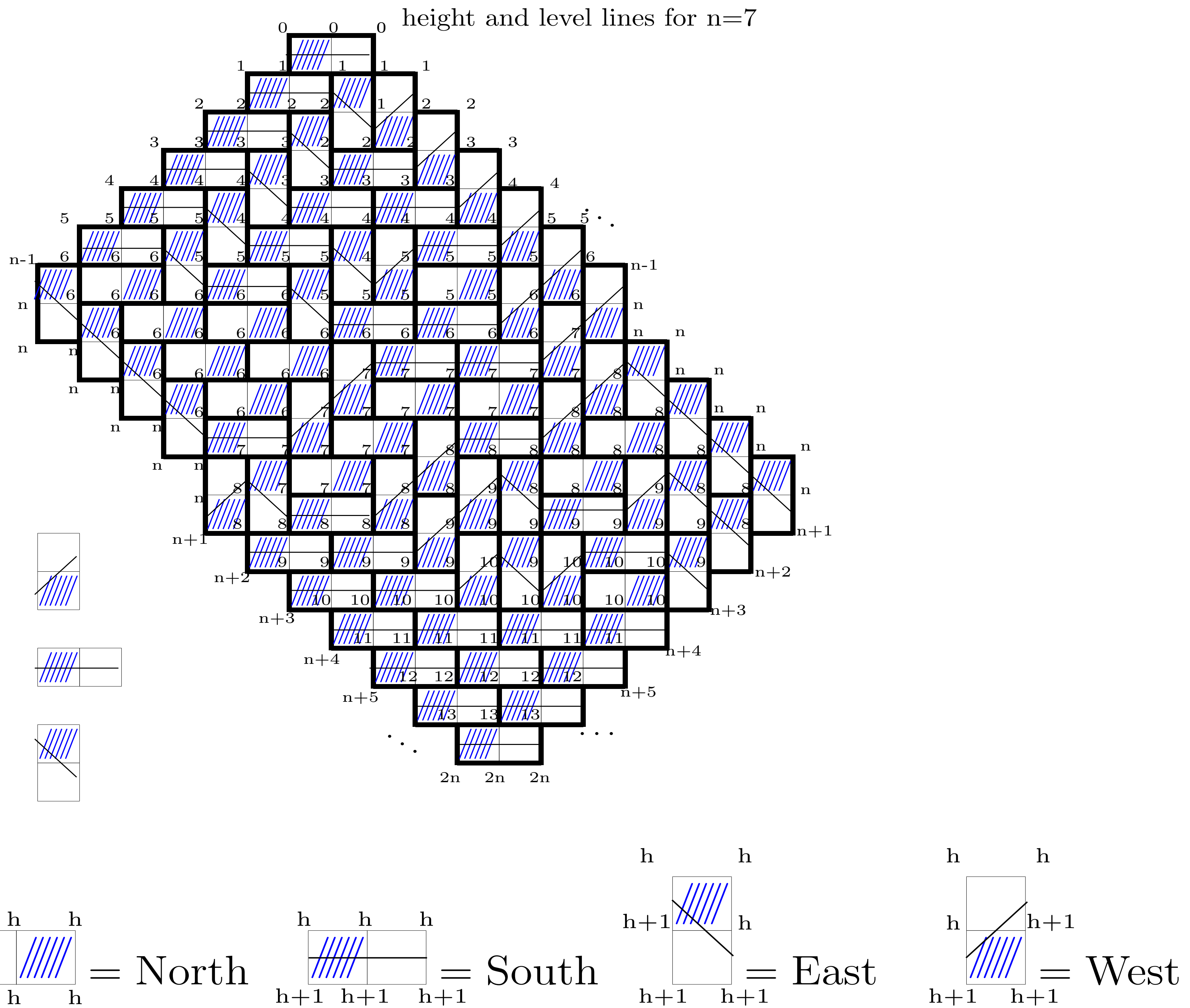


Figure 1: Double Aztec diamond of type $(n, m) = (7, 2)$ with $\#\{\text{inliers}\} = M = 2m + 1 = 5$.

Random cover with domino's: two groups of non-intersecting paths



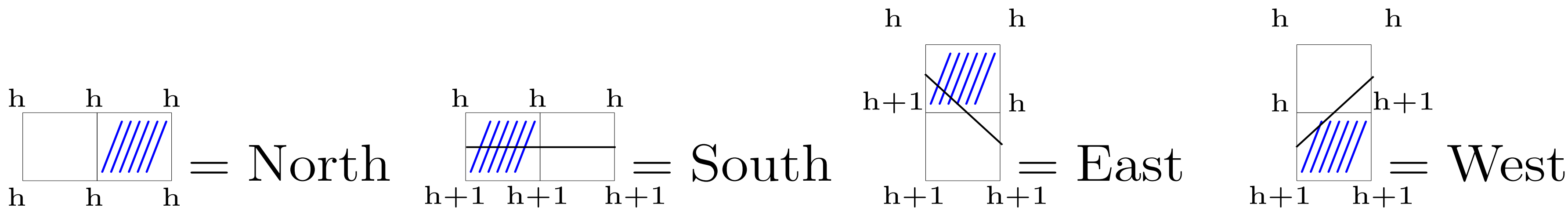


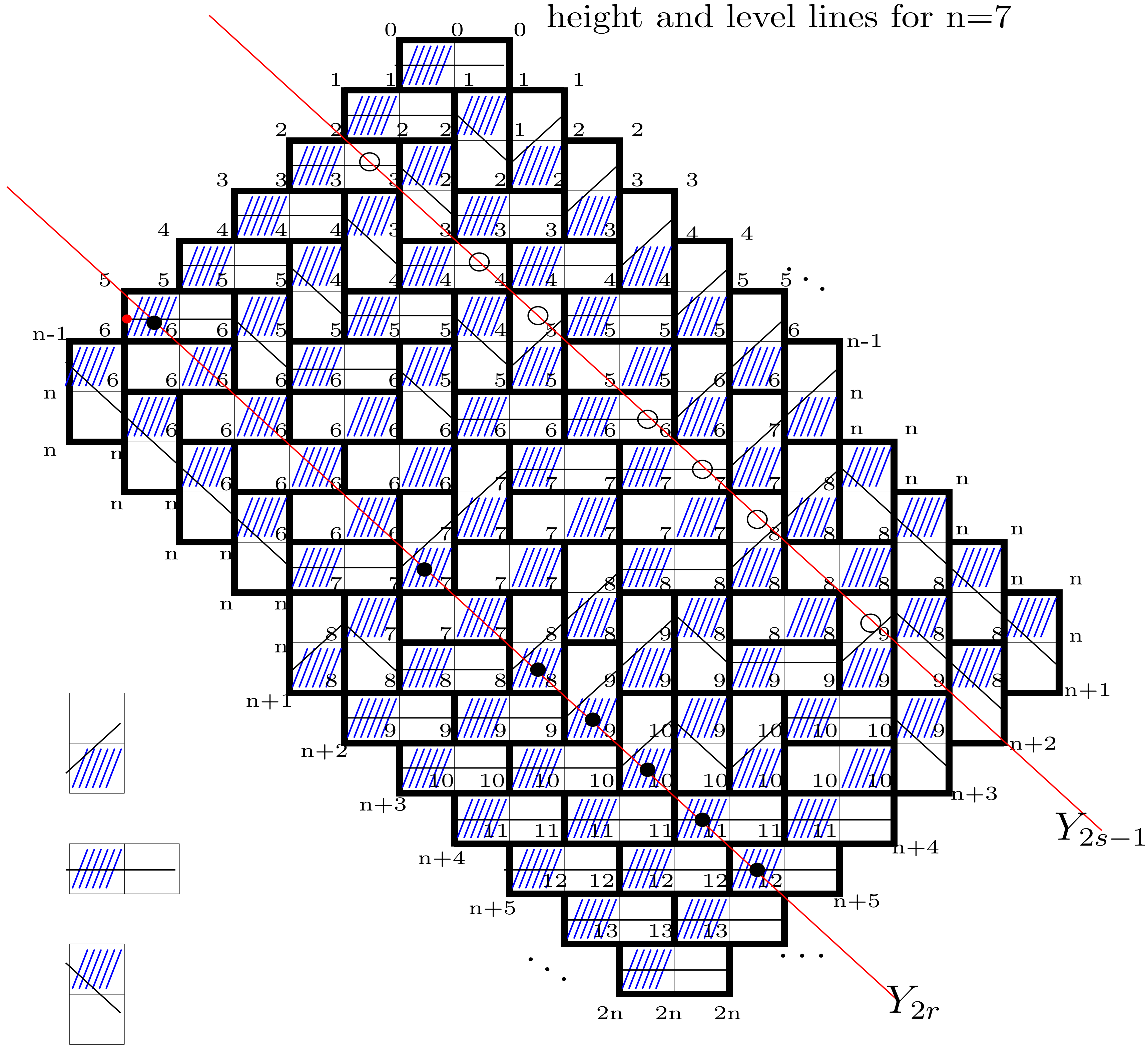
Figure 4. Height function on domino's and level-lines.

Domino-tiling $\Leftrightarrow$ Random Surface

(piecewise-linear)

The level curves for this Random Surface give non-intersecting paths

Implies Fixed boundary condition!


height and level lines for n=7
Y_{2s-1}
Y_{2r}

A weight on domino's and a probability on domino tilings:

- put the weight $0 < a < 1$ on vertical dominoes

- put the weight 1 on horizontal dominoes,

Define:

$$\mathbb{P}(\text{domino tiling } T) = \frac{a^{\#\text{vertical domino's in } T}}{\sum_{\text{all possible tilings } T} a^{\#\text{vertical domino's in } T}} \tag{1}$$

Question:

For an axis Y_{2r} going through black squares only, and an interval $[k, \ell] \subset \mathbb{Z}$,

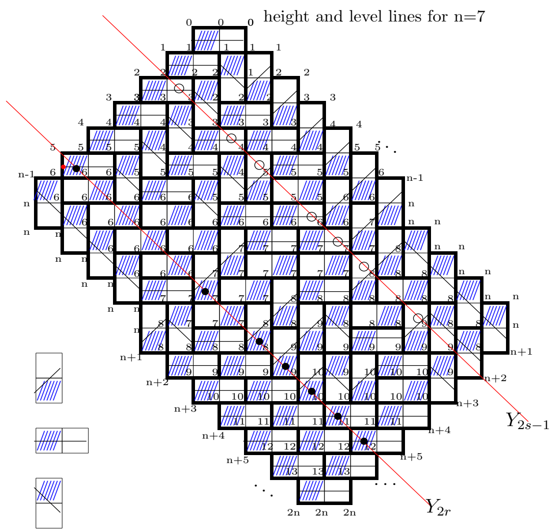
$$\mathbb{P}(\text{height function is flat along the interval } [k, \ell]) = ?$$

$\Longleftrightarrow$

$$\mathbb{P}(\text{no dots along the points of interval } [k, \ell]) = ?$$

$\Longleftrightarrow$

$\mathbb{P}$(domino's are pointing to the left of Y_{2r} along the points of interval $[k, \ell]$)

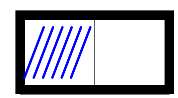


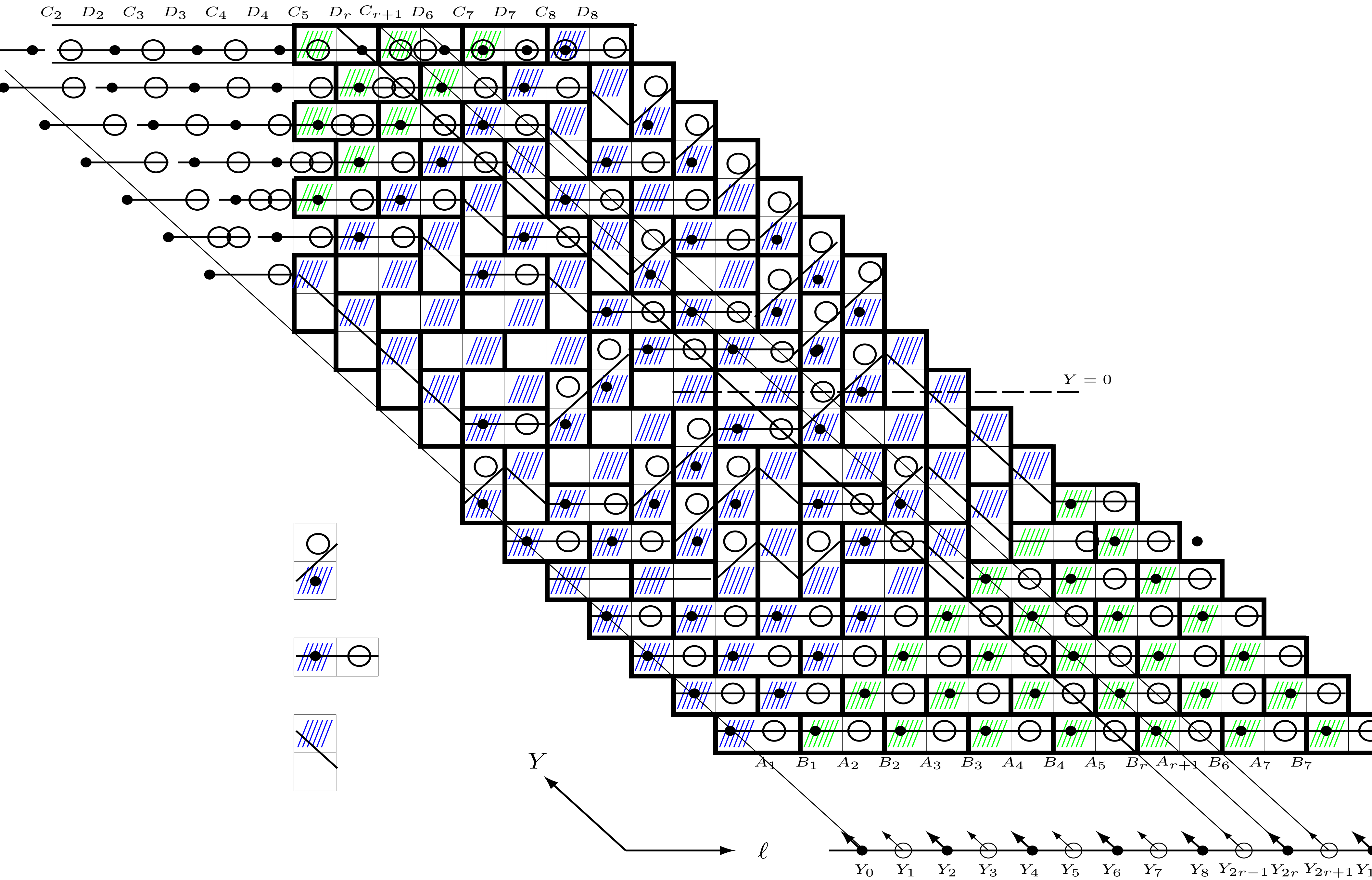
2. A determinantal process

How does one turn this into a determinantal process?

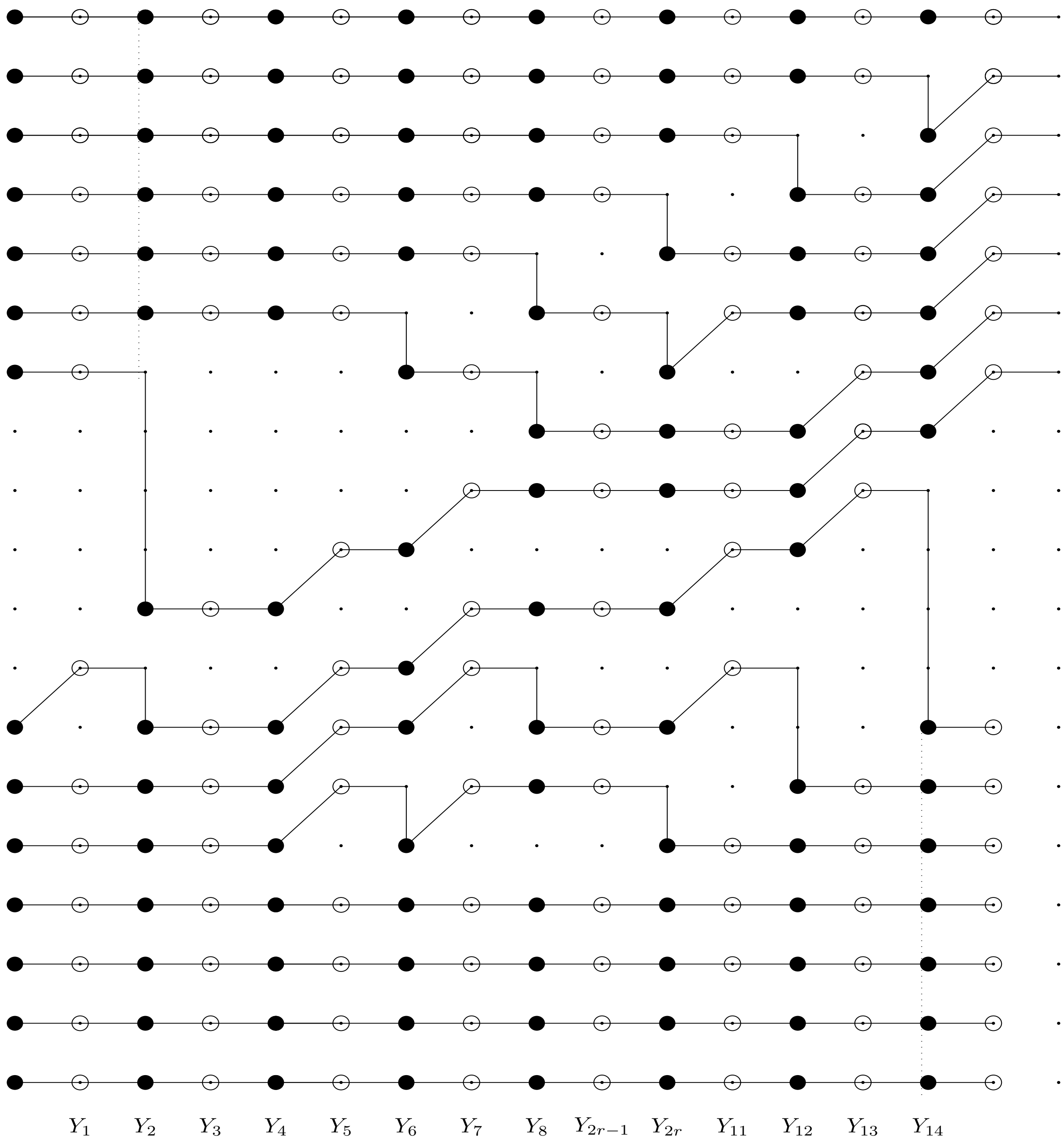
In other terms: How does one turn this into non-intersecting random walks, with synchronized time?

By completing the double Aztec diamond with South domino's (in a trivial way).


(South)



C_2 D_2 C_3 D_3 C_4 D_4 C_5 D_r C_{r+1} D_6 C_7 D_7 C_8 D_8
$Y = 0$
A_1 B_1 A_2 B_2 A_3 B_3 A_4 B_4 A_5 B_r A_{r+1} B_6 A_7 B_7
Y
ℓ
Y_0 Y_1 Y_2 Y_3 Y_4 Y_5 Y_6 Y_7 Y_8 Y_{2r-1} Y_{2r} Y_{2r+1}



$$\mathbb{P}\left(\{\text{non-intersecting random walks}\} \cap [k,\ell] = \emptyset \text{ along } Y_{2r}\right) = ?$$

In order to compute the kernel for the determinantal process, it is easier to first consider **Inliers**, instead of the walks before (**outliers**).

Therefore remember the height function and introduce a dual heigth:

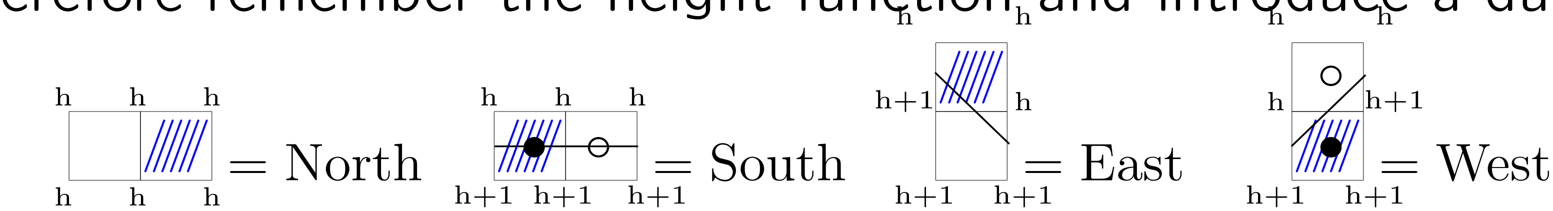


Figure 4. Height function h on domino's and level line.

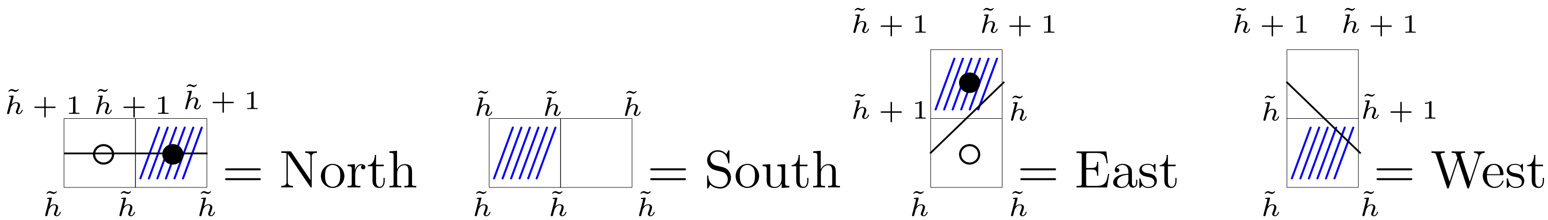
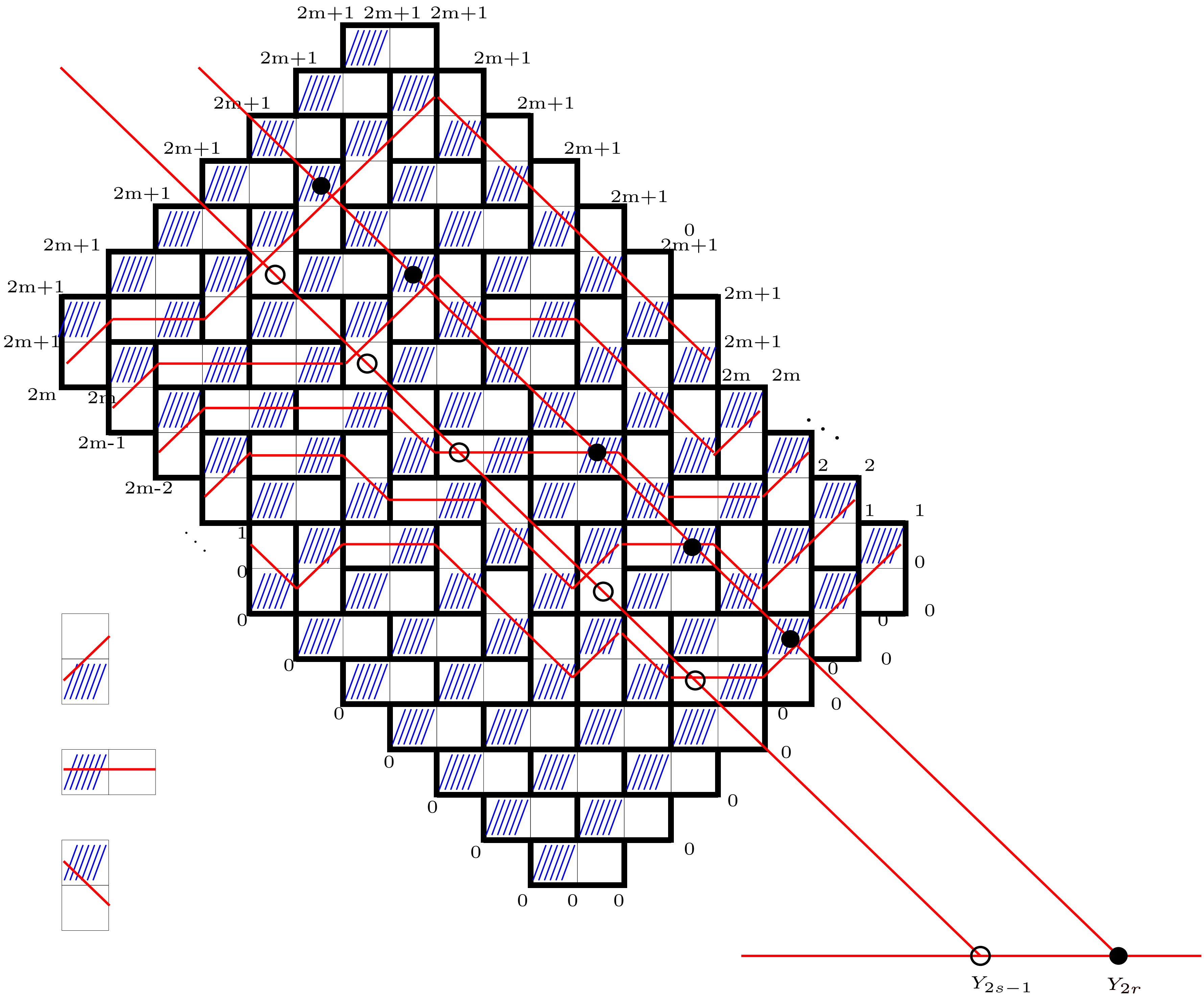
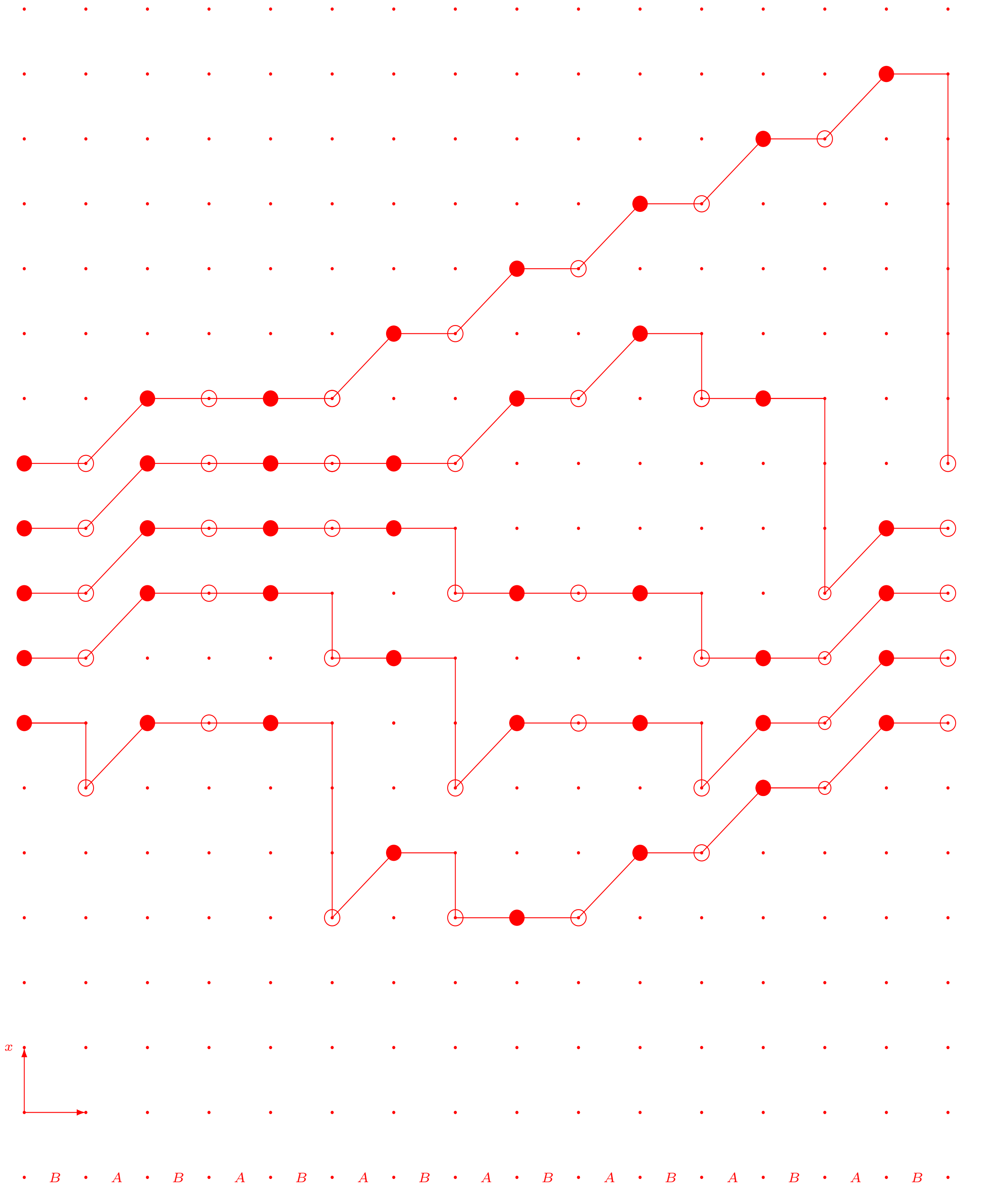


Figure 5. Dual height function $\tilde{h}$ on domino's and level line.


2m+1
2m
2m-1
2m-2
2
1
0
Y_{2s-1}
Y_{2r}


x
B A B A B A B A B A B A B A B

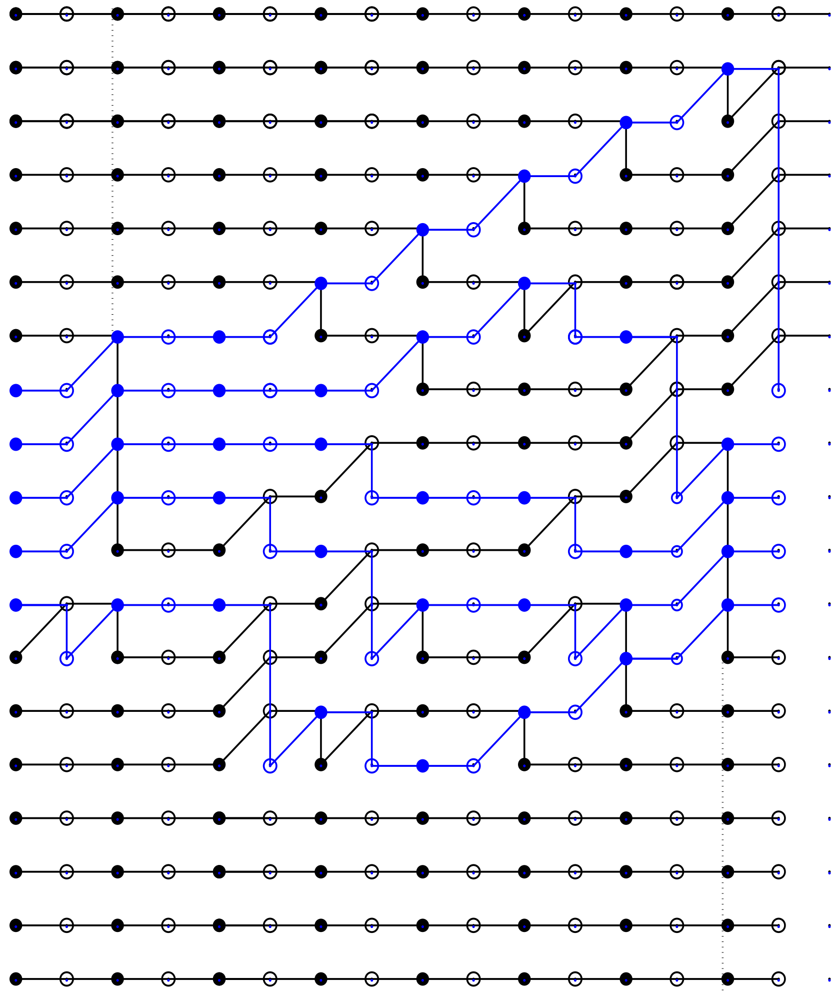
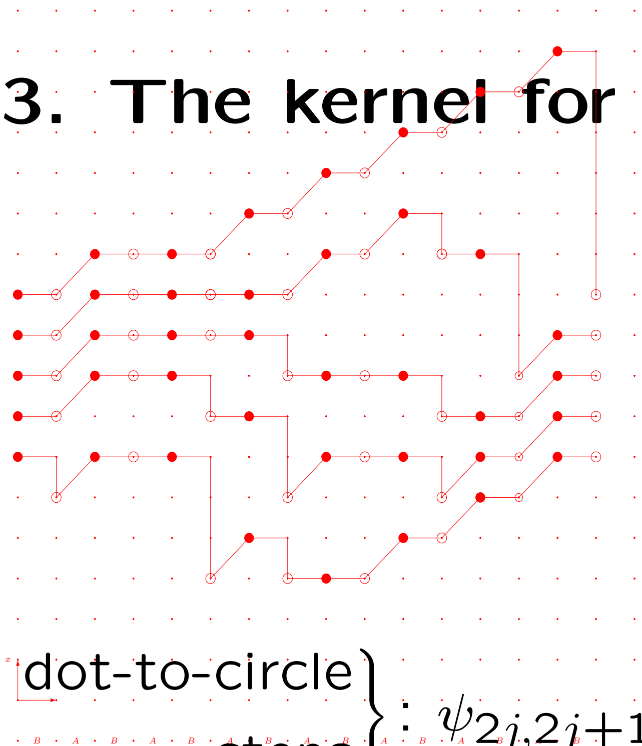


Figure 17: Lattice paths: superimposing in- and outliers.

3. The kernel for the inlier determinantal process



$$\left.\begin{array}{r}\text{dot-to-circle}\\ \text{-steps}\end{array}\right\}: \psi_{2j,2j+1}(x,y) := \left\{\begin{array}{l} a^{x-y},\ \text{if } y-x \le 0 \\ 0 \text{ otherwise}\end{array}\right\} = \oint_{\Gamma_{0,a}} \frac{dz}{2\pi \mathrm{i} z} \frac{z^{x-y}}{1-\frac{a}{z}}$$

$$\left.\begin{array}{r}\text{circle-to-dot}\\ \text{-steps}\end{array}\right\}: \psi_{2j+1,2j+2}(x,y) := \left\{\begin{array}{l} a,\ \text{if } y-x=1 \\ 1,\ \text{if } y-x=0 \\ 0,\ \text{otherwise}\end{array}\right\} = \oint_{\Gamma_0} \frac{dz}{2\pi \mathrm{i} z} z^{x-y}(1+az).$$

Then define in general: $\left\{\begin{array}{l} \psi_{r,s} = \psi_{r,r+1} * \ldots * \psi_{s-1,s},\ \text{if } s>r \\ \qquad = 0, \text{if } s \le r.\end{array}\right.$

where

$$f(x) * g(x) := \sum_{x\in\mathbb{Z}} f(x)g(x)$$

• Lindström-Gessel-Viennot / Karlin-McGregor

$$\mathbb{P}(r; x_1, \ldots, x_{2m+1}) = \frac{1}{Z_{n,m}} \det(\psi_{0,r}(-m+i-1, x_j))_{1\le i,j\le 2m+1} \det(\psi_{r,2n+1}(x_j, -m+i-1))_{1\le i,j}$$

• Via the Eynard-Mehta formula: Expression in terms of **orthogonal polynomials on the circle** (inliers)

$$\mathbb{K}_{n,m}(r,x;s,y) = \sum_{i,j=1}^{2m+1} \psi_{r,2n+1}(x,-m+i-1)([A^{-1}]_{ij})\psi_{0,s}(-m+j-1,y) - \mathbb{1}_{r<s}\psi_{r,s}(x,y),$$

$$= \frac{1}{(2\pi \mathrm{i})^2} \oint_{\gamma_{r_2}} \frac{dw}{w} \rho_a^R(w) \oint_{\gamma_{r_1}} \frac{dz}{z} \rho_a^L(z) \frac{w^{x+m}}{z^{y+m}} \underbrace{\sum_{k=0}^{2m} P_k(z)\widehat{P}_k(w^{-1})}_{\text{bi-orthogonal pol on circle for } \rho_a^L(z)\rho_a^R(z)}, \quad \text{for } r=s=n$$

$$\rho_a^L(z) = \left(\frac{1+az}{1-\frac{a}{z}}\right)^{n/2} \text{ and } \rho_a^R(z) = \frac{\rho_a^L(z)}{1-\frac{a}{z}}$$

4. The kernel for the outlier determinantal process

• **Kernel for the dot-determinantal process (outlier paths)** (Borodin '00)

$$\widetilde{\mathbb{K}}^{\text{doubleAztec}}_{n,m}(n,x;n,y) = \delta_{x,y} - \mathbb{K}_{n,m}(n,x;n,y).$$

• **Extended kernel (outlier paths) :**

$\widetilde{\mathbb{K}}^{\text{doubleAztec}}_{n,m}(2r,x;2s,y)$

$$= -\mathbb{1}_{s<r}\psi_{(2s-2r)}(x,y) + \psi_{n-2r}(x,\cdot) * \widetilde{\mathbb{K}}^{\text{doubleAztec}}_{n,m}(n,\cdot\ ;n,\circ) * \psi_{(2s-n)}(\circ,y),$$

with

$$\psi_{2k}(x,y) := \oint_{\Gamma_{0,a}} \frac{dz}{2\pi i z}\left(\frac{1+az}{1-\frac{a}{z}}\right)^k$$

Two approaches:

(1) Work immediately with

$$\sum_{k=0}^{2m} P_k(w)\widehat{P}_k(z^{-1})$$

(2) Or use the Christoffel-Darboux formula for bi-orthonormal polynomials on the circle,

$$\sum_{k=0}^{2m} P_k(w)\widehat{P}_k(z^{-1}) = \frac{z^{-2m-1}P_{2m+1}(z)w^{2m+1}\widehat{P}_{2m+1}(w^{-1}) - \widehat{P}_{2m+1}(z^{-1})P_{2m+1}(w)}{1-\frac{w}{z}}.$$

Using approach (1), the kernel for the Double Aztec Diamond reads:

$$(-1)^{x-y}\widetilde{\mathbb{K}}^{\text{DoubleAztec}}_{n,m}(2r,x;2s,y)$$

$$= \mathbb{K}^{\text{SingleAztec}}_{n+1}\Big(2(n+1-r),m+1-x;2(n+1-s),m+1-y\Big)$$
$$+ \sum_{k=2m+1}^{\infty} b_{-x,r}(k)[(\mathbb{1}-\mathcal{K})^{-1}a_{-y,s}](k)$$

$$\mathbb{P}\left(\bigcap_{i=1}^{s}\Big\{\text{the line } Y_{2r_i} \text{ has a gap} \supset [k_i,\ell_i]\Big\}\right)$$
$$= \det\left(\mathbb{1} - \Big[\chi_{[k_i,\ell_i]}\widetilde{\mathbb{K}}^{\text{DoubleAztec}}_{n,m}(2r_i,x_i;2r_j,x_j)\chi_{[k_j,\ell_j]}\Big]_{1\le i,j\le s}\right).$$

Using approach (1), the kernel for the Double Aztec Diamond reads:

$$
\begin{aligned}
&(-1)^{x-y}\widetilde{\mathbb{K}}^{\mathsf{DoubleAztec}}_{n,m}(2r,x;2s,y)\\
&\quad = \mathbb{K}^{\mathsf{SingleAztec}}_{n+1}\Big(2(n+1-r),m+1-x;2(n+1-s),m+1-y\Big)\\
&\qquad + \sum_{k=2m+1}^{\infty} b_{-x,r}(k)[(\mathbb{1}-\mathcal{K})^{-1}a_{-y,s}](k)
\end{aligned}
$$

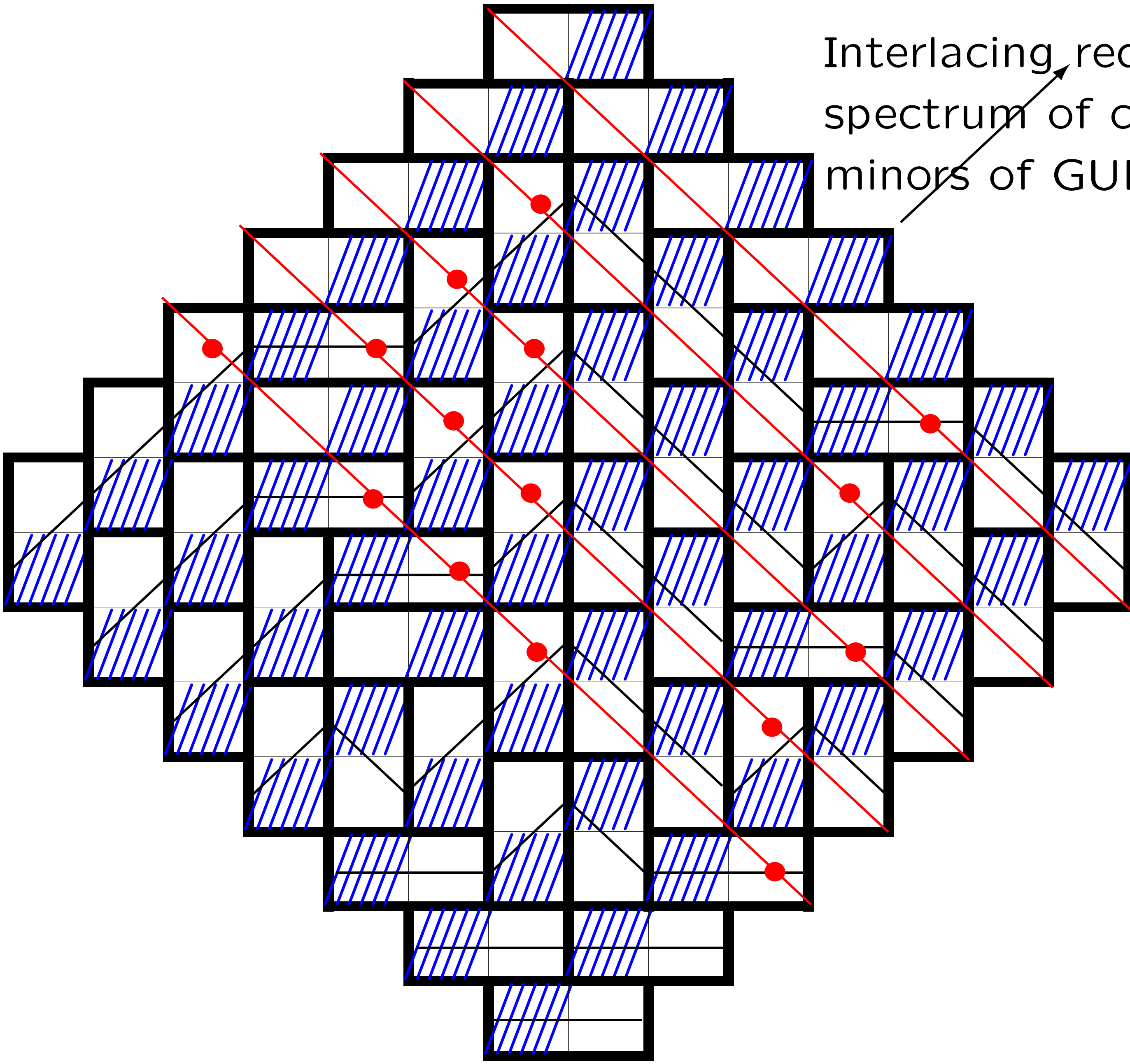
$$
a_{x,s}(k) := \frac{(-1)^{k-x}}{(2\pi \mathrm{i})^2}\oint_{\gamma_{r_1}} du \oint_{\gamma_{r_2}} \frac{dv}{u-v}\,\frac{v^{x+m}}{u^{k+1}}\,\frac{(1+av)^s(1-\frac{a}{v})^{n-s+1}}{\varphi_a(2n;u)}
$$

$$
b_{y,r}(\ell) := \frac{(-1)^{\ell-y}}{(2\pi \mathrm{i})^2}\oint_{\gamma_{r_1}} du \oint_{\gamma_{r_2}} \frac{dv}{u-v}\,\frac{v^{\ell}}{u^{y+m+1}}\,\frac{\varphi_a(2n;v)}{(1+au)^r(1-\frac{a}{u})^{n-r+1}}
$$

$$
\mathcal{K}_{k,\ell} := \frac{(-1)^{k+\ell}}{(2\pi \mathrm{i})^2}\oint_{\gamma_{r_1}} du \oint_{\gamma_{r_2}} \frac{dv}{v-u}\,\frac{u^{\ell}}{v^{k+1}}\,\frac{\varphi_a(2n,u)}{\varphi_a(2n,v)}
$$

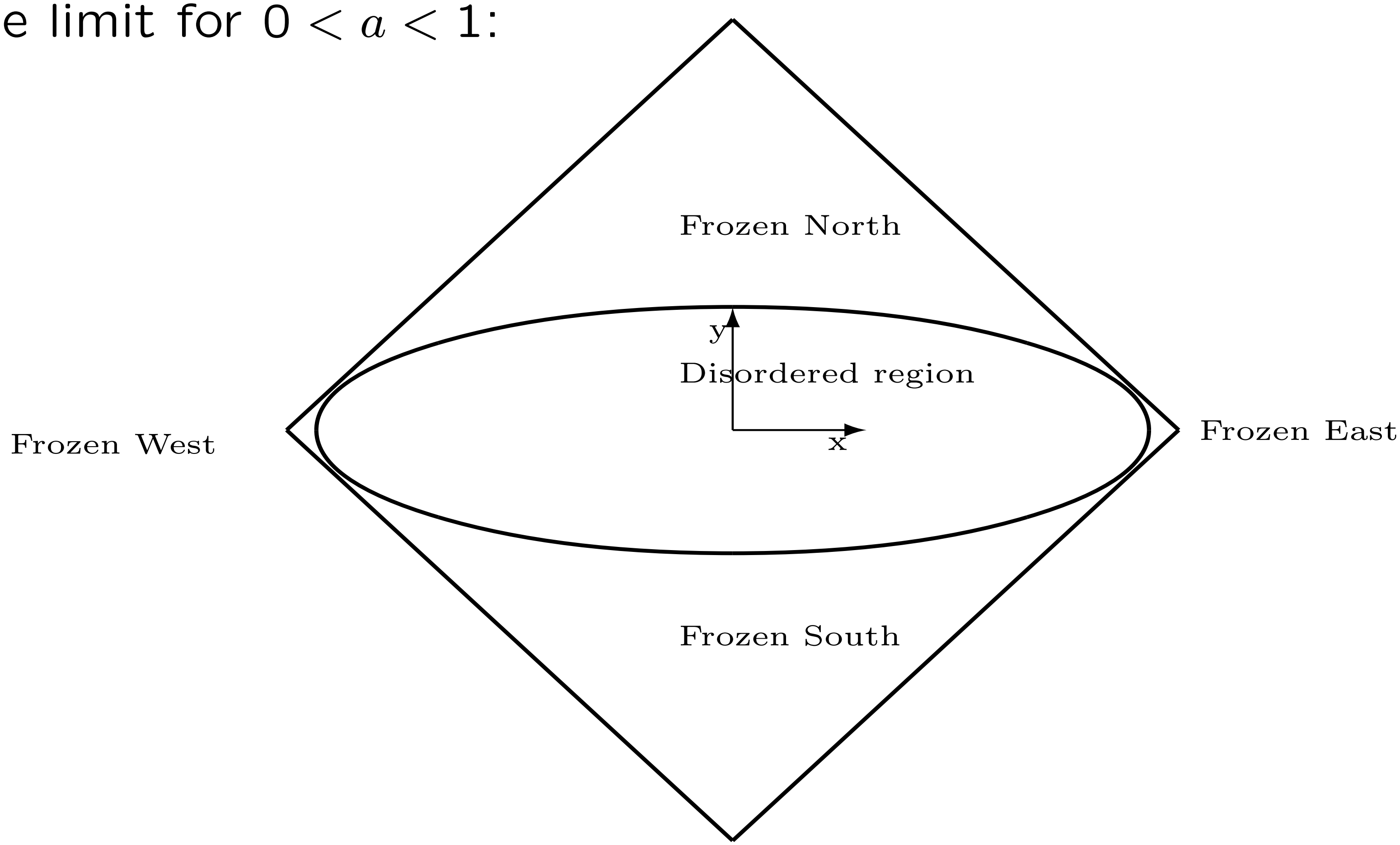
with $\varphi_a(2n;z) := (1+az)^n(1-\frac{a}{z})^{n+1}$

5. Determinantal point process for single Aztec diamond: Interlacing red dots on the red lines (Johansson '05)



Interlacing red dots for $n \to \infty \simeq$ spectrum of consecutive principal minors of GUE-matrices

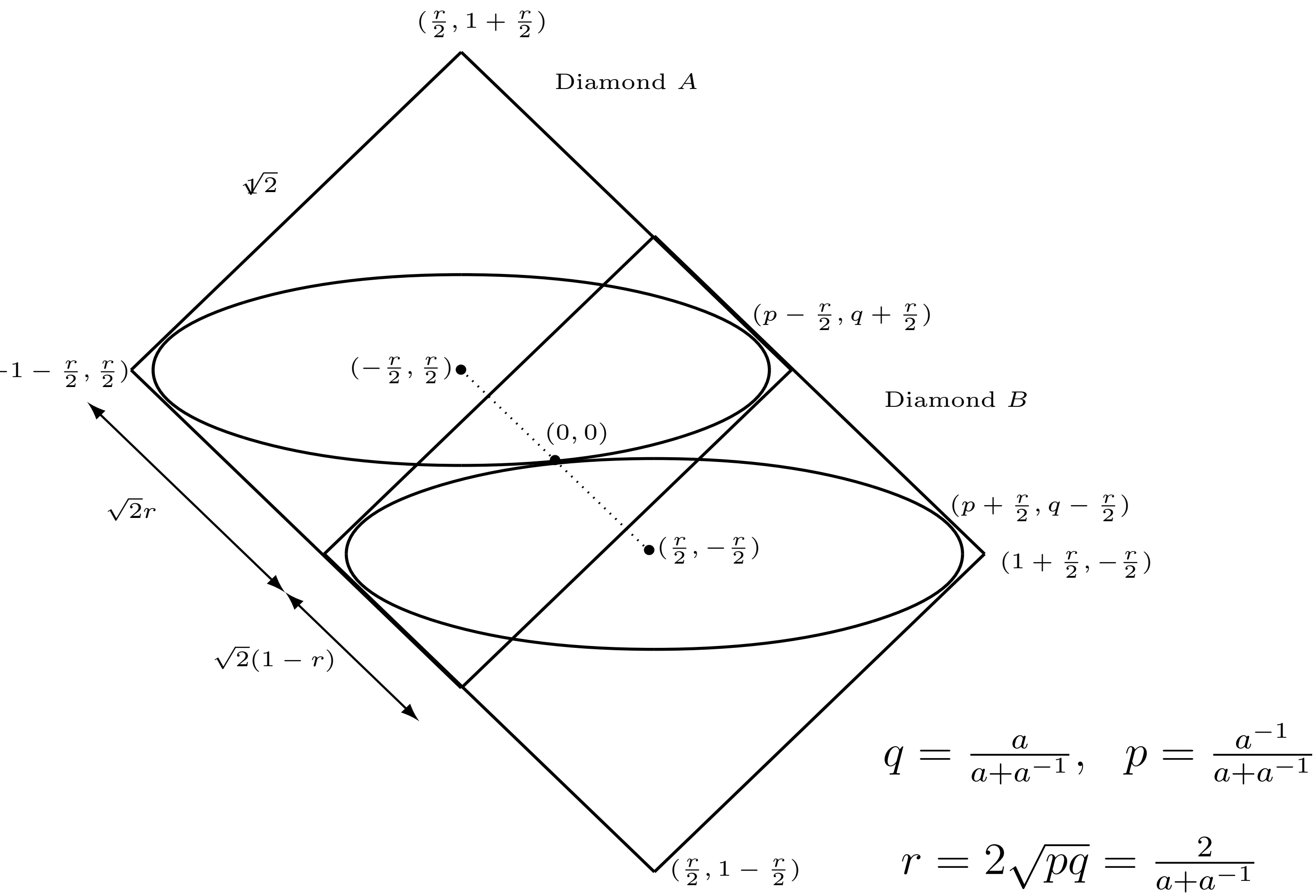
In the limit for $0 < a < 1$:



6. Limiting behavior for the double Aztec diamond: the Tacnode Process, for $n \to \infty$.

Bringing Arctic ellipses together (tacnode):

$$\frac{(x \pm \frac{r}{2})^2}{p} + \frac{(y \pm \frac{r}{2})^2}{q} = 1, \quad \text{with } q = \frac{a}{a + a^{-1}} \text{ and } p = 1 - q = \frac{a^{-1}}{a + a^{-1}},$$



SCALING: From old to new variables: $\xi_i,\ \tau_i$

$$(r,x;\ s,y)\mapsto(\tau_1,\xi_1;\ \tau_2,\xi_2)$$

$$n=2t, \qquad \frac{2m}{n}=\frac{2}{a+a^{-1}}+\sigma\rho t^{-2/3}$$

$$\frac{x}{n}=a^2\theta\tau_1 t^{-1/3}+\tfrac{1}{2}\xi_1\rho t^{-2/3}, \qquad \frac{y}{n}=a^2\theta\tau_2 t^{-1/3}+\tfrac{1}{2}\xi_2\rho t^{-2/3}$$

$$\frac{r}{n}=\frac{1}{2}+\tfrac{\theta}{2}(1+a^2)\tau_1 t^{-1/3}, \qquad \frac{s}{n}=\frac{1}{2}+\tfrac{\theta}{2}(1+a^2)\tau_2 t^{-1/3},$$

Given the weight $0<a<1$ on vertical dominoes, define:

$$v_0:=-\frac{1-a}{1+a}<0,\quad A^3:=\frac{a(1+a)^5}{(1-a)(1+a^2)},\quad \rho:=-Av_0>0,\ \theta:=\sqrt{\rho(a+a^{-1})}$$

$$\lim_{t\to\infty}(-v_0)^{y-x+r-s}(-1)^{y-x}\widetilde{\mathbb{K}}^{\mathsf{DoubleAztec}}_{n,m}(2r,x;2s,y)\rho t^{1/3}=\mathbb{K}^{\mathsf{tac}}(\tau_1,\xi_1;\tau_2,\xi_2)$$

For $\tau_2>\tau_1$, one has the **tacnode process**:

$$\mathbb{K}^{\mathsf{tac}}(\tau_1,\xi_1;\tau_2,\xi_2)=\mathbb{K}^{\mathsf{AiryProcess}}(\tau_2,\sigma-\xi_2+\tau_2^2;\tau_1,\sigma-\xi_1+\tau_1^2)$$

$$+\frac{g(\tau_1,\xi_1)}{g(\tau_2,\xi_2)}2^{1/3}\int_{\tilde\sigma}^{\infty}\Big((\mathbb{1}-K_{\mathsf{Ai}})^{-1}_{\tilde\sigma}\mathcal{A}^{\tau_1}_{\xi_1-\sigma}\Big)(\lambda)\mathcal{A}^{(-\tau_2)}_{\xi_2-\sigma}(\lambda)d\lambda.$$

where

$$\mathcal{A}^{\tau}_{\xi}(\kappa):=\mathsf{Ai}^{(\tau)}(\xi+2^{1/3}\kappa)-\int_0^{\infty}\mathsf{Ai}^{(\tau)}(-\xi+2^{1/3}\beta)\mathsf{Ai}(\kappa+\beta)d\beta$$

$$\mathsf{Ai}^{(\tau)}(x):=e^{\tau x+\frac{2}{3}\tau^3}\mathsf{Ai}(x+\tau^2).$$