Stochastic calculus of variations on the diffeomorphisms group

Warwick, April 2012

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The diffeomorphisms group

M compact finite-dim. Riemannian manifold

Here $M = \mathbb{T}^2$

$G^s = \{g \in H^s(M,M) : g \ \ bijective\ , \ \ g^{-1} \in H^s(M,M)\}$

If $s > 2$, $H^s \subset C^1$ and is a (infinite dim.) Hilbert manifold, locally diffeomorphic to

$H^s_g(TM) = \{X \in H^s(M;TM) : \pi o X = g\}, \pi : TM \to M$

chart at g given by:

$\varphi : H^s_g(TM) \to \{diffeom.on \ \ M\}$

$\varphi(X)(.) = exp \ oX(.)$

G^s is a group for composition of maps
Lie algebra:
$\mathcal{G}^s = H^s(TM)(= H^s_e(TM))$
$(e = \mathrm{id})$
On G^s we consider the Riemannian metric

$$< X_g, Y_g >_{L^2} = \int_M < X_g(x), Y_g(x) > dm(x)$$

for $X, Y \in T_g(G^s(M)) = H^s_g(TM)$ (weak Riemannian structure, Ebin-Marsden)

Volume preserving counterparts:
$G^s_V = \{g \in G^s : (g)_*(dm) = dm\}$
$\mathcal{G}^s_V = \{X \in H^s : div\ X = 0\}$

G^s_V submanifold of G^s

There exists a right-invariant Levi-Civita connection ∇^0:

$$\nabla^0_X Y = P_e(\nabla_X Y)$$

where P_e orth. projection into the divergence free part in the Hodge decomposition,

$$H^s(TM) = div^{-1}(\{0\}) \oplus_{L^2} grad\ H^{s+1}(M)$$

Hydrodynamics:

Geodesic equation for ∇^0 = Euler equation

$$\frac{\partial u}{\partial t} + (u.\nabla u) = -\nabla p$$

equaivalent to

$$\frac{\partial u}{\partial t} + \nabla^0_u(u) = -\nabla p$$

Brownian motions

On the Lie algebra we consider

$$dx(t) = \sum_k (A_k dx_k^1(t) + B_k dx_k^2(t))$$

x_k^i i.i.d. real valued Br. motions, A_k, B_k L^2 o.n. basis:

$$A_k = \frac{1}{|k|}[(k_2 \cos k.\theta)\partial_1 - (k_1 \cos k.\theta)\partial_2)]$$

$$B_k = \frac{1}{|k|}[(k_2 \sin k.\theta)\partial_1 - (k_1 \sin k.\theta)\partial_2)]$$

$k \in \tilde{Z}^2 - \{(0,0)\}$,
$|k|^2 = k_1^2 + k_2^2$,
$\partial_i = \frac{\partial}{\partial \theta^i}$

Brownian motions on the group G_V:

$$dg(t) = o\rho(x(t))\, g(t), \quad g(0) = e$$

for ρ an Hermitian operator that diagonalizes in the basis A_k and B_k with the same eigenvalues λ_k.

Theorem.

The process is well defined iff $\sum_k \lambda_k^2 < \infty$

Proof. Use S. Fang's methodology, e.g.

Generator:

$$\Delta_\rho = \frac{1}{2} \sum_k \lambda_k^2 (\partial_{A_k}^2 + \partial_{B_k}^2)$$

$$\partial_Z f(g) = \frac{d}{d\epsilon}|_{\epsilon=0} f(exp\,(\epsilon Z) g$$

In particular no canonical L^2 Brownian motion.

Constants of structure:

$$[A_k, A_l] = \frac{[k,l]}{2|k||l|}(|k+l|B_{k+l} + |k-l|B_{k-l})$$

$$[B_k, B_l] = -\frac{[k,l]}{2|k||l|}(|k+l|B_{k+l} - |k-l|B_{k-l})$$

$$[A_k, B_l] = -\frac{[k,l]}{2|k||l|}(|k+l|A_{k+l} - |k-l|A_{k-l})$$

$$[\partial_i, A_k] = -k_i B_k$$

$$[\partial_i, B_k] = k_i A_k$$

For

$$\alpha_{k,l} := \frac{1}{2|k||l||k+l|}(l \mid (l+k))$$

$$\beta_{k,l} := \alpha_{-k,l} = \frac{1}{2|k||l||k-l|}(l \mid (l-k))$$

$$[k,l] = k_1 l_2 - k_2 l_1$$

Christoffel symbols:

$$\nabla^0_{A_k,A_l} = [k,l](\alpha_{k,l} B_{k+l} + \beta_{k,l} B_{k-l}), \;\; \nabla^0_{B_k,B_l} = [k,l](-\alpha_{k,l} B_{k+l} + \beta_{k,l} B_{k-l})$$

$$\nabla^0_{A_k,B_l} = [k,l](-\alpha_{k,l} A_{k+l} + \beta_{k,l} A_{k-l}), \;\; \nabla^0_{B_k,A_l} = [k,l](-\alpha_{k,l} A_{k+l} - \beta_{k,l} A_{k-l})$$

Remarks:

1. The Christoffel symbols give rise to unbounded antihermitian operators on $\mathcal{G}$.

2. Since $\nabla^0_{A_k,A_k} = \nabla^0_{B_k,B_k} = 0$, Stratanovich = Itô in the equation for $g(t)$.

3. We do not want use the metric $< U, V >_\rho = < \rho(U), V >$.

Lifting to the frame bundle

Orthonormal frames above G_V:
$r : T_g(G_V) \to \mathcal{G}_V$ isometric isomorphism

$O(G_V)$= collection of o.n. frames (frame bundle) above G_V

it can be identified with $S = \mathcal{U}(\mathcal{G}_V) \times G_V$
where $\mathcal{U}$ stands for unitary group.

Lie algebra of S= $\mathcal{S} = su\,(G_V) \times \mathcal{G}_V$.

Denote (σ, ω) the parallelism in S defined by the Levi-Civita connection.

Lift of a vector filed Z:

$$< \tilde{Z}, \sigma >_{U,g} = UZ, \ldots \quad , < \tilde{Z}, \omega > = 0$$

We have $[\partial_Z f] \, o \, \pi = \partial_{\tilde{Z}}(fo\pi), \pi : \mathcal{S} \to G_V$

Lifted Laplacian:

$$\tilde{\Delta}_\rho = \frac{1}{2} \sum_k \lambda_k^2 (\partial^2_{\tilde{A}_k} + \partial^2_{\tilde{B}_k})$$

Then $[\Delta_\rho f] \, o \, \pi = \tilde{\Delta}_\rho(fo\pi)$ generates the lifted ρ-Brownian motion $r_x(t)$
$\pi(r_x(t)) = g(t)$
$< odr_x(t), \omega > = 0$

Stochastic calculus of variations

Derivation of the Itô map $x \to r_x(t)$:

Theorem. For $r_0 \in S$ and given a semimartingale ξ with values in $T_{r_0}(S)$ with an antisymmetric diffusion coefficient, we have,

$$< \frac{d}{d\tau}_{|_{\tau=0}} r_x(r_0 + \tau\xi)(t), \sigma >= \xi'_{x,t}$$

$$< \frac{d}{d\tau}_{|_{\tau=0}} r_x(r_0 + \tau\xi)(t), \omega >= \gamma_{x,t}$$

where

$$d\xi'(t) = (\Gamma_{\xi'_{x,t}}\rho - \rho\Gamma_{\xi'_{x,t}}) \circ dx(t) + \gamma_{x,t}(\rho \circ dx(t))$$

$$d\gamma(t) = \Omega(\xi'(t), \rho \circ dx(t))$$

with $\gamma_{x,0} = 0$ and $\xi'_{x,0} =< \xi, \sigma >$

Difficulty:

We cannot choose $\rho = Id$ and

$$\Gamma_{\xi'_{x,t}} \rho - \rho \Gamma_{\xi'_{x,t}}$$

is not antisymmetric.

Truncated diffusions

Consider the case $\lambda_k = 1$ for $|k| \leq N$ and $\lambda_k = 0$ for $|k| > N$, ρ^N the corresponding operator;
Denote by $r_x^N(t)$ the S-valued process for such this choice and call it *truncated* diffusion.
(r_x^N not finite dimensional projections: still infinite-dimensional, but driven by a finite number of Brownian motions)

$$\pi(\tilde{r}^N_{x,t}) = g^N_{x,t}$$

$$< \circ d\tilde{r}^N_{x,t}, \omega >= 0$$

where

$$dg^N_{x,t} = odx^N(t) g^N_{x,t}, \qquad g^N_{x,0} = e$$

and

$$dx^N(t) = \sum_{|k| \leq N} (A_k dx^1_k(t) + B_k dx^2_k(t))$$

Theorem.
For $r_0 \in S$ and given a semimartingale ξ with values in $T_{r_0}(S)$, with an antisymmetric diffusion coefficient, we have

$$< \frac{d}{d\tau}_{|_{\tau=0}} r_x^N(r_0 + \tau\xi)(t), \sigma >= \xi_{x,t}^N$$

$$< \frac{d}{d\tau}_{|_{\tau=0}} r_x^N(r_0 + \tau\xi)(t), \omega >= \gamma_{x,t}^N$$

where, for components k such that $|k| \leq N$ we have

$$d(\xi^N(t))^k = (\gamma_{x,t}^N(\circ dx^N(t))^k$$

$$d\gamma^N(t) = \Omega(\xi^N(t), \circ dx^N(t))$$

where $\gamma_{x,0}^N = 0$ and $\xi_{x,0}^N =< \xi, \sigma >$.

Proof.
As Γ antisymmetric, matrix $(\Gamma\rho^N - \rho^N\Gamma)_{k,j}$, with $|k|, |j| \leq N$ equal to zero.

Among the consequences

Corollary.(Bismut formula)

If Φ is a cylindrical functional on G_V, we have

$$\frac{d}{d\tau}|_{\tau=0} E(\Phi(\pi(\tilde{r}_x^{N,N}(r_0+\tau\xi)))) = E(< [\tilde{r}_x^{N,N}]^{-1}(\zeta_{x,t}^N), D\Phi >_{g_{x,t}^N})$$

where ζ^N satisfies

$$(d\zeta^N)^k = (\gamma_{x,t}^N dx^N(t))^k - \frac{1}{2}(Ricci^N(\zeta^N(t)))^k dt$$

$|k| \leq N$, and where $Ricci^N$ is the operator defined by

$$Ricci^N(Z) = -\sum_{|k|\leq N}(\Omega(A_k, Z, A_k) + \Omega(B_k, Z, B_k))$$

Expression of the Ricci tensor:

$$Ricci^N(A_j) = -\sum_{|k|\leq N} [k,j]^4 \frac{|k|^2+|j|^2}{|k|^2|j|^2|k-j|^2|k+j|^2} A_j$$

$$Ricci^N(B_j) = -\sum_{|k|\leq N} [k,j]^4 \frac{|k|^2+|j|^2}{|k|^2|j|^2|k-j|^2|k+j|^2} B_j$$

Weitzenbock formulae:

$$d(\Delta f)_l - \Delta(df)_l - Ricci_\rho(df)_l$$

$$= \sum_{k,i} \rho(k)^2 \Gamma^i_{l,k}(\partial_k \partial_i f + \partial_i \partial_k f) + \sum_{k,m,j} \rho(k)^2 (\Gamma^l_{k,m}[e_j, e_m]^k + \Gamma^l_{m,k}[e_k, e_j]^m)\partial_j$$

($e_k = A_k$ or B_k).

Back to Hydrodynamics:

The equation

$$\frac{\partial u}{\partial t} + \nabla^0_u(u) + \nu \sum_{|k|\leq N} \nabla^0_k \nabla^0_k(u) + \nu Ricci^N(u) = -\nabla p$$

is equivalent to

$$\frac{\partial u}{\partial t} + (u.\nabla u) + \nu c(N)\Delta u = -\nabla p$$

(Navier-Stokes equation)

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