THE UNIVERSITY *of* EDINBURGH
SCHOOL OF MATHEMATICS

Accelerated spatial approximations for time discretized SPDE: **report of results**

Stochastic Analysis and Stochastic PDEs
Warwick University · April 2012

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Orientation

- Space-time finite difference scheme for second order linear SPDE of parabolic type
- Rate of convergence known, for example see I. Gyöngy and A. Millet (2009)
- Give sufficient conditions for accelerating the rate of convergence with respect to the spatial approximation
- Extend results of I. Gyöngy and N. Krylov (2010)
- In general, cannot also accelerate in time: A. M. Davie and J. G. Gaines (2000)

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The setting

Given

- integers $d \geqslant 1$, $d_1 \geqslant 1$ and real-valued $T \geqslant 0$
- $(\Omega, \mathcal{F}, P)$ probability space equipped with filtration $(\mathcal{F}(t))_{t \geqslant 0}$
- $(w^\rho)_{\rho=0}^{d_1}$ sequence of independent $\mathcal{F}(t)$-Wiener processes

The equation

We consider the Cauchy problem for

$$\begin{aligned} du(t,x) =& (\mathcal{L}u(t,x) + f(t,x))dt \\ &+ \sum_{\rho=1}^{d_1} (\mathcal{M}^\rho u(t,x) + g^\rho(t,x))dw^\rho(t) \end{aligned} \tag{Eq}$$

on $\Omega \times [0,T] \times \mathbf{R}^d$ with initial condition $u_0 = u(0,x)$

- $\mathcal{L}(t) := a^{\alpha\beta}(t)D_\alpha D_\beta, \quad a^{\alpha\beta} = a^{\beta\alpha}$
- $\mathcal{M}^\rho(t) := b^{\alpha\rho}(t)D_\alpha$

for $\alpha, \beta \in \{0, \dots, d\}$.

The equation

We consider the Cauchy problem for

$$du(t,x) = (\mathcal{L}u(t,x) + f(t,x))dt + \sum_{\rho=1}^{d_1} (\mathcal{M}^\rho u(t,x) + g^\rho(t,x))dw^\rho(t) \qquad \text{(Eq)}$$

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Temporal discretization

For $\tau \in (0, 1)$ we define the time-grid

$$T_\tau := \left\{t_i = i\tau; i \in \{0, 1, \dots, n\}, \tau n = T\right\}$$

and write u_i in place of $u(t_i)$ and in particular define

$$\xi_i^\rho := \Delta w^\rho(t_{i-1}) = w^\rho(t_i) - w^\rho(t_{i-1})$$

for $\rho \in \{1, \dots, d_1\}$.

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Implicit Euler scheme

Together with (Eq) we consider

$$u_i^\tau = u_{i-1}^\tau + \left(\mathcal{L}_i u_i^\tau + f_i\right)\tau + \sum_{\rho=1}^{d_1}\left(\mathcal{M}_{i-1}^\rho u_{i-1}^\tau + g_{i-1}^\rho\right)\xi_i^\rho \qquad (Eq_\tau)$$

for $i \in \{1, \dots, n\}$ and $(\omega, x) \in \Omega \times \mathbf{R}^d$ with initial condition. The solution u^τ will be the leading term in our asymptotic expansion.

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Spatial discretization

For $h \in \mathbb{R}\setminus\{0\}$ and finite subset $\Lambda \subset \mathbb{R}^d$ containing the origin define the space-grid

$$G_h := \left\{\lambda_1 h + \cdots + \lambda_p h; \lambda_1, \ldots, \lambda_p \in \Lambda \cup (-\Lambda)\right\}$$

and spatial differences

$$\delta_{h,\lambda}\phi(x) := \frac{\phi(x + h\lambda) - \phi(x)}{h}$$

for $\lambda \in \Lambda$.

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Space-time difference scheme

Together with (Eq) and (Eq_τ) we consider for each fixed τ

$$u_i^{\tau,h} = u_{i-1}^{\tau,h} + \left(L_i^h u_i^{\tau,h} + f_i\right)\tau + \sum_{\rho=1}^{d_1}\left(M_{i-1}^{h\rho} u_{i-1}^{\tau,h} + g_{i-1}^{\rho}\right)\xi_i^{\rho} \qquad (Eq_\tau^h)$$

for $i \in \{1, \dots, n\}$ and $(\omega, x) \in \Omega \times G_h$ with initial condition where

- $L_i^h := \mathfrak{a}^{\lambda\mu}\delta_{h,\lambda}\delta_{-h,\mu}, \quad \mathfrak{a}^{\lambda\mu} = \mathfrak{a}^{\mu\lambda}$
- $M_i^{h\rho} := \mathfrak{b}_i^{\lambda\rho}\delta_{h,\lambda}$

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for $\lambda, \mu \in \Lambda$.

Assumptions

Assumption (consistency)

For $i \in \{0, \dots, n\}$ $\quad \mathfrak{a}_i^{00} = a_i^{00}$,
$\sum_{\lambda \in \Lambda_0} \mathfrak{a}_i^{\lambda 0} \lambda^\alpha + \sum_{\mu \in \Lambda_0} \mathfrak{a}_i^{0\mu} \mu^\alpha = a_i^{\alpha 0} + a_i^{0\alpha}$,
$\sum_{\lambda, \mu \in \Lambda_0} \mathfrak{a}_i^{\lambda\mu} \lambda^\alpha \mu^\beta = a_i^{\alpha\beta}$, $\mathfrak{b}_i^{0\rho} = b_i^{0\rho}$, *and*
$\sum_{\lambda \in \Lambda_0} \mathfrak{b}_i^{\lambda\rho} \lambda^\alpha = b_i^{\alpha\rho}$ *for all* $\alpha, \beta \in \{1, \dots, d\}$ *and* $\rho \in \{1, \dots, d_1\}$.

Assumption (parabolicity)

There exists $\kappa > 0$ *such that*
$\sum_{\alpha,\beta=1}^{d} (2a^{\alpha\beta} - b^{\alpha\rho} b^{\beta\rho}) z^\alpha z^\beta \geqslant \kappa |z|^2$ *and*
$\sum_{\lambda,\mu \in \Lambda_0} (2\mathfrak{a}^{\lambda\mu} - \mathfrak{b}^{\lambda\rho} b^{\mu\rho}) z_\lambda z_\mu \geqslant \kappa \sum_{\lambda \in \Lambda_0} |z|^2$.

Assumptions

Assumption (regularity initial condition, free terms)

The $u_0 \in L^2(\Omega, \mathcal{F}_0, W_2^{m+1})$, *the* f *and* g^ρ *are predictable processes in* W_2^m *and* W_2^{m+1}. *Moreover*
$\mathbb{E}\int_0^T (\|f(t)\|_m^2 + \|g(t)\|_{m+1}^2)\,dt + \mathbb{E}\|u_0\|_{m+1}^2 < \infty$.

Assumption (regularity coefficients)

The $a^{\alpha\beta}$ *and* $\mathfrak{a}^{\alpha\beta}$ *and their derivatives are* m *times continuously differentiable in* x *and bounded in magnitude. The* b^α *and* $\mathfrak{b}^\alpha$ *and their derivatives are* $m+1$ *times continuously differentiable in* x *and bounded in magnitude.*

Expansion results

Theorem

If the assumptions hold with $m > k + 1 + d/2$ *for* $k \geqslant 0$ *then*

$$u_i^{\tau,h}(x) = u_i^{\tau}(x) + \sum_{j=i}^{k} \frac{h^j}{j!} u_i^{\tau(j)}(x) + R_i^{\tau,h}(x) \qquad \text{(A)}$$

holds almost surely for $i \in \{1, \ldots, n\}$ *and* $x \in G_h$ *where*

$$\mathbb{E} \max_{i \leqslant n} \sup_{x \in G_h} |R_i^{\tau,h}(x)|^2 \leqslant N h^{2(k+1)} \mathcal{K}_m^2$$

for $\mathcal{K}_m^2 := \mathbb{E}\|u_0\|_{m+1}^2 + \mathbb{E}\tau \sum_{i=0}^{n} (\|f_i\|_m^2 + \|g_i\|_{m+1}^2) < \infty$ *and a constant* N *independent of* τ *and* h.

Acceleration results

Fix an integer $k \geqslant 0$ and let

$$\bar{u}^{\tau,h} := \sum_{j=0}^{k} \beta_j u^{\tau,2^{-j}h}$$

where $u^{\tau,2^{-j}h}$ solves, with $2^{-j}h$ in place of h, the space-time scheme (Eq_τ^h) with initial condition. Here β is given by $(\beta_0, \beta_1, \ldots, \beta_k) := (1, 0, \ldots, 0)V^{-1}$ where V^{-1} is the inverse of the Vandermonde matrix with entries $V^{ij} := 2^{-(i-1)(j-1)}$ for $i, j \in \{1, \ldots, k+1\}$.

Acceleration results

Theorem

Under the assumptions of the theorem,

$$\mathbb{E}\max_{i\leqslant n}\sup_{x\in G_h}|\bar{u}_i^{\tau,h}(x)-u_i^{\tau}(x)|^2\leqslant N|h|^{2(k+1)}\mathcal{K}_m^2$$

for a constant N *that is independent of* τ *and* h.

This work

E. J. Hall. *Accelerated spatial approximations for time discretized stochastic partial differential equations*, submitted, http://arxiv.org/abs/1201.5769.