A STATISTICAL APPROACH TO DATA ASSIMILATION FOR HEMODYNAMICS

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Joint work with A. Veneziani

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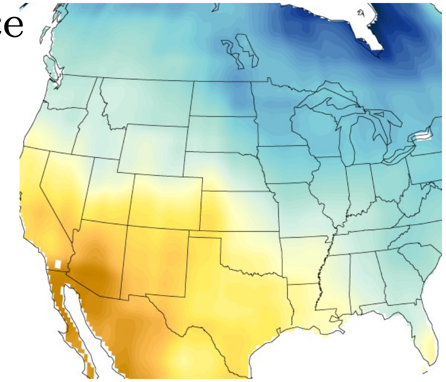
why data assimilation?

1. **scientific computing** (SC) has an increasing role in engineering, science and society → reliability of numerical results is a crucial issue for
   - investigation/ranking of methods
   - assessing the **impact** of numerical simulations



precipitation simulation

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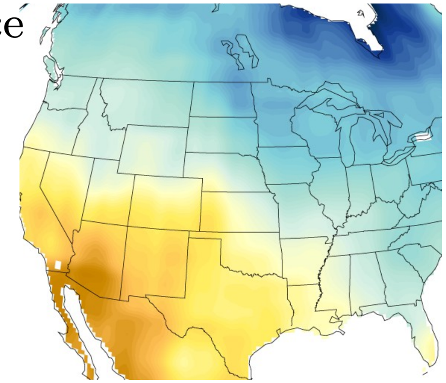
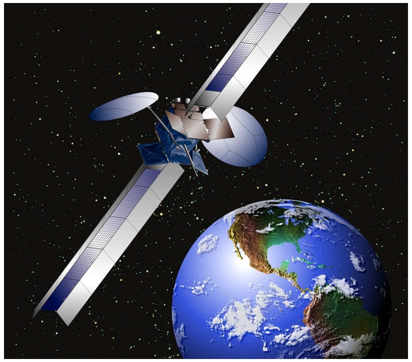


precipitation simulation

2. many application fields experience a **tremendous increment** of the amount of available data



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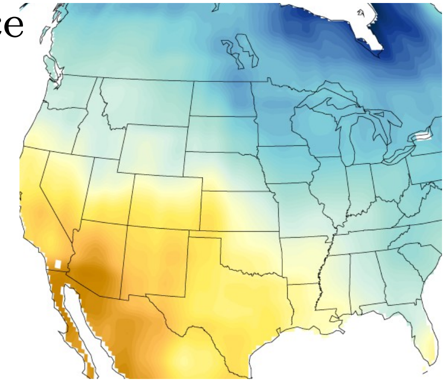

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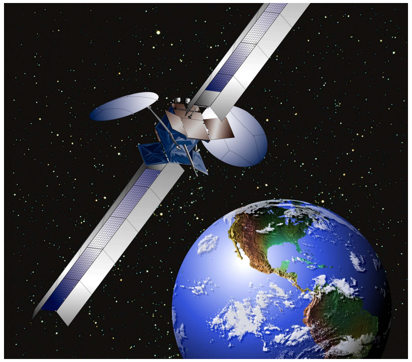
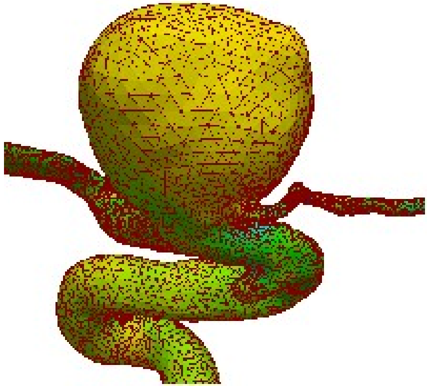


3. cardiovascular mathematics is an emerging field in SC
   - development of **numerical models**
   - development of **diagnostic devices**

   → decision supporting in clinical practice

   → reliability is mandatory



how to use measures?

- validation: new **benchmark** for numerical simulations
- **merging** into numerical simulations to obtain reliable results

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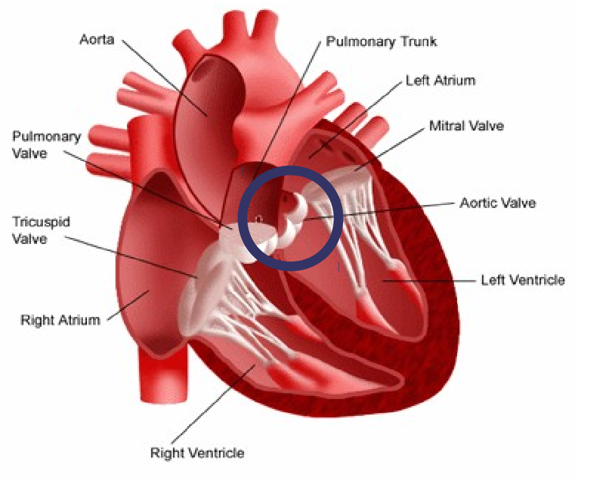
data assimilation

ensemble of methods for merging sparse and noisy information into a numerical model based on the approximation of physical and constitutive laws

goal: link together heterogeneous (in nature, quality, and density) sources of information in order to retrieve a consistent state for phenomena of interest

an application

- **CHOA** project – investigation of the bicuspid aortic valve, a congenital hearth disease



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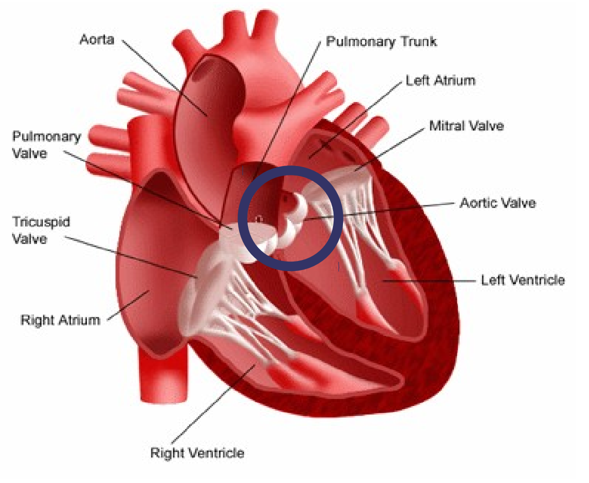
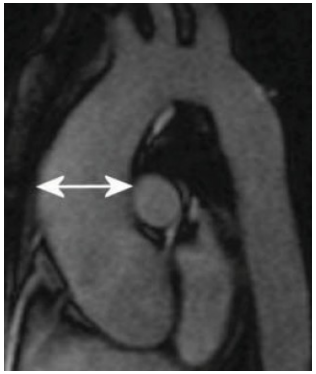


- main symptom of development of serious complications is the **dilatation of the aorta –** clinical methods **fail to guide decisions** for early intervention



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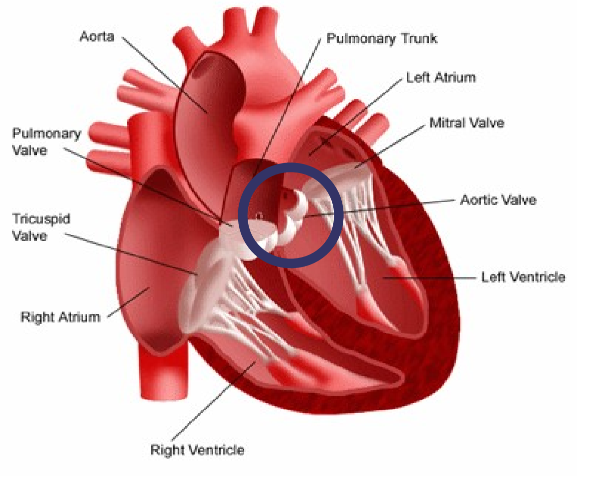

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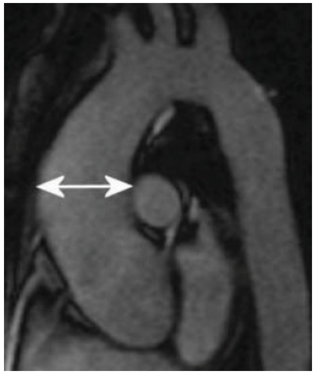
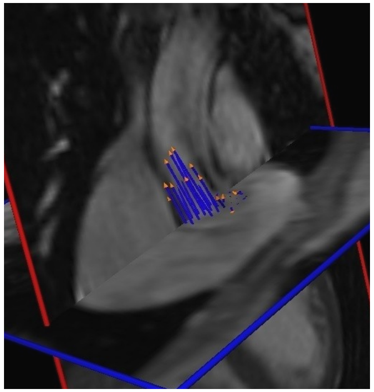


- using 4D MRI, determine and analyze the **blood flow patterns in the aortic root –** flow reconstruction by image processing is not accurate enough



Dr. M.Brummer Emory CHOA

outline

1. **deterministic formulation** of the continuous and discrete problem
   - optimality result and alternative regularization
   - consistency and validation results

2. **statistical formulation** of the discrete problem
   - Bayesian inversion: point and spread estimators
   - comparison with deterministic estimates
   - confidence intervals for velocity and wall shear stress

3. future work

1. deterministic formulation [1,2,3]

[1] M. DE, A. Veneziani, *Methods for assimilating blood velocity measures in hemodynamics simulations: Preliminary results,* Procedia Computer Science, **2010**

[2] M. DE et al., *A variational data assimilation procedure for the incompressible Navier-Stokes Equations in hemodynamics*, to appear on Journal of Scientific Computing, **2011.**

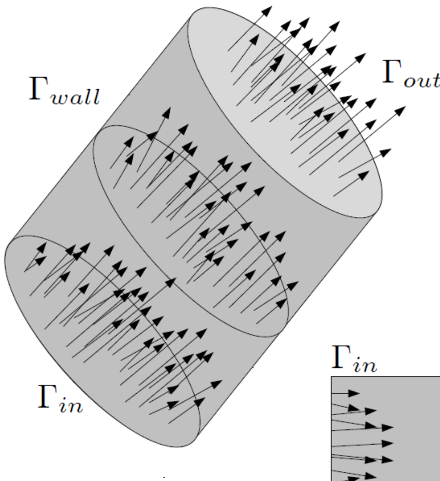
[3] M. DE et al., *Applications of variational data assimilation in computational hemodynamics*, Chapter in Modeling of Physiological Flows, Springer, **2011.**

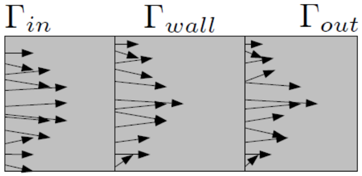
mathematical problem

vessel: domain $\Omega \subset \mathbb{R}^2,\ \mathbb{R}^3$, with boundaries Γ_{in}, Γ_{out}, Γ_{wall}

variables: velocity $\mathbf{u}$ and pressure p

data: $\mathbf{d} \in \mathbb{R}^{N_s}$, vector of measured velocities





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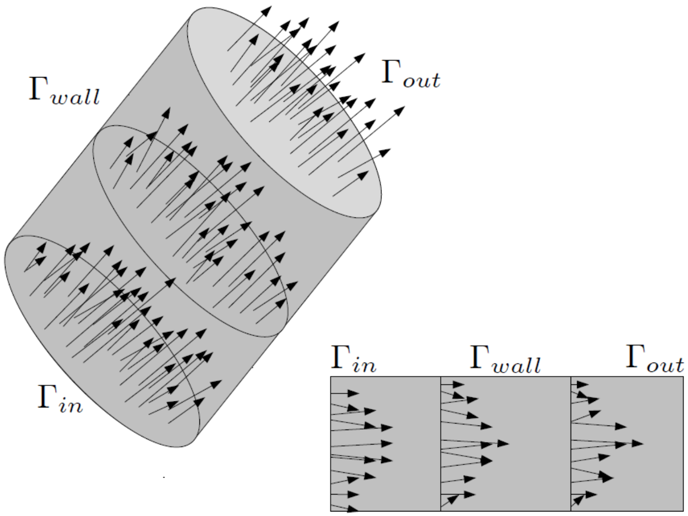

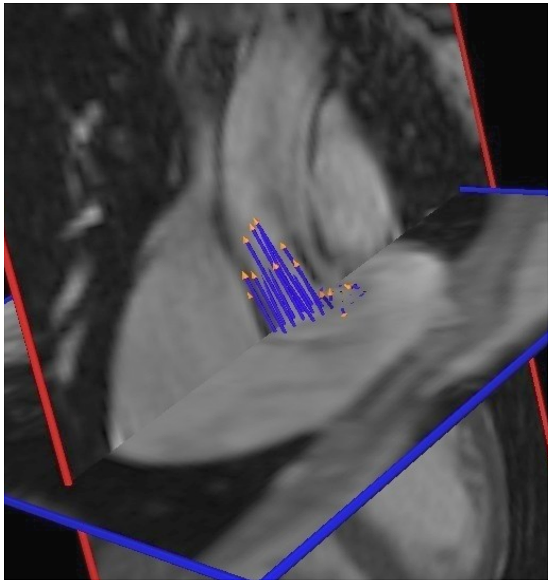

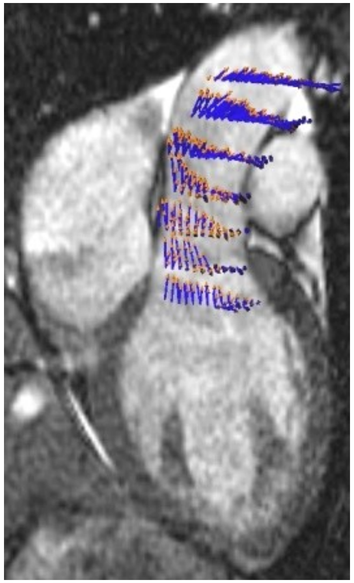

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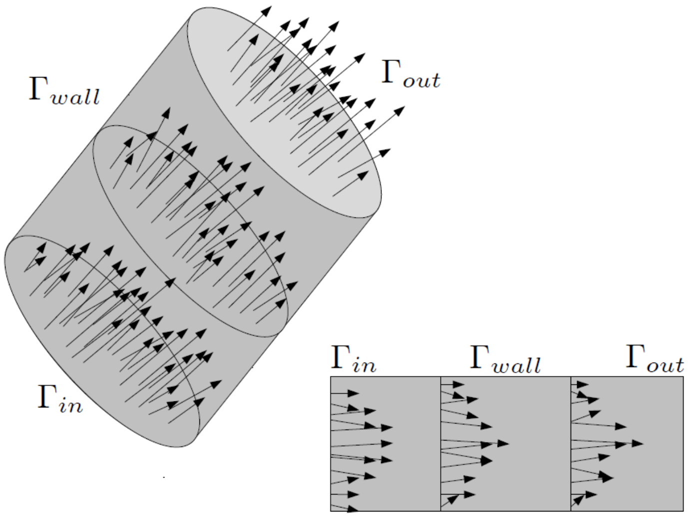

state equations:

$$\begin{cases} -\nu\, \nabla \cdot (\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}}) + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{s} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_{wall}, \\ -\nu\,(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}})\mathbf{n} + p\mathbf{n} = \mathbf{h} & \text{on } \Gamma_{in}, \\ -\nu\,(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}})\mathbf{n} + p\mathbf{n} = \mathbf{g} & \text{on } \Gamma_{out}. \end{cases}$$

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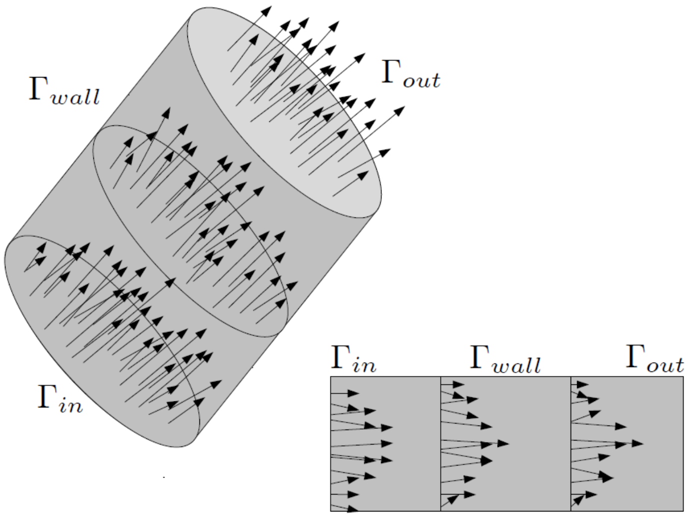

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assimilation:

$$\min_{\mathbf{h}} \mathcal{J}(\mathbf{u}, \mathbf{h}) = dist(f(\mathbf{u}), \mathbf{d}) + \mathcal{R}(\mathbf{h})$$

s.t. **state equations**

discrete formulation

$(\mathbf{u} \cdot \nabla)\mathbf{u} \Rightarrow (\boldsymbol{\beta} \cdot \nabla)\mathbf{u}$

discretize using the finite element (FE) method [1]

$$\min_{\mathbf{H}} \mathcal{J}(\mathbf{V}, \mathbf{H}) = \frac{1}{2}\|\mathrm{D}\mathbf{V} - \mathbf{d}\|_2^2 + \frac{\alpha}{2}\|\mathrm{L}\mathbf{H}\|_2^2$$
$$\text{s.t. } \mathrm{S}\mathbf{V} = \mathrm{R}_{in}^{\mathrm{T}}\mathrm{M}_{in}\mathbf{H} + \mathbf{F}.$$

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notation

- $\mathbf{V} = \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix}$, $\mathbf{U}$: discretized velocity, $\mathbf{P}$: discretized pressure
- $\mathrm{S} = \begin{bmatrix} \mathrm{C} + \mathrm{A} & \mathrm{B}^{\mathrm{T}} \\ \mathrm{B} & \mathrm{O} \end{bmatrix}$,
- C, A, B: discrete diffusion, advection and divergence operators
- R_{in}: restriction matrix, M_{in}: boundary mass matrix
- Q: selection matrix, s.t. $[\mathrm{Q}\mathbf{U}]_i$ = solution evaluated at the data sites
  $\mathrm{D} = [\mathrm{Q} \;\; \mathrm{O}]$, extension of Q to pressure degrees of freedom
- L: discretized differential operator, here discrete gradient

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discrete formulation

optimize

solving the KKT system induced by the Lagrangian with the Reduced Hessian method:

$$\mathcal{L}(\mathbf{V}, \mathbf{H}, \mathbf{\Lambda}) = \frac{1}{2}\|\mathrm{D}\mathbf{V} - \mathbf{d}\|_2^2 + \frac{\alpha}{2}\ \|\mathrm{L}\mathbf{H}\|_2^2 + \mathbf{\Lambda}^{\mathrm{T}}(\mathrm{S}\mathbf{V} - \mathrm{R}_{in}^{\mathrm{T}}\mathrm{M}_{in}\mathbf{H} - \mathbf{F})$$

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adjoint equation: $\mathrm{D}^{\mathrm{T}}(\mathrm{D}\mathbf{V} - \mathbf{d}) + \mathrm{S}^{\mathrm{T}}\mathbf{\Lambda} = \mathbf{0}$

residual equation: $\alpha\mathrm{L}^{\mathrm{T}}\mathrm{L}\mathbf{H} - \mathrm{M}_{in}^{\mathrm{T}}\mathrm{R}_{in}\mathbf{\Lambda} = \mathbf{0}$

state equation: $\mathrm{S}\mathbf{V} - \mathrm{R}_{in}^{\mathrm{T}}\mathrm{M}_{in}\mathbf{H} - \mathbf{F} = \mathbf{0}$.

$$\Downarrow$$

reduced system: $\mathrm{W}\mathbf{H} = \mathrm{Z}^{\mathrm{T}}(\mathbf{d} - \mathrm{D}\mathrm{S}^{-1}\mathbf{F})$

reduced Hessian: $\mathrm{W} = \mathrm{Z}^{\mathrm{T}}\mathrm{Z} + \alpha\,\mathrm{L}^{\mathrm{T}}\mathrm{L}$

sensitivity matrix: $\mathrm{Z} = \mathrm{D}\mathrm{S}^{-1}\mathrm{R}_{in}^{\mathrm{T}}\mathrm{M}_{in}$

optimality result

- sufficient conditions for an equality PDE constrained opt.pb: positive definite Hessian
- the regularized formulation **always** satisfies necessary and sufficient conditions
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Proposition sufficient conditions for the existence of a unique minimizer are:

1. $\alpha > 0$, or
2. $\alpha = 0$ and $Null(\mathrm{D}) \cap Range(\mathrm{S}^{-1}\mathrm{R}_{in}^{\mathrm{T}}\mathrm{M}_{in}) = \{\mathbf{0}\}$

this condition is satisfied by choosing D such that its restriction to rows corresponding to sites on Γ_{in} has rank N_{in} (degrees of freedom of $\mathbf{U}$ on Γ_{in})

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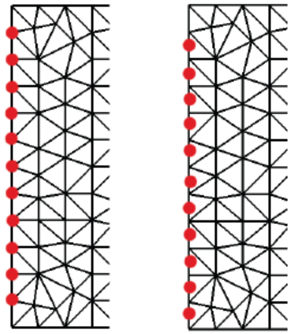

left: sites on grid nodes
right: sparse sites on the inflow boundary

(using P1bubble-P1 FE pair)

interpolation

- given sparse measurements on inflow not satisfying sufficient conditions
- recover those conditions with approximated data on grid nodes on **inflow**

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approximation: piecewise linear **interpolation** of each velocity component

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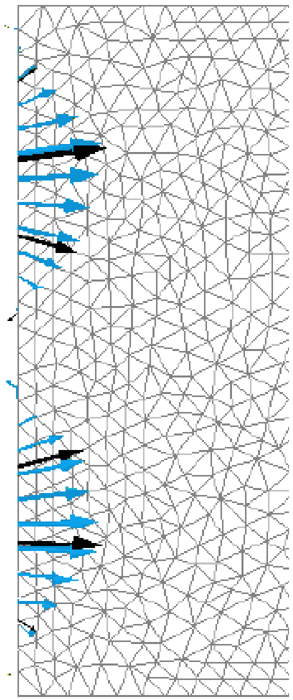

- black data: $\mathbf{d}$
- blue data: $\widetilde{\mathbf{d}}$

- original data $\mathbf{d} = \mathbf{u}_{ex} + \boldsymbol{\varepsilon}$
- interpolated data $\widetilde{\mathbf{d}} = \Pi\,\mathbf{d} = \mathbf{u}_{ex} + \boldsymbol{\eta}$

  where Π is the interpolation matrix

nonlinear formulation

non-linear constraint Navier-Stokes momentum equation, $-\nu\,\nabla\cdot(\nabla\mathbf{u}+\nabla\mathbf{u}^{\mathrm{T}})+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p$

algorithm 1. iterative procedure exploiting the **Picard** (or **Newton**) method:

`given` $\mathbf{V}_k$`, a guess for the velocity at step k+1, solve`

$$\begin{aligned}\min_{\mathbf{H}_{k+1}}\ & \tfrac{1}{2}\|\mathrm{D}\mathbf{V}_{k+1}(\mathbf{H}_{k+1})-\mathbf{d}\|_2^2+\tfrac{\alpha}{2}\|\mathrm{L}\mathbf{H}_{k+1}\|_2^2\\ \text{s.t.}\quad & \mathrm{S}_k\mathbf{V}_{k+1}=\mathrm{R}_{in}^{\mathrm{T}}\mathrm{M}_{in}\mathbf{H}_{k+1}+\mathbf{F}\end{aligned}$$

`where` $\mathrm{S}_k=\begin{bmatrix}\mathrm{C}+\mathrm{A}_k & \mathrm{B}^{\mathrm{T}}\\ \mathrm{B} & \mathrm{O}\end{bmatrix}$

`up to fulfillment of a convergence criterion`

numerical results

- the deterministic procedure is an effective and robust noise filtering method

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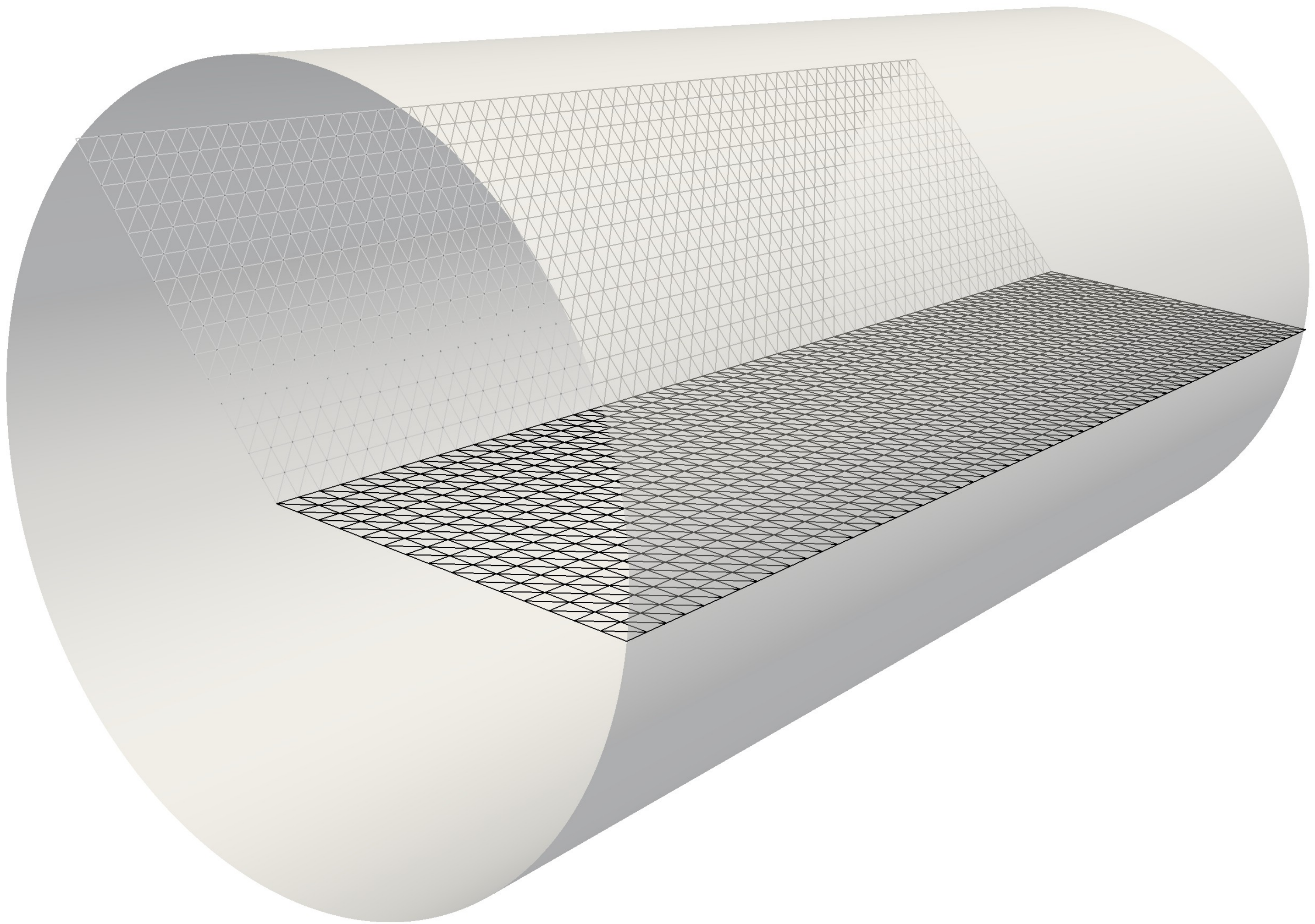
- the discretization error decreases as more data are available: it is proportional $N_s^{-0.5}$.

- the sample mean of the computed velocity over N_r noise realizations converges to the noise-free solution with rate $N_r^{-0.5}$

- the discretization error is proportional to the amount of noise

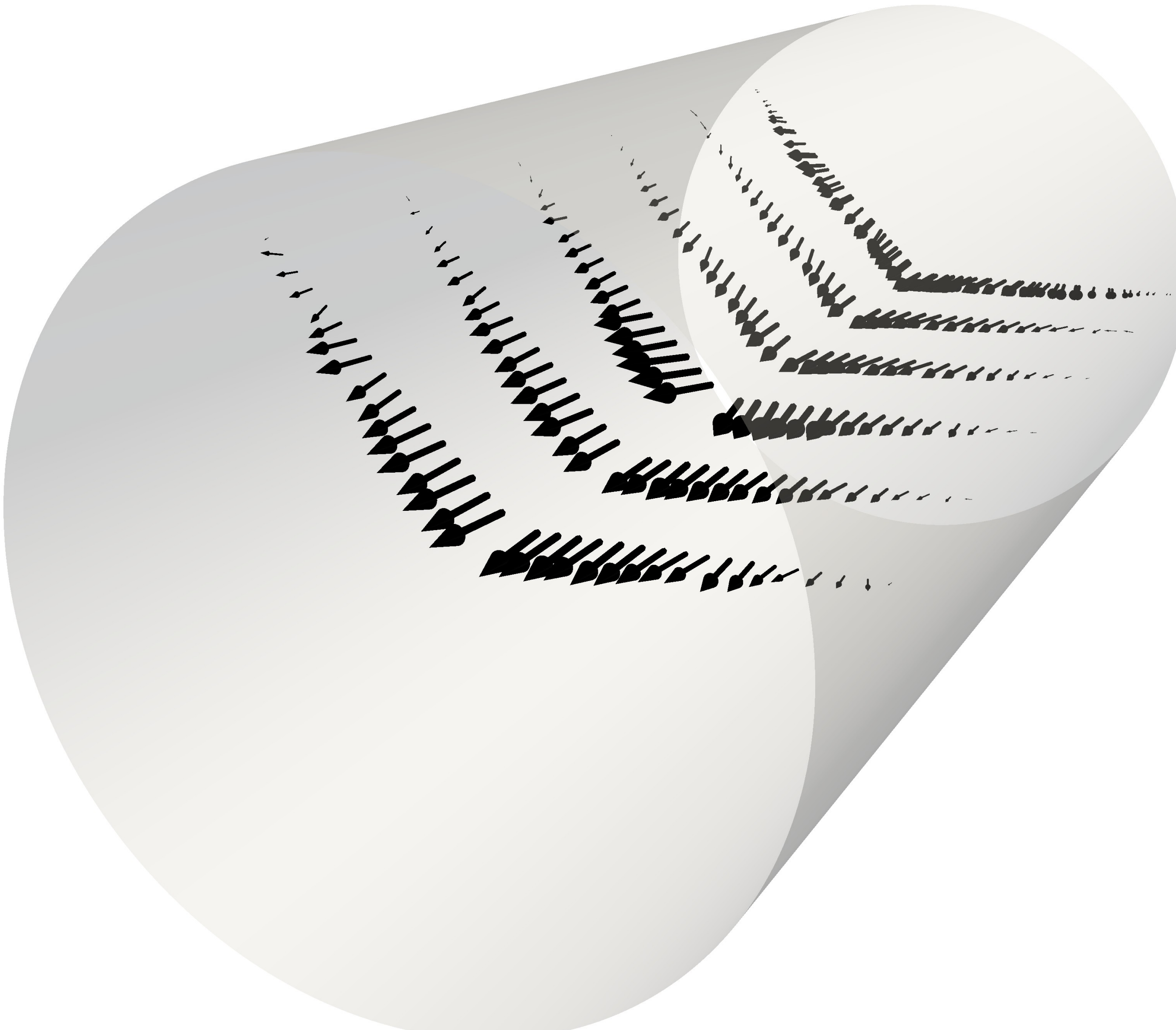
numerical results

domain: rectangular domain representing a slice of a cylinder



numerical results

data generation: analytical solution with additional noise, data on the inflow boundary do not satisfy sufficient conditions → piece-wise linear interpolation



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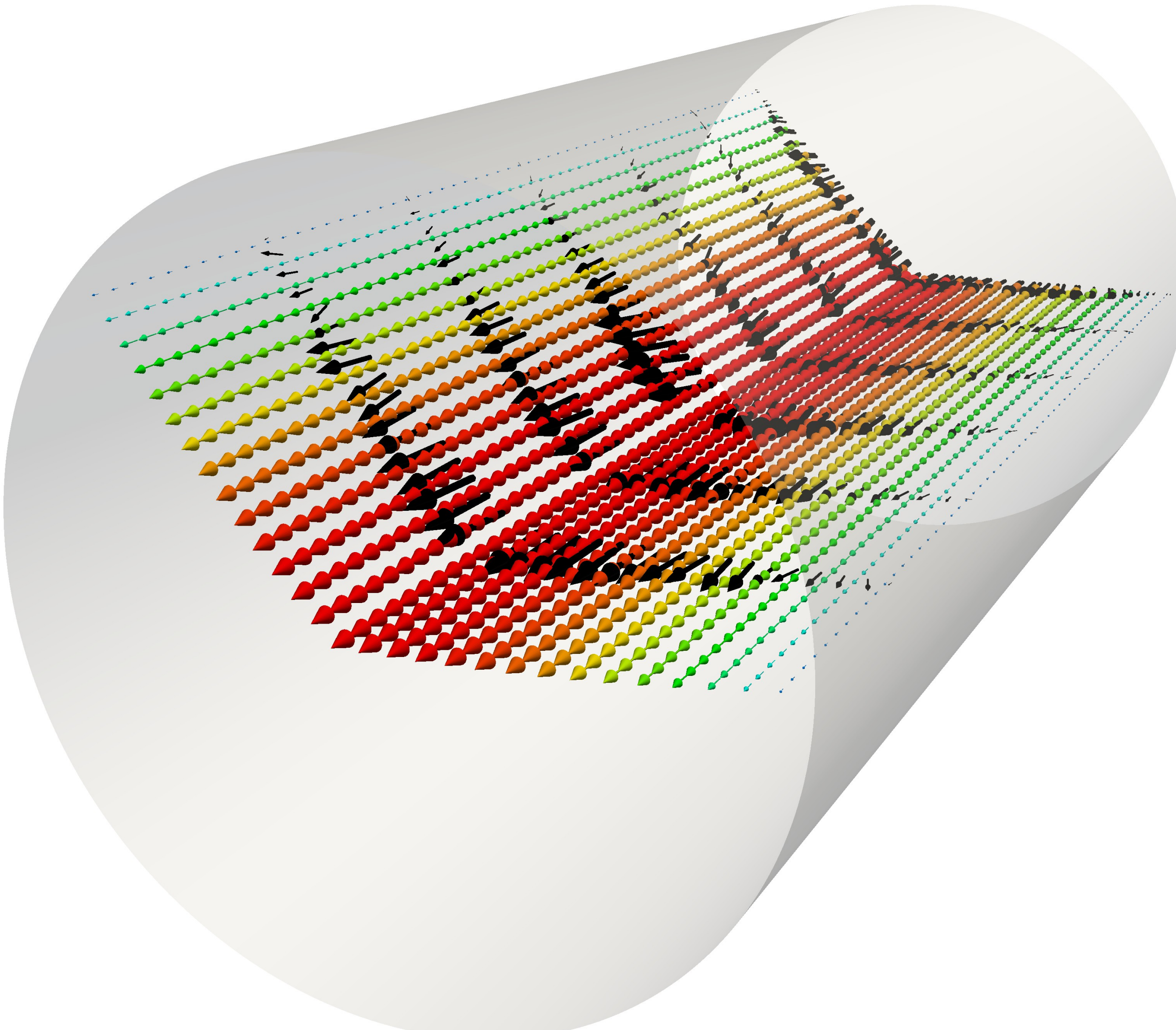

2. statistical formulation [4]

[4] M.DE,. Veneziani, *Uncertainty Quantification for the incompressible Navier-Stokes equations in Hemodynamics,* in preparation.

statistical inversion

goal: estimate the reliability of results → quantification of the uncertainty

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idea: we predict stochastic features of the variables of interest

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main features:

- all discretized variables are treated as **random**

  the randomness is in the degree of information of their realizations

  such degree resides in the **probability distributions**

- the entities involved are probability density functions (PDFs)
- the method delivers a **distribution**

(deterministic methods produce a **single** estimate)

statistical inversion – notation

random variables (RV):

- $\mathbf{H}$: RV for normal stress of the fluid at the inflow section
- $\mathbf{M}$: RV for the measures
- $\boldsymbol{\varepsilon}$: RV for the noise perturbing the measurements

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probability density functions (PDFs):

- $\pi_{pr}(H)$: PDF of $\mathbf{H}$, the *prior*
- $\pi_{noise}(\varepsilon)$: PDF of $\boldsymbol{\varepsilon}$
- $\pi(M|H)$: PDF of $\mathbf{M}$ conditioned on a realization of $\mathbf{H}$; the *likelihood*
- $\pi_{post}(H) = \pi(H|M)$: PDF of $\mathbf{H}$ conditioned on a realization of $\mathbf{M}$, the *posterior*

statistical inversion – notation

statistical properties of $\mathbf{M}$ are determined by the distribution of $\mathbf{H}$ and $\boldsymbol{\varepsilon}$

$$\mathrm{Z}\mathbf{H} + \boldsymbol{\varepsilon} = \mathbf{M} \qquad (\textit{additive noise}\ \text{model})$$

- linear (or linearized) **deterministic** model that relates $\mathbf{H}$ and $\mathbf{M}$
- $\mathrm{Z} = \mathrm{D}\mathrm{S}^{-1}\mathrm{R}_{in}\mathrm{M}_{in}$ is the “Neumann-to-Dirichlet” map

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assumption: independence of $\mathbf{H}$ and $\boldsymbol{\varepsilon}$

consequence: $M|H$ is distributed like $\boldsymbol{\varepsilon}$ with density function translated by ZH

$$\Rightarrow \pi(M|H) = \pi_{noise}(M - ZH)$$

statistical inversion

objective: estimate the posterior exploiting the *Bayes theorem*

$$\pi_{post}(H) = \frac{\pi(M|H)\pi_{pr}(H)}{\pi(M)}$$

we are interested in $\mathbf{H} \Rightarrow$ the denominator does not affect the solution

when $M = \mathbf{d}$ is a specific realization of $\mathbf{M}$, $\underline{\pi_{post}(H) \propto \pi_{noise}(\mathbf{d} - \mathrm{Z}H)\pi_{pr}(H)}$

Gaussian assumption: $H \sim \mathcal{N}(H_0, \Sigma_{pr})$, *prior*

$\varepsilon \sim \mathcal{N}(\varepsilon_0, \Sigma_{noise})$, *likelihood*

⇓

$H|M \sim \mathcal{N}(H_{post}, \Sigma_{post})$, *posterior*

$$H_{post} = (\Sigma_{pr}^{-1} + \mathrm{Z}^{\mathrm{T}}\Sigma_{noise}^{-1}\mathrm{Z})^{-1}(\mathrm{Z}^{\mathrm{T}}\Sigma_{noise}^{-1}(\mathbf{d} - \varepsilon_0) + \Sigma_{pr}^{-1}H_0)$$

$$\Sigma_{post} = (\Sigma_{pr}^{-1} + \mathrm{Z}^{\mathrm{T}}\Sigma_{noise}^{-1}\mathrm{Z})^{-1}$$

statistical inversion

using this result one can calculate **point** or **interval** estimates

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maximum a posteriori (MAP) estimator:

the most likely value of $\mathbf{H}$ given $\mathbf{d}$: $H_{MAP} = \arg\max_{H} \pi_{post}(H)$

Gaussian assumption $\Rightarrow H_{MAP} = H_{post}$: expected value of the posterior

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maximum likelihood (ML) estimator:

value of $\mathbf{H}$ which is most likely to produce the data $\mathbf{d}$: $H_{ML} = \arg\max_{H} \pi(M|H)$

point estimators

comparison

$$
\begin{array}{llll}
\mathbf{1}. & H_{det} = (\alpha\,\mathrm{L}^{\mathrm{T}}\mathrm{L} + \mathrm{Z}^{\mathrm{T}}\mathrm{Z})^{-1} & \mathrm{Z}^{\mathrm{T}} & (\mathbf{d} - \mathrm{D}\mathrm{S}^{-1}\mathbf{F}) \\
\mathbf{2}. & H_{MAP} = (\Sigma_{pr}^{-1} + \mathrm{Z}^{\mathrm{T}}\Sigma_{noise}^{-1}\mathrm{Z})^{-1} & \mathrm{Z}^{\mathrm{T}}\Sigma_{noise}^{-1} & (\mathbf{d} - \mathrm{D}\mathrm{S}^{-1}\mathbf{F}) \\
\mathbf{3}. & H_{ML} = (\mathrm{Z}^{\mathrm{T}}\Sigma_{noise}^{-1}\mathrm{Z})^{-1} & \mathrm{Z}^{\mathrm{T}}\Sigma_{noise}^{-1} & (\mathbf{d} - \mathrm{D}\mathrm{S}^{-1}\mathbf{F})
\end{array}
$$

the choice between **2.** and **3.** depends on the **level of prior knowledge**

2. corresponds to moving the estimate towards the prior

3. corresponds to not trusting our prior belief on $\mathbf{H}$: "$\Sigma_{pr} \to 0$"

point estimators

comparison

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Gaussian smoothness priors: prior with encoded **structural information**

example: assumption of differentiability for **H**, $\Sigma_{prior}^{-1} \propto \mathrm{L}^{\mathrm{T}}\mathrm{L}$

point estimators

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likelihood parameters

likelihood function: Gaussian PDF

- **expected value**: $\varepsilon_0 = \mathbf{0}$, (*personal communication of Dr. Brummer, CHOA*)
- **correlation**: exponential decay w.r.t. the square of the mutual distance

$$[\Sigma_{noise}]_{ij} = [\Sigma_\varepsilon]_{ij} = exp\left\{-\frac{1}{l^2}\|\mathbf{x}_i^m - \mathbf{x}_j^m\|_2^2\right\}, \qquad l = \text{reference distance}$$

numerical results

discretization: compatible finite element (FE) spaces for velocity and pressure P1bubble-P1

C++ finite element solver `lifeV` finite element library, see www.lifev.org

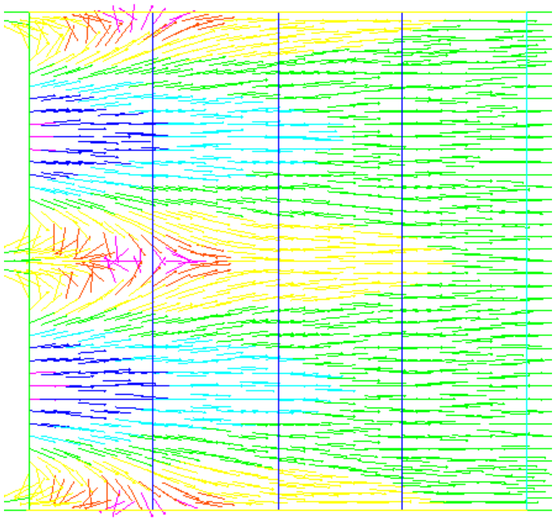
analytic solution:

$$\Omega = [-0.5,\ 1.5] \times [0,\ 2]$$

$$[\mathbf{u}]_1(x,y) = 1 - e^{\lambda x}\cos(2\pi y)$$

$$[\mathbf{u}]_2(x,y) = \frac{\lambda}{2\pi} e^{\lambda x}\sin(2\pi y)$$

$$p(x,y) = \frac{1}{2} e^{2\lambda x} + C$$

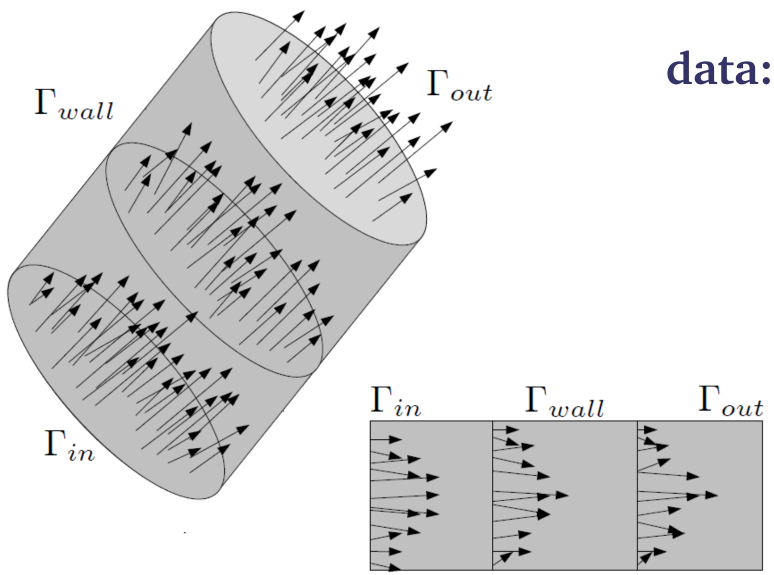


data: - generated adding to the analytic solution Gaussian noise

- located on grid nodes, i.e. discretization step Δ is s.t. $\Delta \propto N_s^{-1}$.
  with layers: $\{(x,y) \mid x \in \{0.5,\ 0,\ 0,5,\ 1.5\},\ \ y \in [0,\ 2]\ \}$.

- OR sparse on Γ_{in} and in Ω

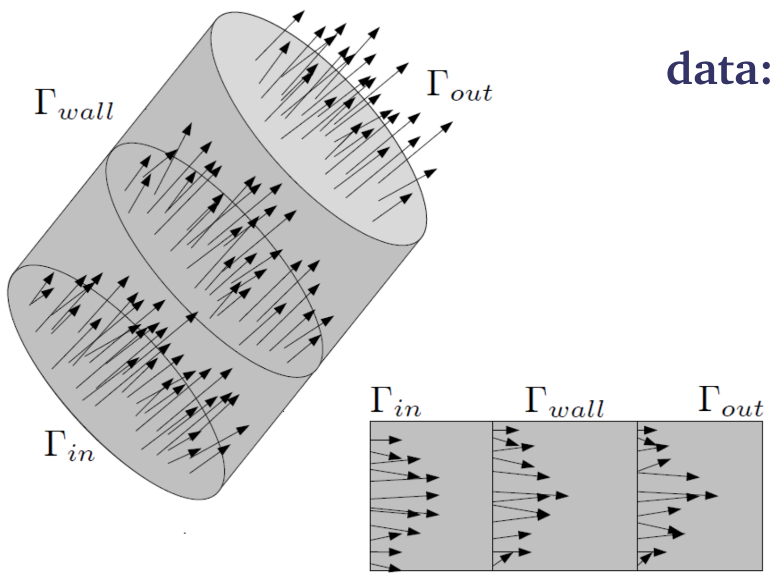
point estimators – numerical results



tests: compare MAP and ML with deterministic estimates

data:
- on Γ_{in} not satisfying conditions for optimality
- on internal slices parallel to Γ_{in}

point estimators – numerical results



tests: compare MAP and ML with deterministic estimates

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indexes of accuracy:

- $\bar{E}_{\mathbf{U}} = \frac{1}{n}\sum_{i=1}^{n} E_{\mathbf{U},i}$, for $n = 30$ noise realizations
- $\%\ \text{gain} = \gamma = 1 - \dfrac{\overline{E}_{\mathbf{U},MAP}}{\overline{E}_{\mathbf{U},det}} \quad \text{or} \quad 1 - \dfrac{\overline{E}_{\mathbf{U},ML}}{\overline{E}_{\mathbf{U},det}}$

point estimators – numerical results

<u>**test case A**</u>

- linearized formulation
- H_{det} versus H_{MAP}
- three different values of α
- interpolation not active

SNR	α	$\overline{E}_{\mathbf{U},det}$	$\overline{E}_{\mathbf{U},MAP}$	γ
20	0.5	0.0665	0.0530	24%
20	0.05	0.0666	0.0550	17%
20	0.005	0.0706	0.0579	18%
10	0.5	0.1272	0.0946	26%
10	0.05	0.1514	0.1032	32%
10	0.005	0.1256	0.1059	28%

$\mathtt{it}(H_{MAP}) \propto 1.3\ \mathtt{it}(H_{det})$, (due to the presence of Σ_{noise}^{-1})

point estimators – numerical results

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test case B

- linearized formulation
- H_{det} versus H_{ML}
- $\alpha = 0$
- interpolation active

SNR	$\overline{E}_{\mathbf{U},det}$	$\overline{E}_{\mathbf{U},ML}\,(mod)$	γ
20	0.0709	0.0552	22%
10	0.1518	0.1256	17%

$\texttt{it}(H_{ML}) \propto 1.5\ \texttt{it}(H_{det})$, (due to the presence of Σ_{noise}^{-1})

point estimators – numerical results

test case C

- nonlinear formulation
- H_{det} versus H_{MAP}
- $\alpha = 0.5$ (see linearized case)
- interpolation not active

SNR	$\overline{E}_{\mathbf{U},det}$	$\overline{E}_{\mathbf{U},MAP}$	γ
20	0.0822	0.07371	10%
10	0.1394	0.1041	25%

point estimators – numerical results

test case C

- nonlinear formulation
- H_{det} versus H_{MAP}
- $\alpha = 0.5$ (see linearized case)
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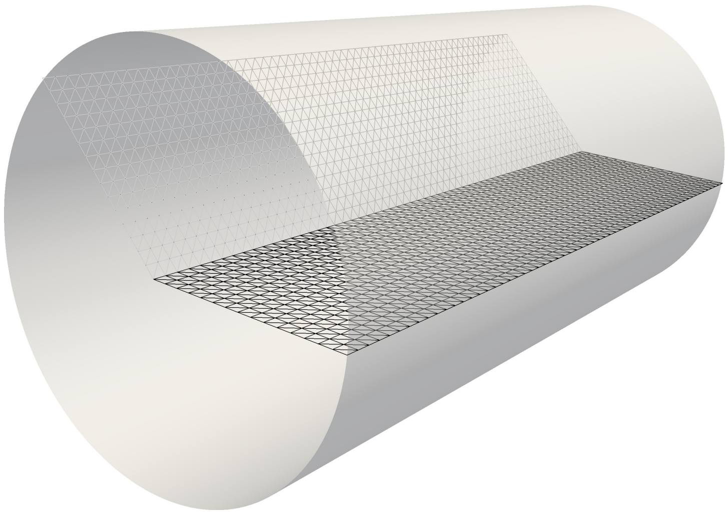
SNR	$\overline{E}_{\mathbf{U},det}$	$\overline{E}_{\mathbf{U},MAP}$	γ
20	0.0822	0.07371	10%
10	0.1394	0.1041	25%

test case D

- nonlinear formulation
- H_{det} versus H_{ML}
- $\alpha = 0$ (see linearized case)
- interpolation active

SNR	$\overline{E}_{\mathbf{U},det}$	$\overline{E}_{\mathbf{U},ML}$	γ
20	0.0855	0.0579	6%
10	0.1675	0.1363	18%

point estimators – numerical results



test case E

- axisymmetric formulation
- H_{det} versus H_{MAP}
- $\alpha = 10^{-7}$
- interpolation active

SNR	$\overline{E}_{\mathbf{U},det}$	$\overline{E}_{\mathbf{U},MAP}$	γ
20	0.0396	0.0308	22%
10	0.1423	0.0978	31%

spread estimators – mathematical background

the multivariate normal distribution

probability density function of a random vector $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$:

$$f(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^d det(\Sigma)}} exp\left\{-(\mathbf{x} - \boldsymbol{\mu})^{\mathrm{T}} \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu})\right\} \quad \forall\, x_i \in (-\infty, \infty),\ i = 1, ..., d$$

$\boldsymbol{\mu} \in \mathbb{R}^d$: expected value, $\Sigma \in \mathbb{R}^{d,d}$: s.p.d. covariance matrix.

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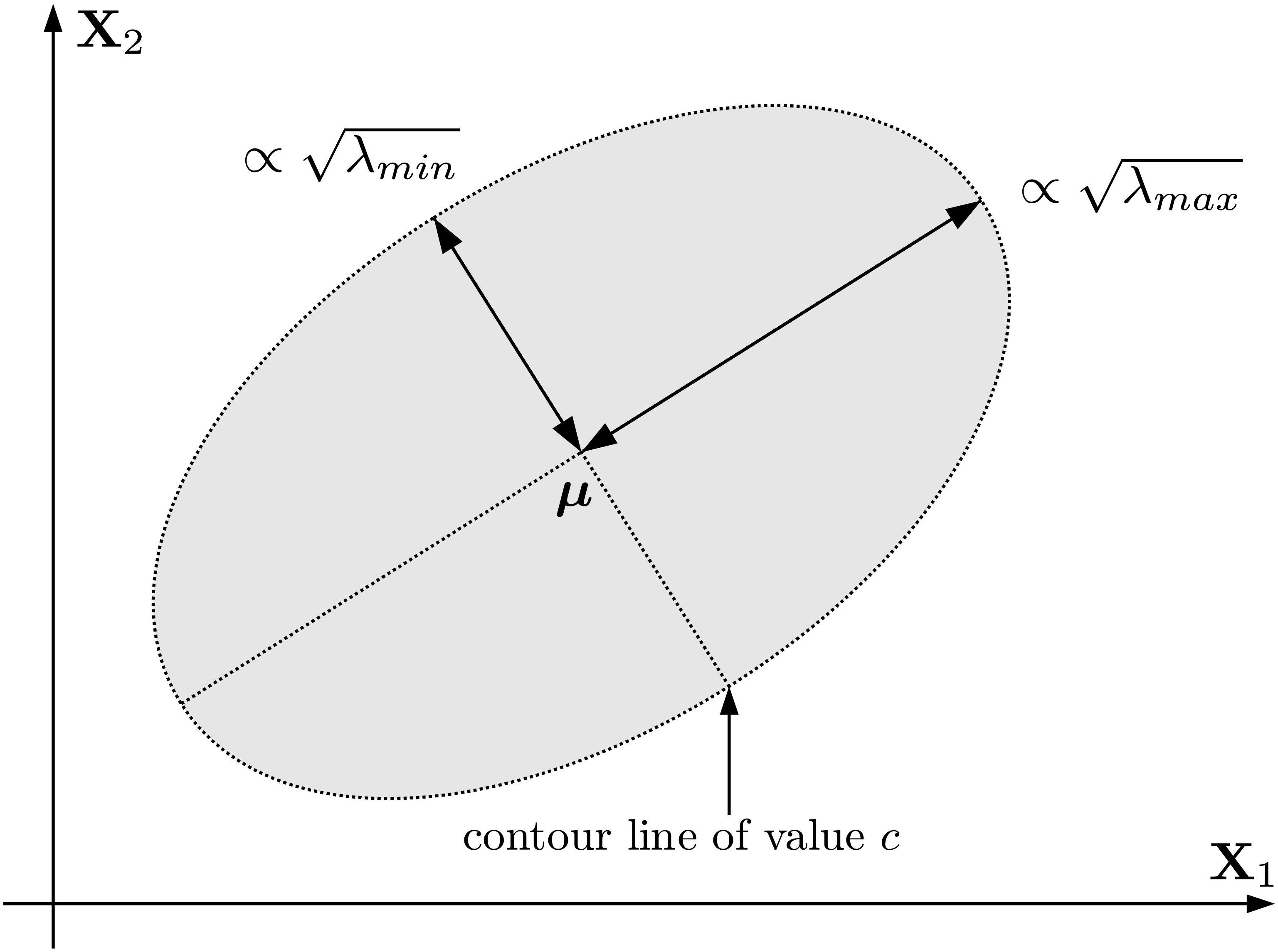
contour lines of constant density c are ellipsoids generated by $(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu}) = c^2$

2D example

λ_i, $i = 1, 2$, are e-values of Σ



spread estimators – mathematical background

properties:

P1 Affine transformations of $\mathbf{X}$ are normally distributed.

P2 All subsets of the components of $\mathbf{X}$ have normal distribution.

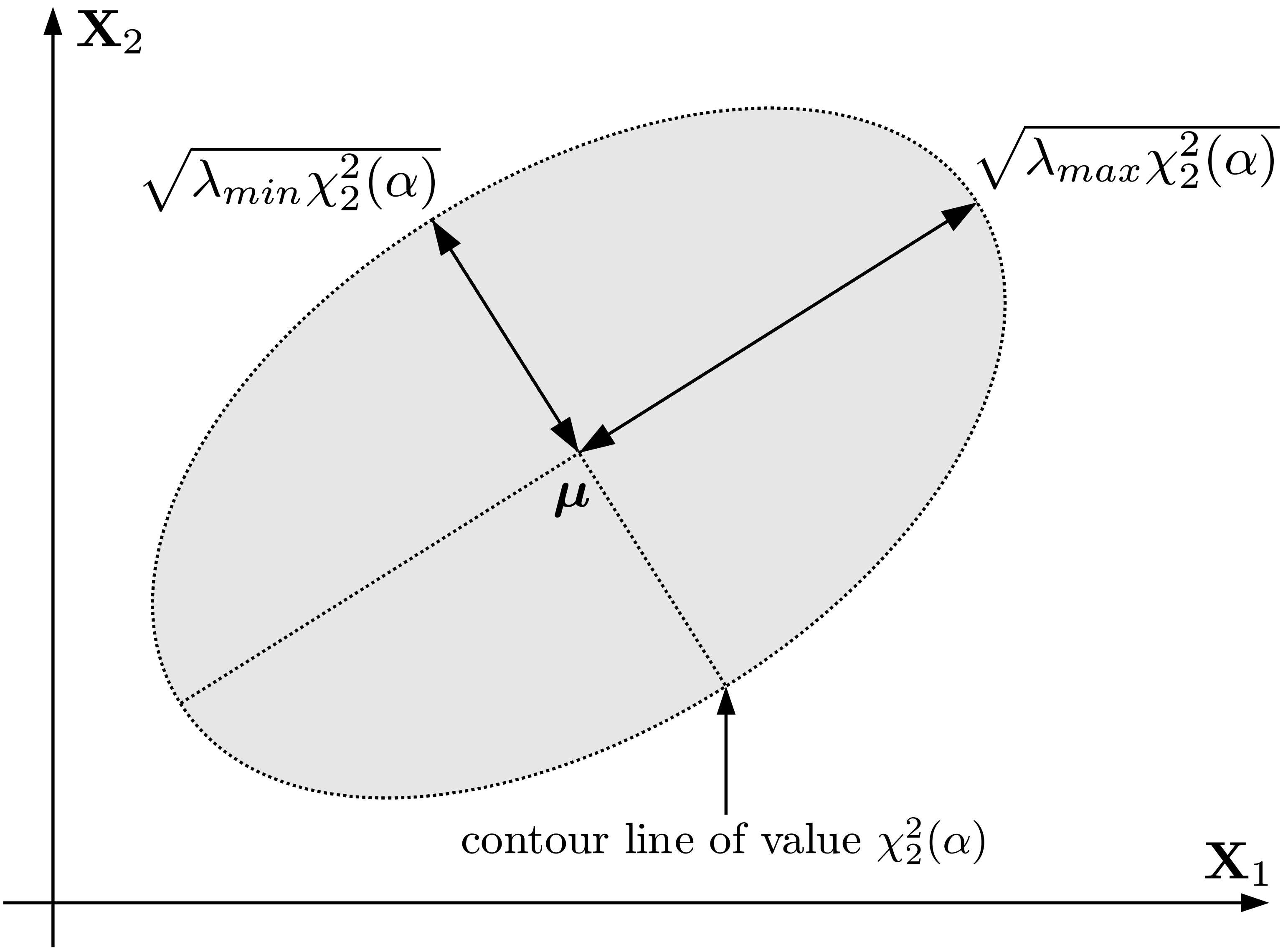
P3

- $(\mathbf{X} - \boldsymbol{\mu})^{\mathrm{T}} \Sigma^{-1} (\mathbf{X} - \boldsymbol{\mu})$ is distributed as χ^2_d
  where χ^2_d denotes the chi-squared distribution with d DOFs
- The $\mathcal{N}(\boldsymbol{\mu}, \Sigma)$ distribution assigns probability $(1 - \alpha)$ to the ellipsoid $\{\mathbf{x} : \ (\mathbf{x} - \boldsymbol{\mu})^{\mathrm{T}} \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}) < \chi^2_d(\alpha)\}$

2D example

probability for $\mathbf{X}$ to be inside the shaded region is $1-\alpha$

λ_i, $i = 1, 2$, are e-values of Σ



spread estimators - velocity

goal: quantify how likely velocity and flow related variables are inside an interval of (critical, significant) values → predict vessel dilatation

spread estimators - velocity

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velocity distribution

- deterministic model: affine transformation $\mathbf{V} = \mathrm{S}^{-1}\mathrm{R}_{in}^{\mathrm{T}}\mathrm{M}_{in}\mathbf{H} + \mathrm{S}^{-1}\mathbf{F}$

$$\mathbf{U} = \mathrm{E}(\mathrm{S}^{-1}\mathrm{R}_{in}^{\mathrm{T}}\mathrm{M}_{in}\mathbf{H} + \mathrm{S}^{-1}\mathbf{F}) = \mathrm{T}\mathbf{H} + \mathrm{E}\mathrm{S}^{-1}\mathbf{F}$$

- velocity distribution: $\mathbf{U} \sim \mathcal{N}(U, \Sigma_U)$
- expectation value $U = \mathrm{T}H_{post} + \mathrm{E}\mathrm{S}^{-1}\mathbf{F}$
- correlation matrix: $\Sigma_U = \mathrm{T}\Sigma_{post}\mathrm{T}^{\mathrm{T}}$

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velocity confidence regions

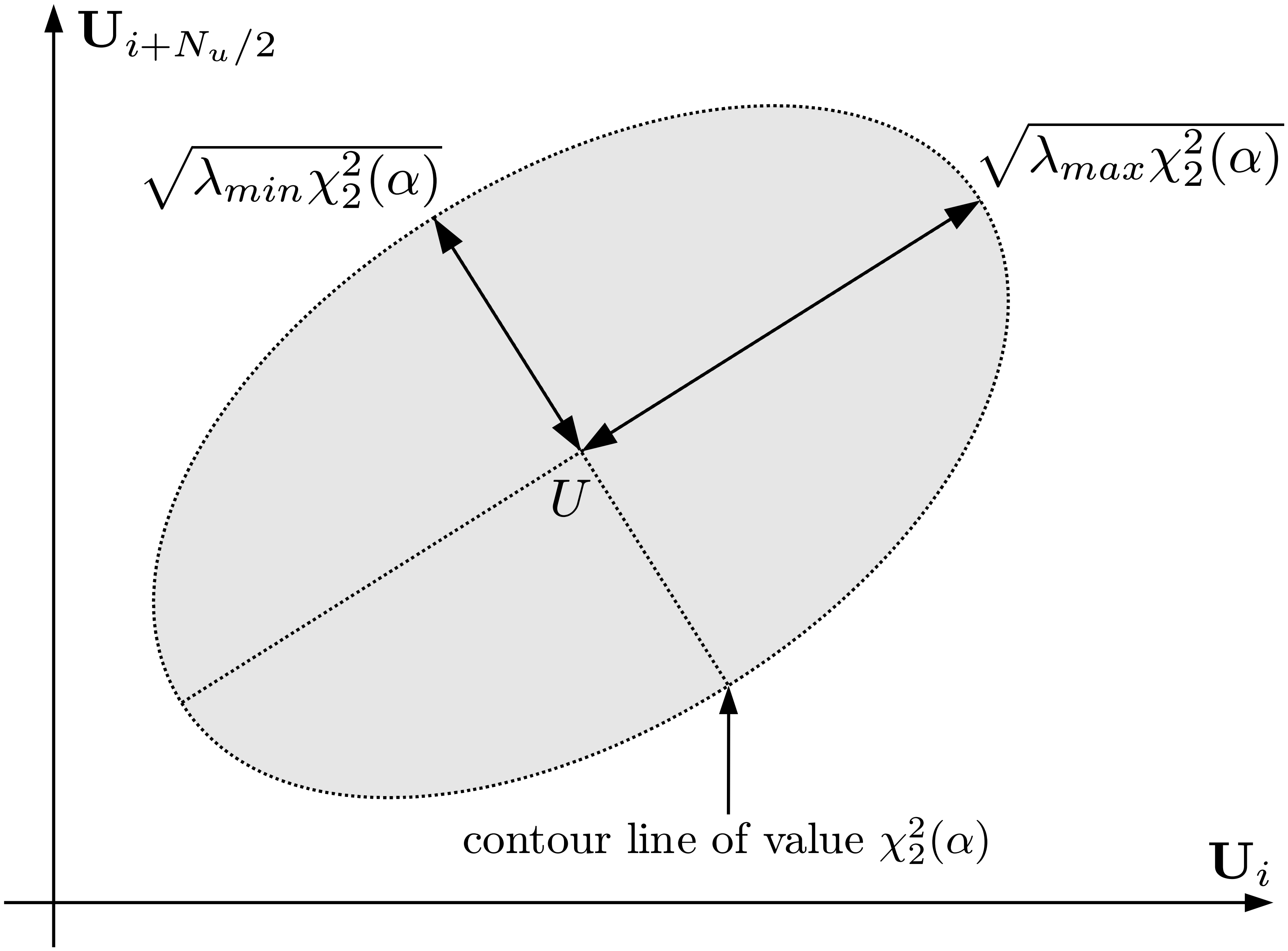
horizontal and vertical velocity in the i-th DOF, $[\mathbf{U}_i \;\; \mathbf{U}_{i+N_u/2}]^{\mathrm{T}} \in \mathbb{R}^2$:

subset of the components of $\mathbf{U}$ $\Rightarrow$ 2D Gaussian random vector

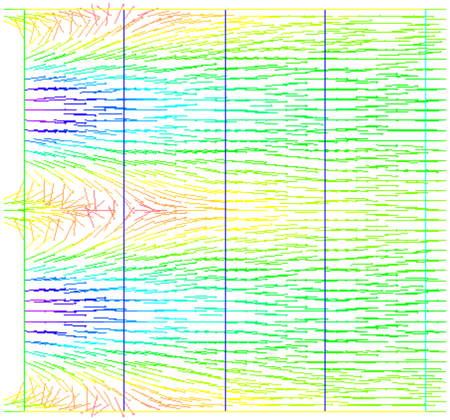
$\Rightarrow$ we can draw credibility regions

$1-\alpha$ credibility region for $\mathbf{U}$ on the i-th DOF

λ_i, $i = 1, 2$, are e-values of $\Sigma_{U,i}$



spread estimators – numerical results

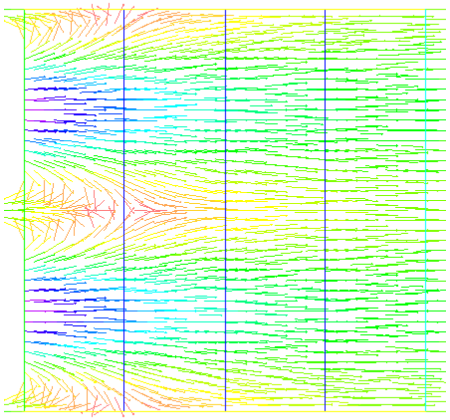


test case: same analytic solution, square domain, SNR $= 20$

map of the maximum deviation from the mean
in a 60% confidence region: $\sqrt{\lambda_{max}}$

$$(\mathbf{U}_j - U_j)^{\mathrm{T}} \Sigma_{U,j}^{-1} (\mathbf{U}_j - U_j) < \chi_2^2(40\%) \cong 1,$$

spread estimators – numerical results



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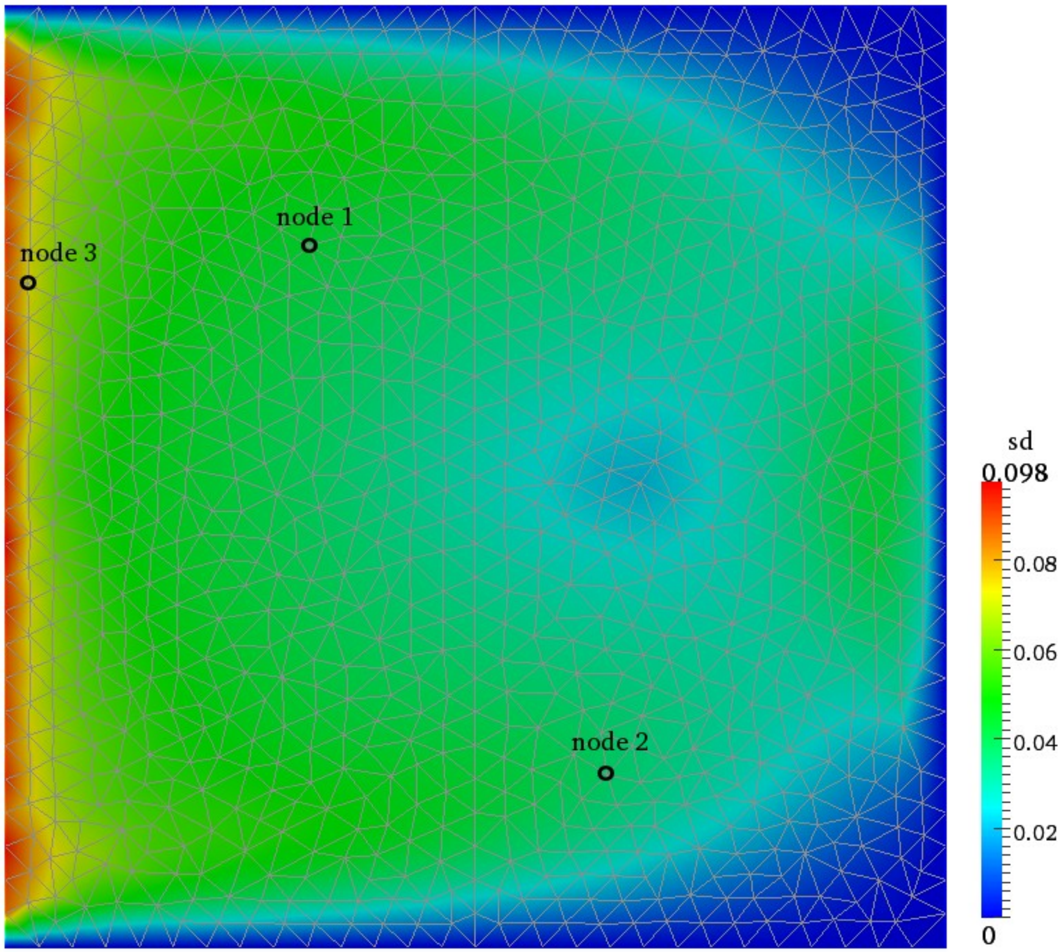
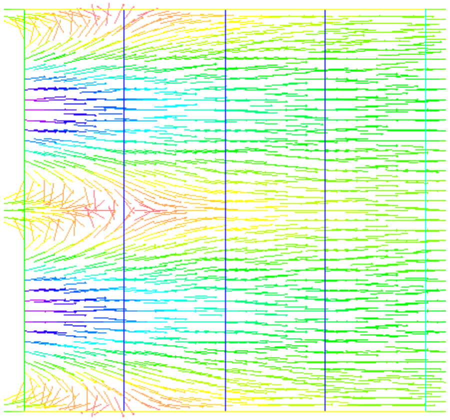


input noise: std $= 0.1467$

output noise: max std $= 0.098$

spread estimators – numerical results



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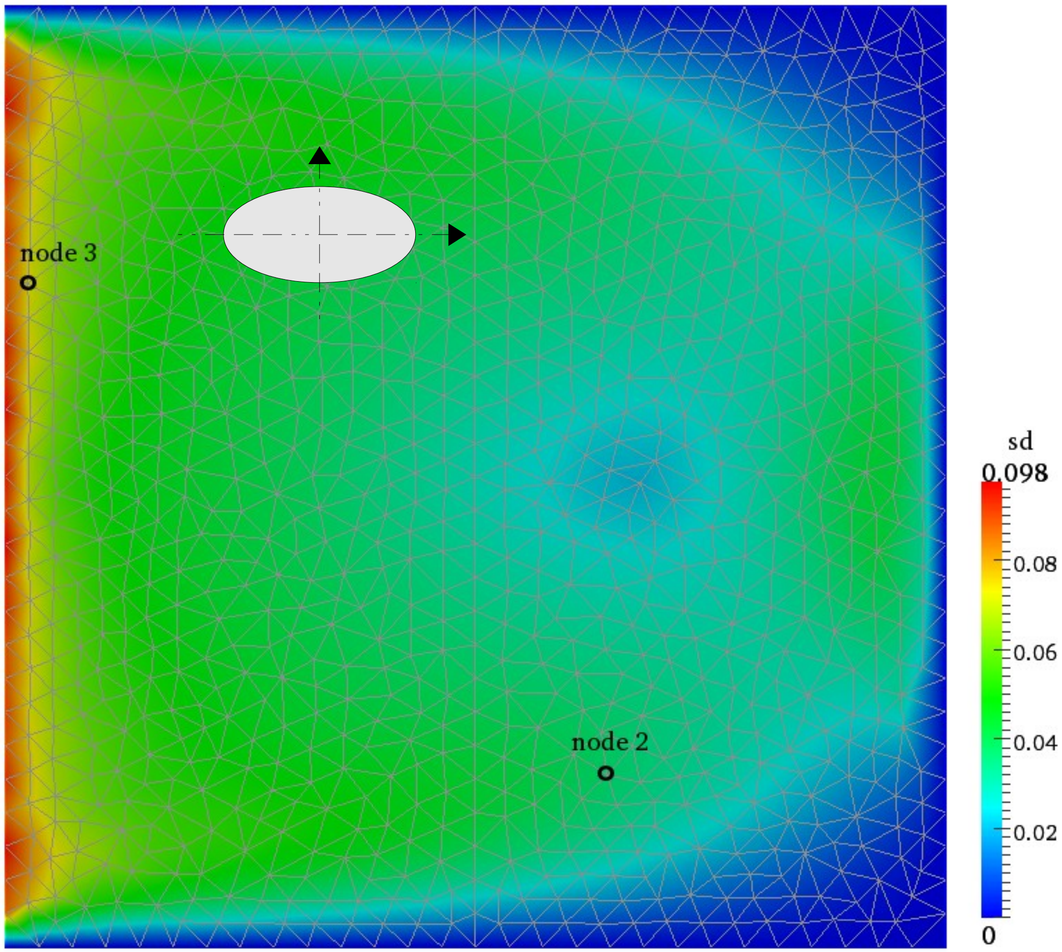

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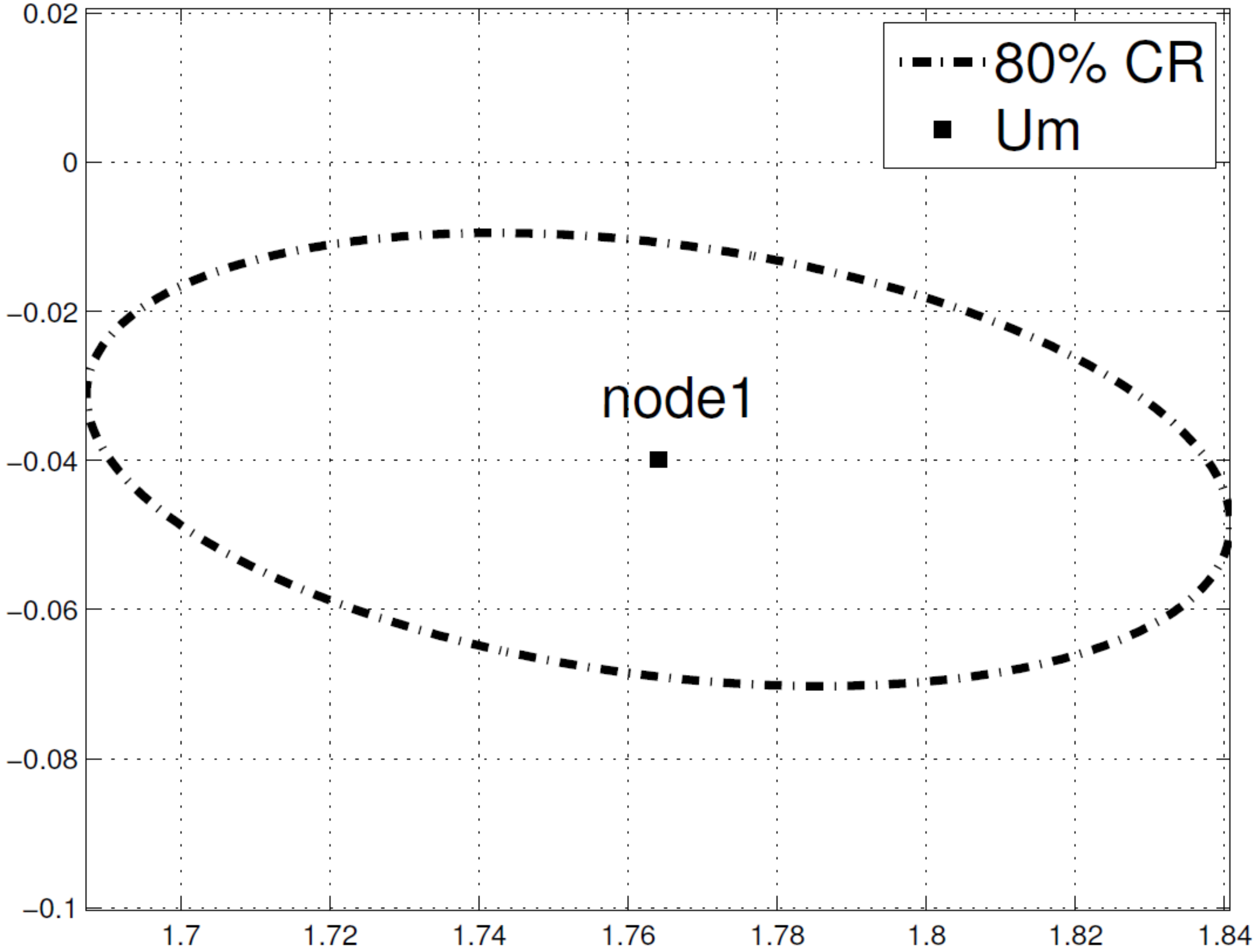
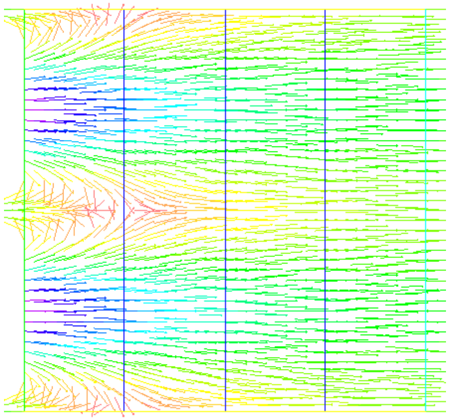


spread estimators – numerical results



test case: same analytic solution, square domain, SNR = 20

map of the maximum deviation from the mean in a 60% confidence region: $\sqrt{\lambda_{max}}$

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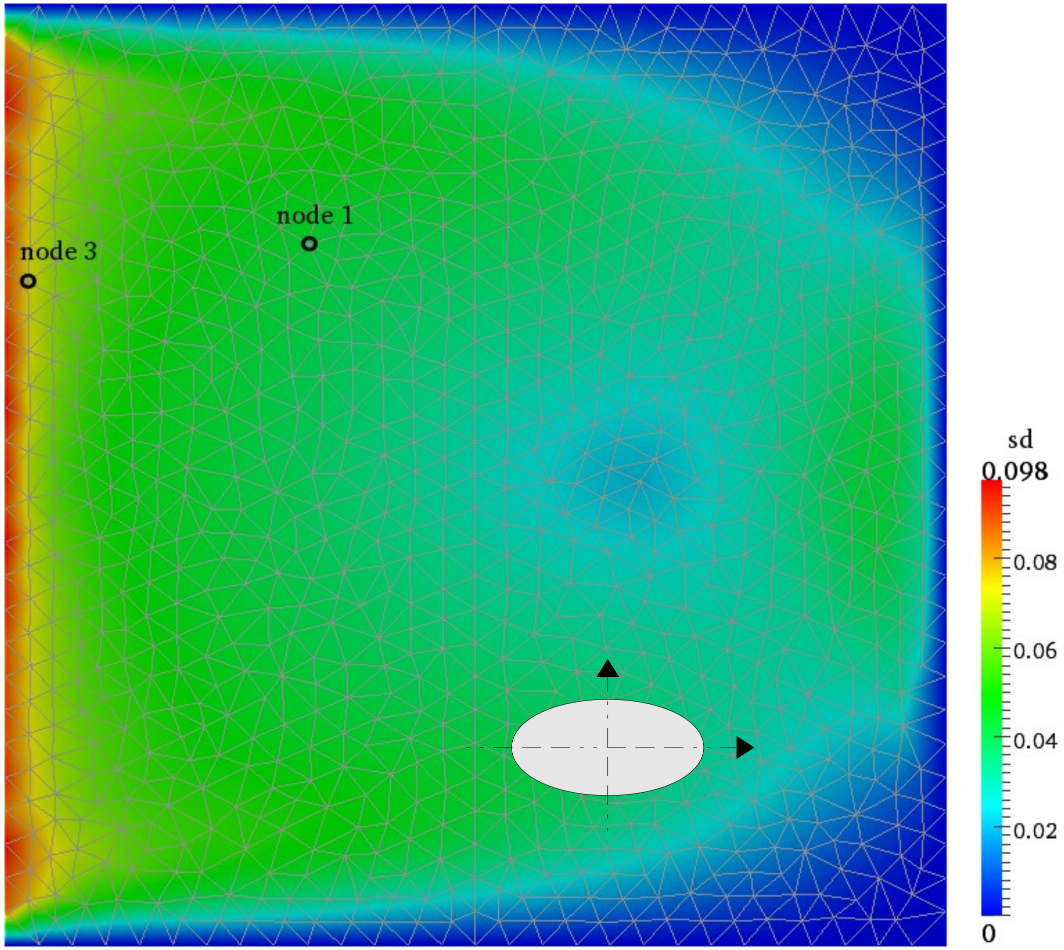

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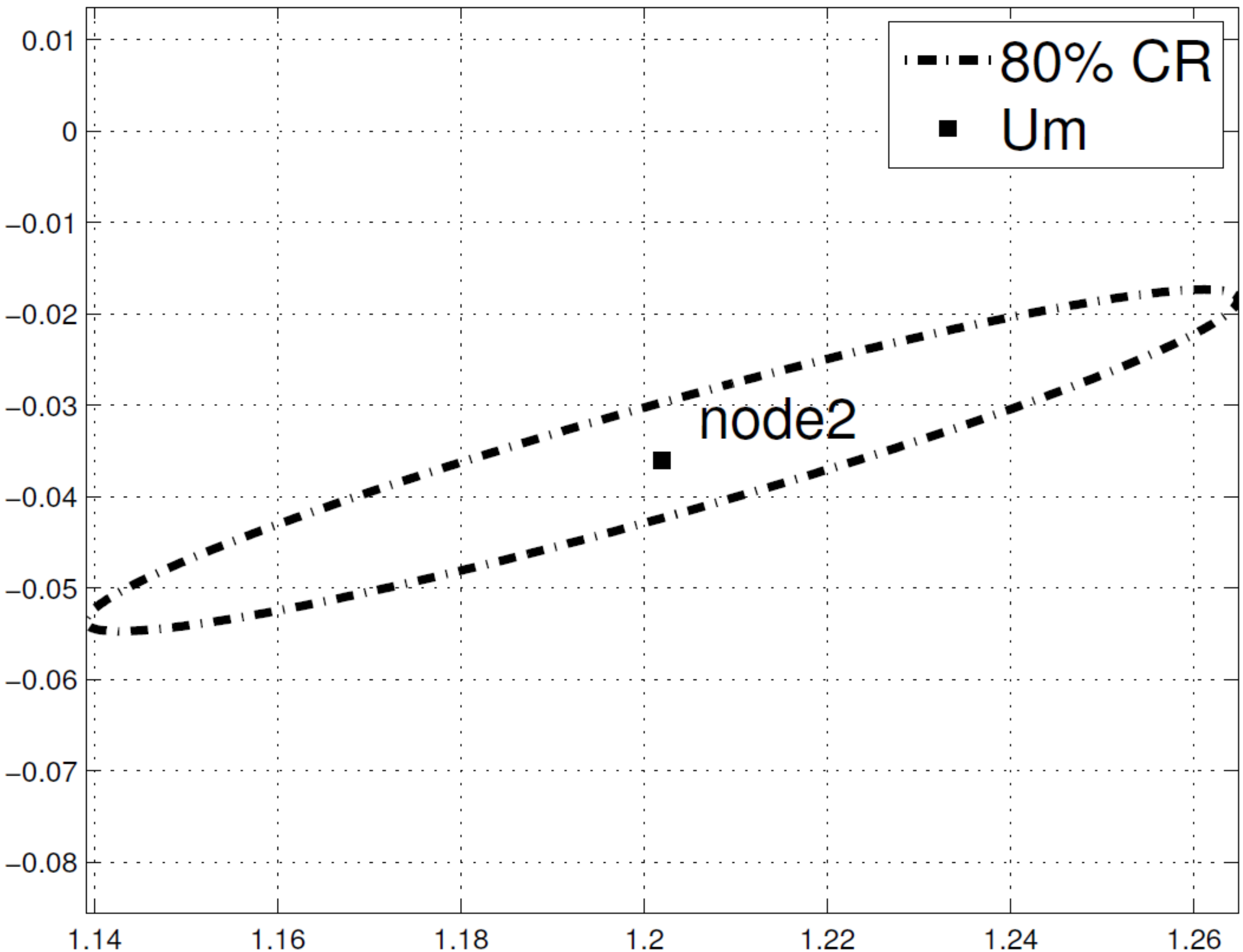
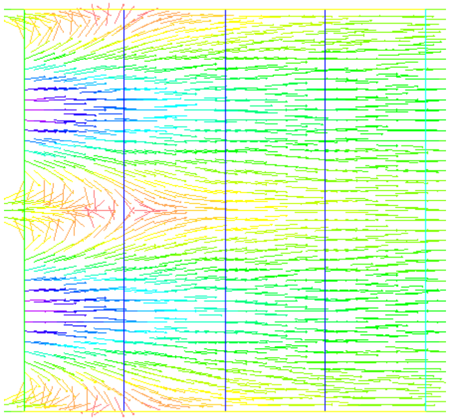
spread estimators – numerical results



spread estimators – numerical results



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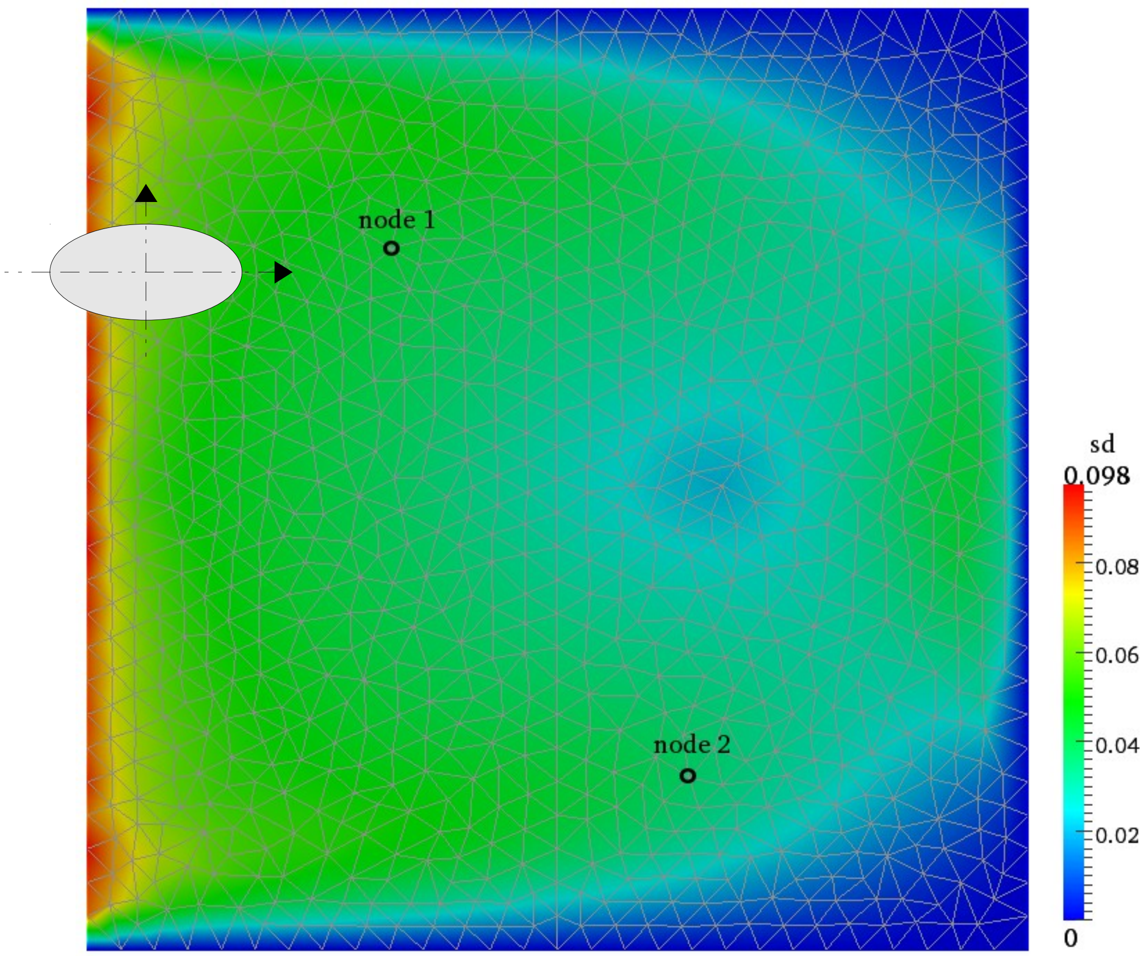

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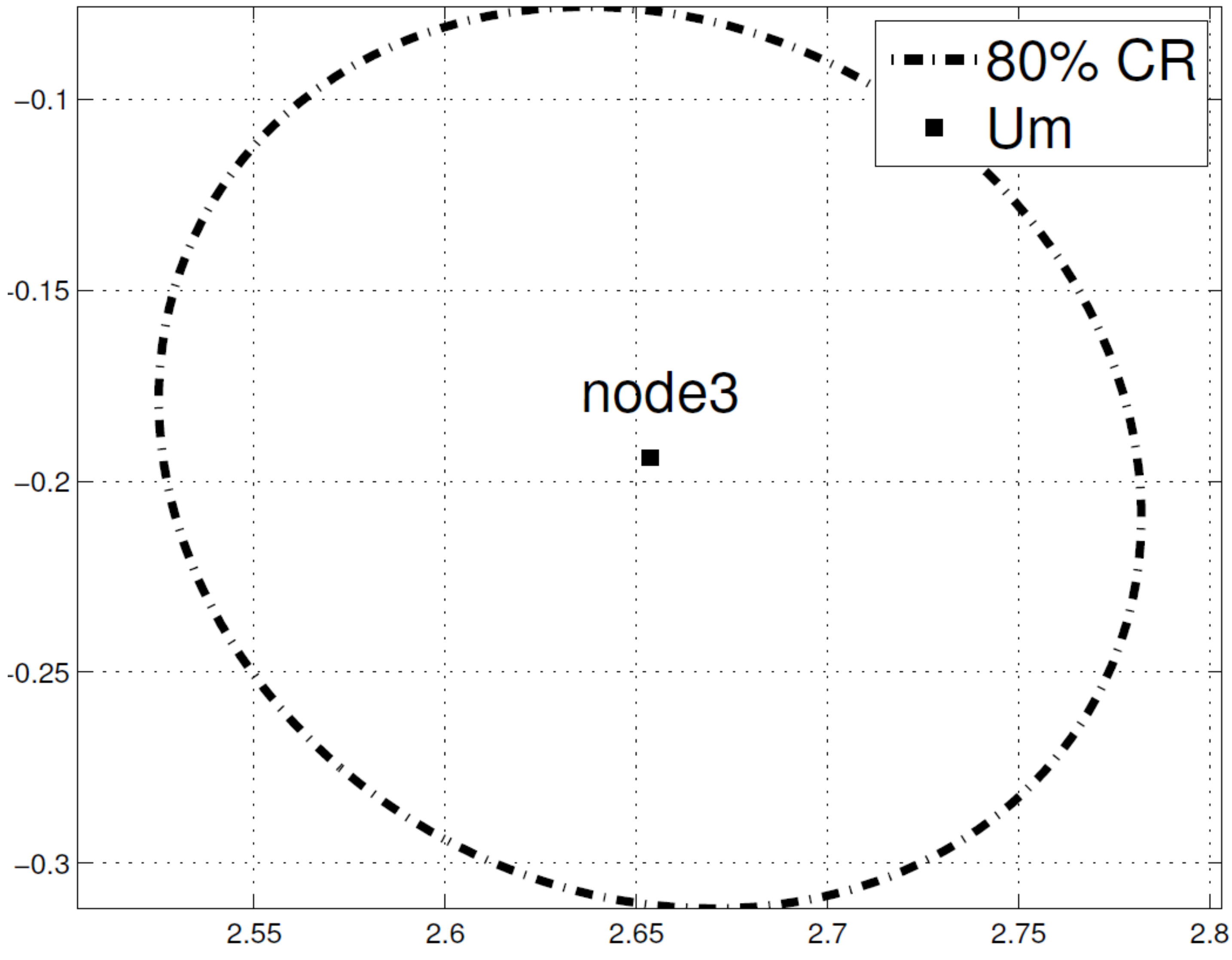
spread estimators – numerical results



spread estimators – numerical results

test case: axisymmetric case, cylindrical square domain, SNR $= 20$

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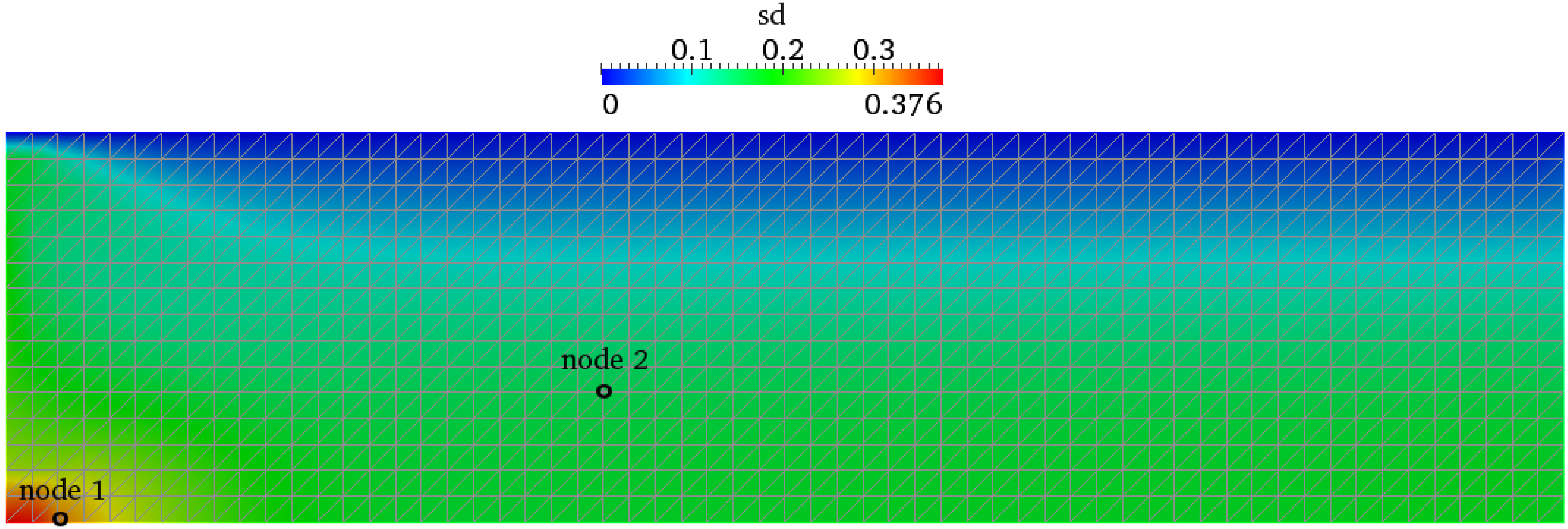

input noise: std $= 0.325$, all over the domain

output noise: max std $= 0.376$, in a restricted area

spread estimators – numerical results

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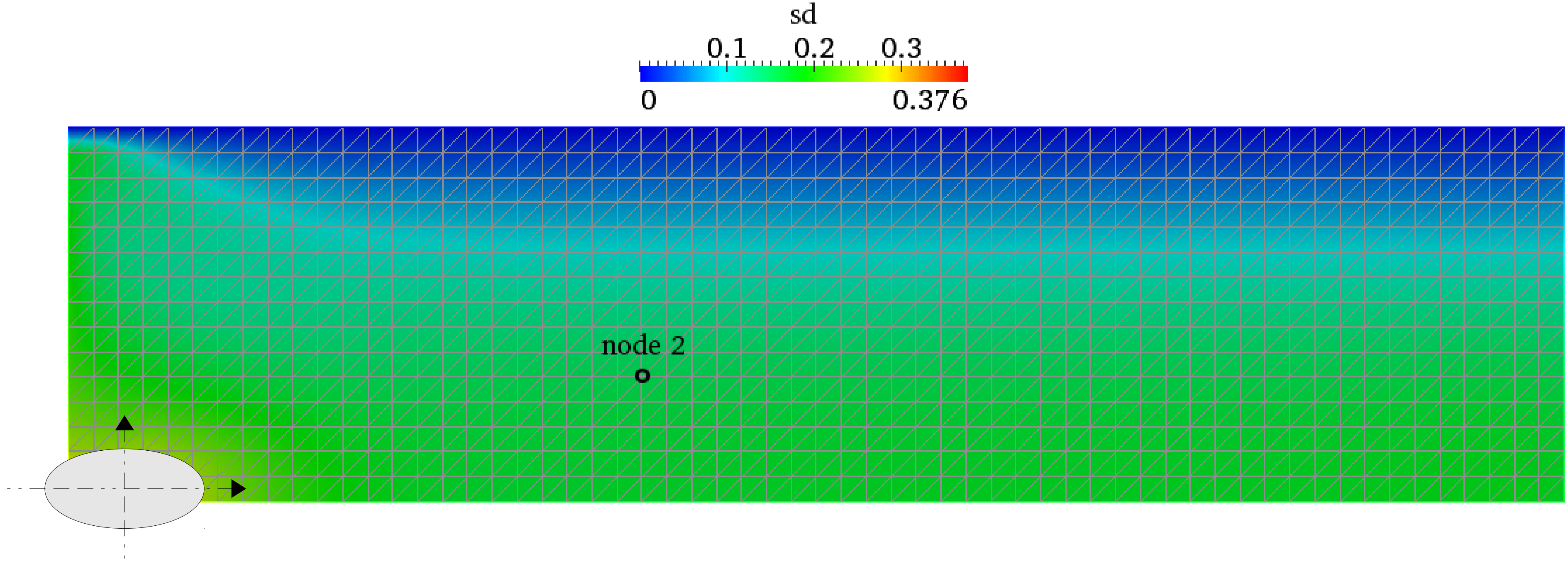

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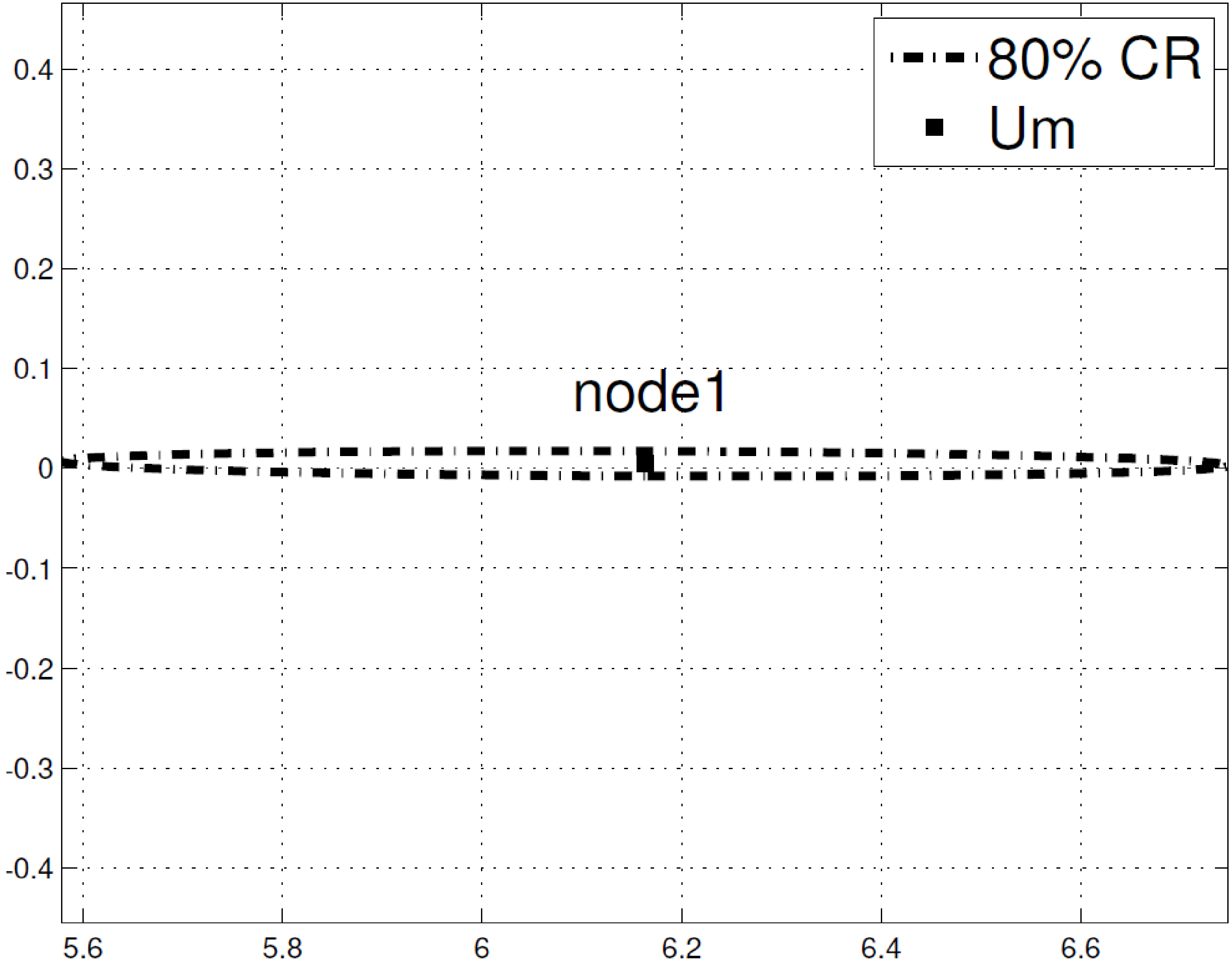

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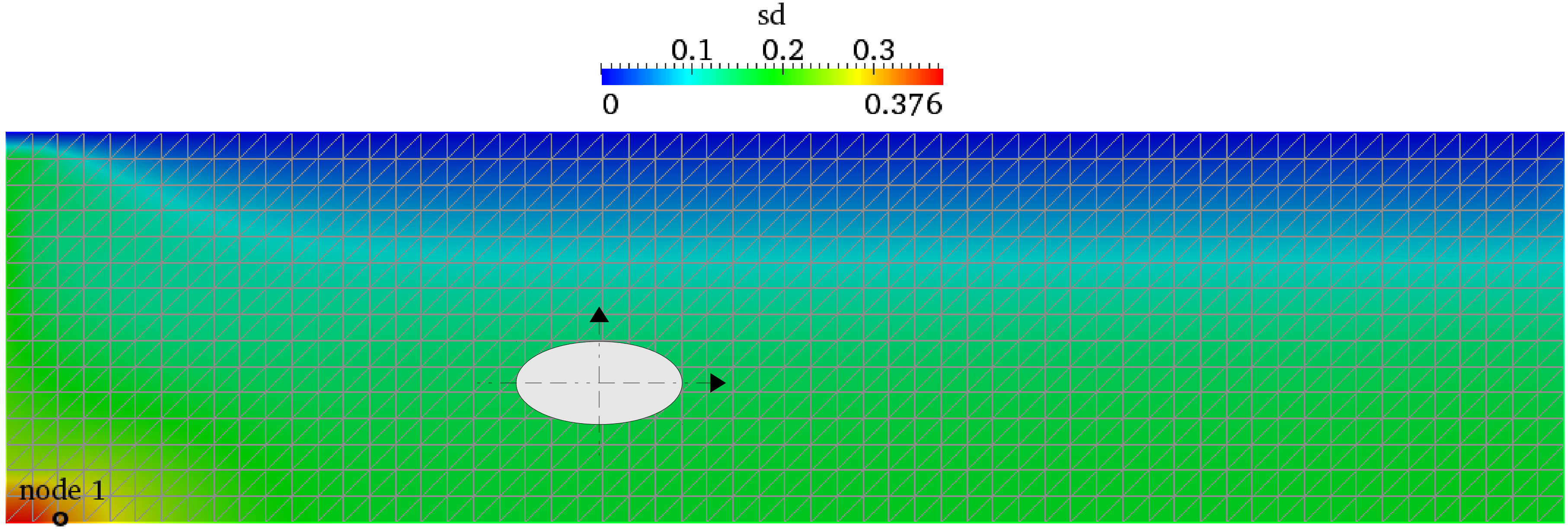

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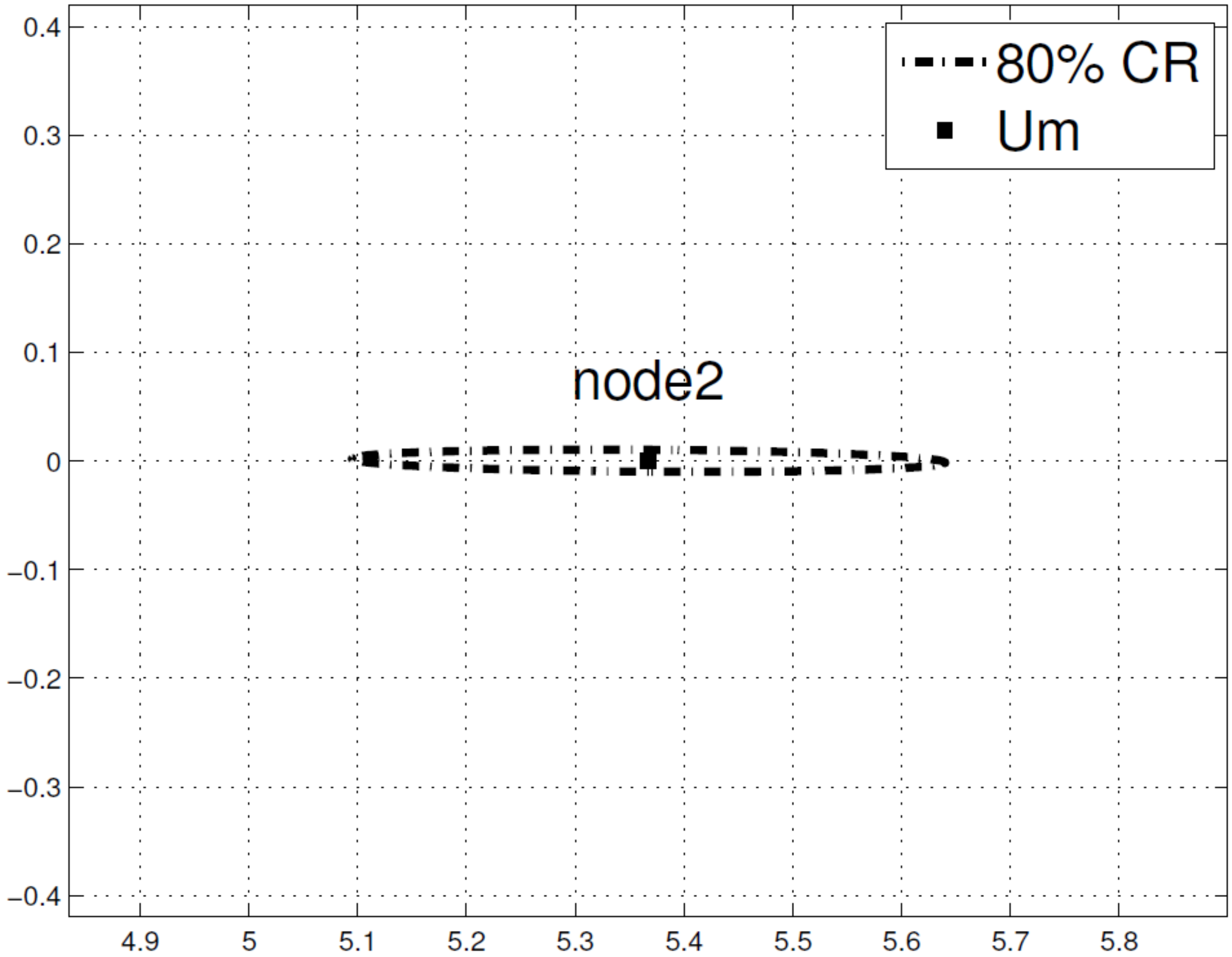
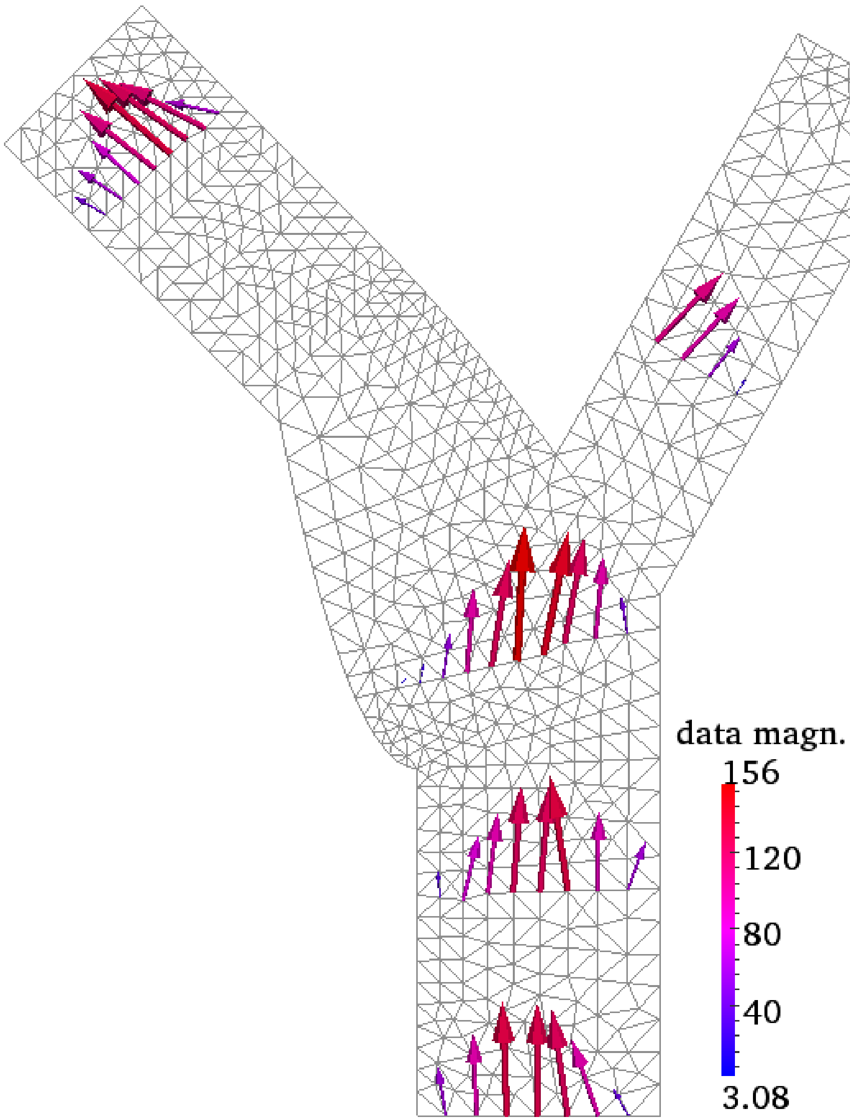
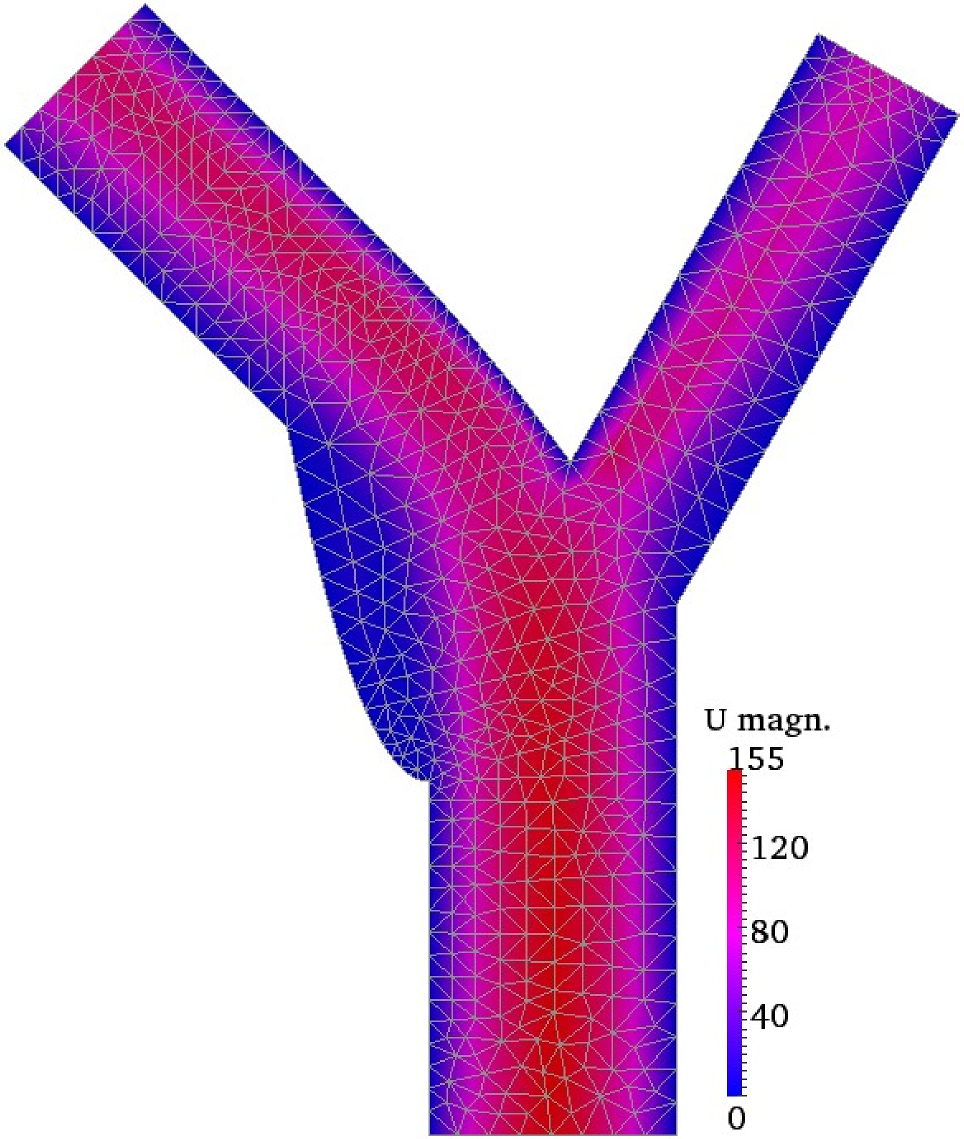


towards real geometries - carotid

SNR	n	$\overline{E}_{U,det}$	$\overline{E}_{U,ML}$	γ
20	20	0.05273	0.03617	31%

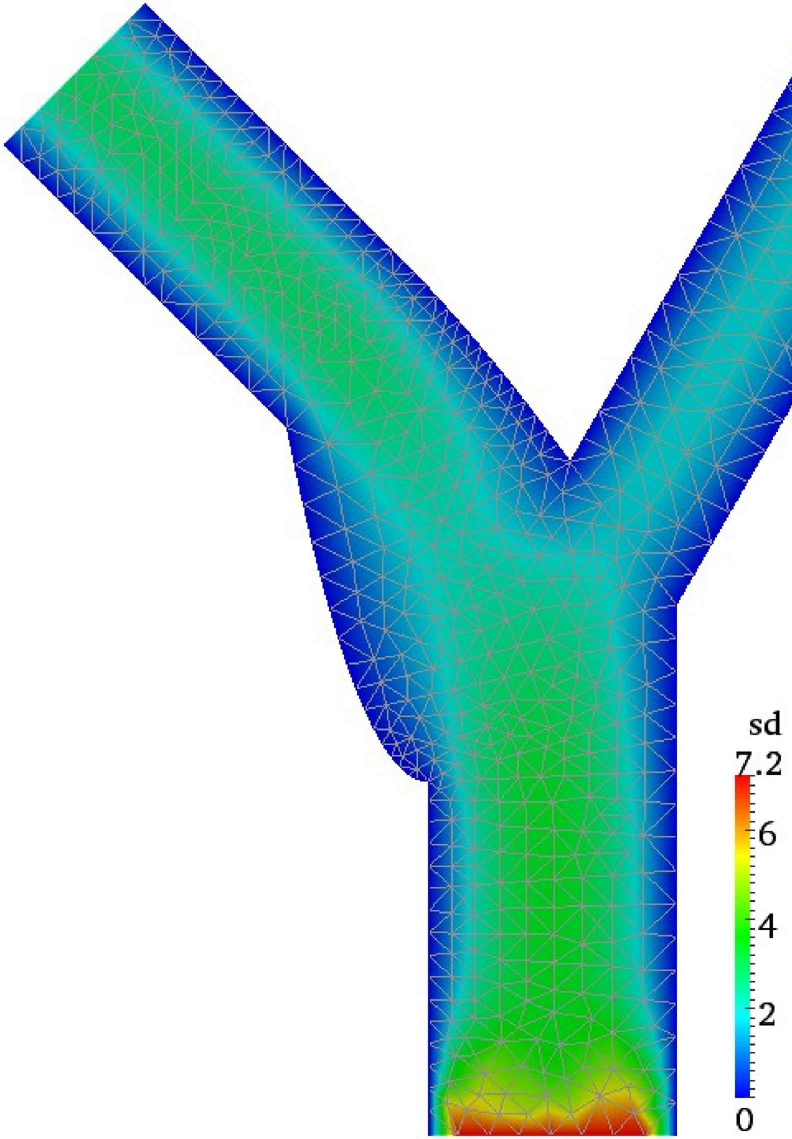




towards real geometries - carotid

map of the maximum deviation from the mean in a 60% confidence region: $\sqrt{\lambda_{max}}$

$$(\mathbf{U}_j - U_j)^{\mathrm{T}} \Sigma_{U,j}^{-1} (\mathbf{U}_j - U_j) < \chi_2^2(40\%) \cong 1,$$



spread estimators – the wall shear stress (WSS)

WSS distribution

- <u>deterministic model</u>: linear transformation $\mathbf{WSS} = \mathrm{T}_w \mathbf{U}$

  $\Rightarrow \mathbf{WSS} \sim \mathcal{N}(WSS, \Sigma_{WSS})$

  T_w maps the discretized velocity into the discretized WSS

- expectation value $WSS = \mathrm{T}_w U$, covariance $\Sigma_w = \mathrm{T}_w \Sigma_U \mathrm{T}_w^{\mathrm{T}}$

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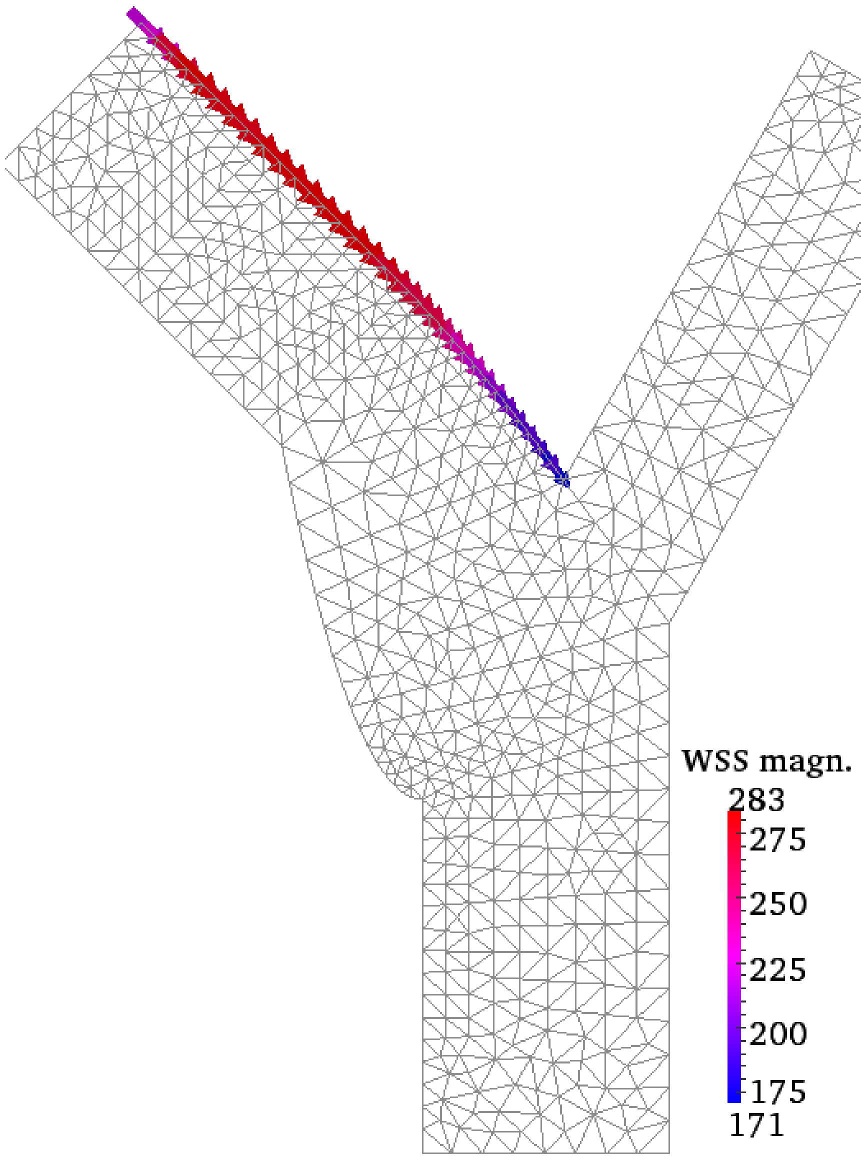
WSS confidence regions

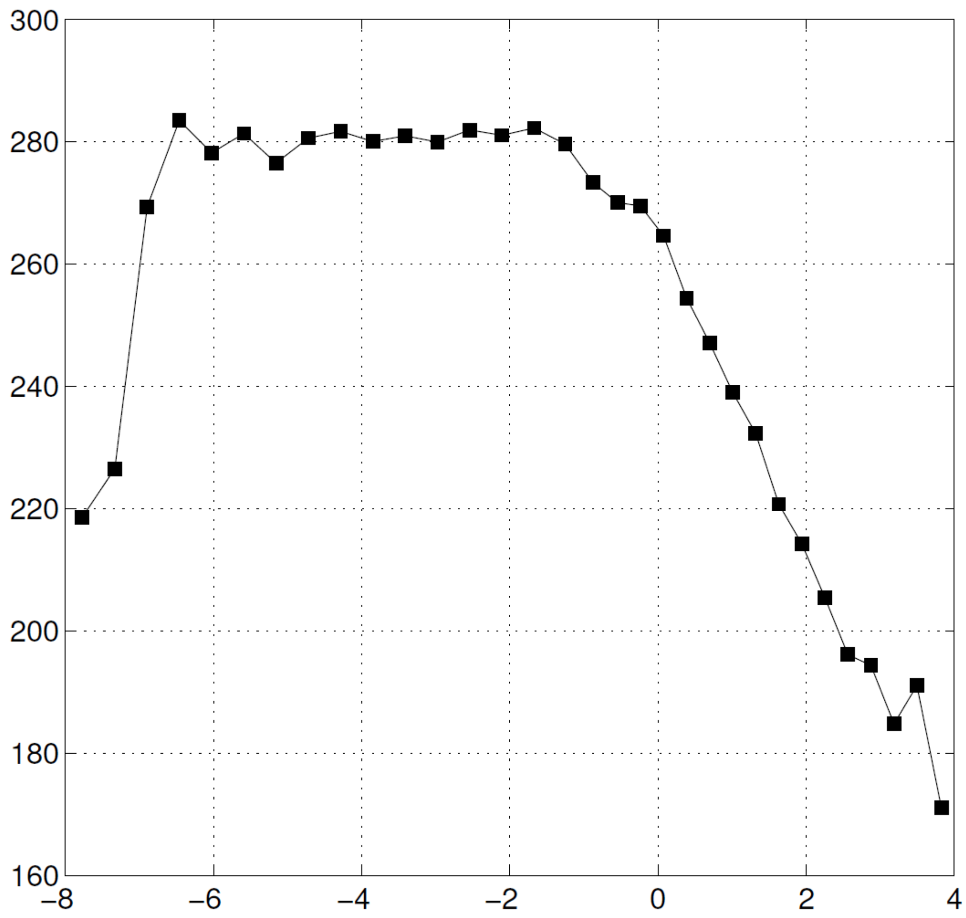
horizontal and vertical WSS in the i-th DOF, $[\mathbf{WSS}_i \;\; \mathbf{WSS}_{i+N_w/2}]^{\mathrm{T}} \in \mathbb{R}^2$:

subset of the components of $\mathbf{WSS}$ $\Rightarrow$ 2D Gaussian random vector

$\Rightarrow$ we can draw credibility regions

spread estimators – the wall shear stress (WSS)





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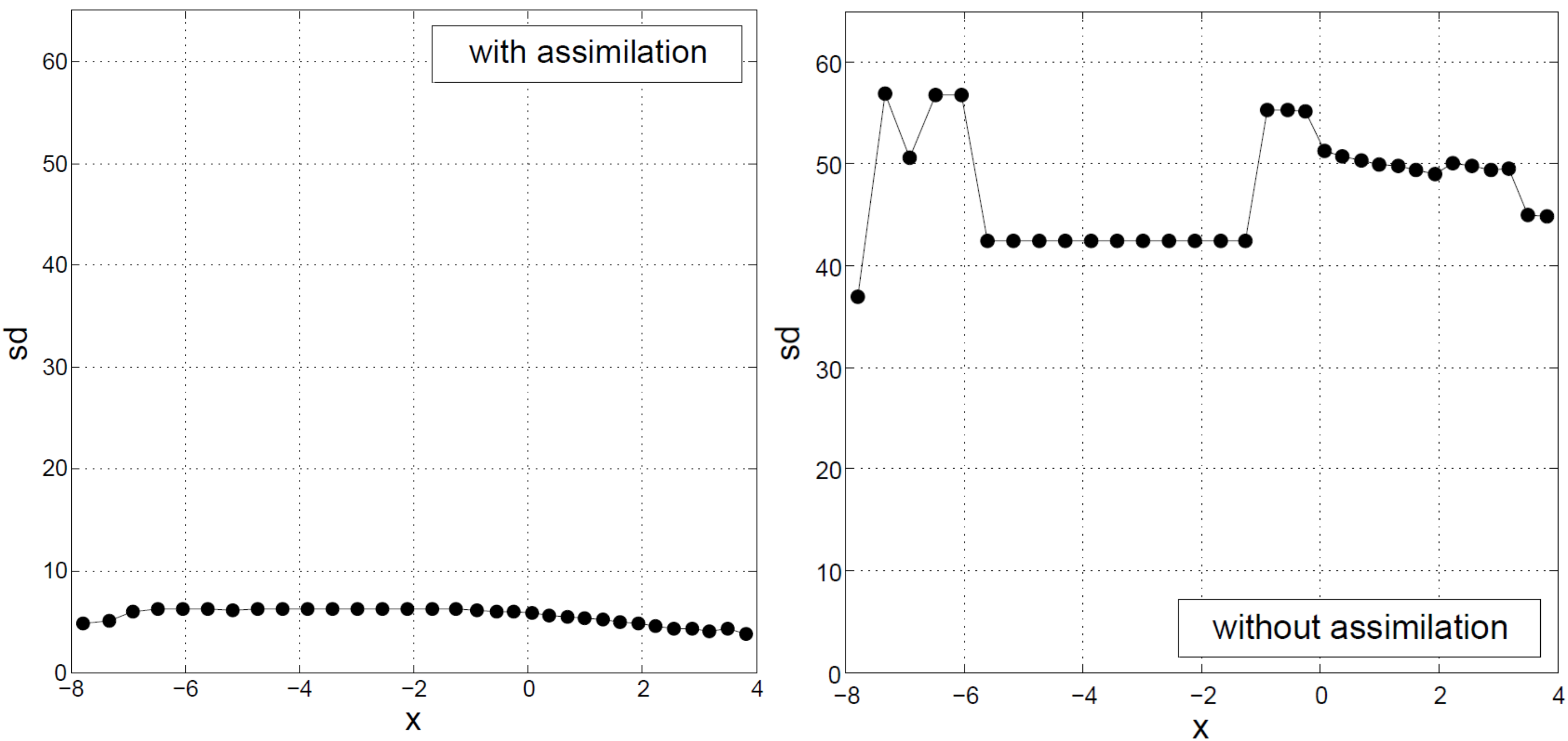
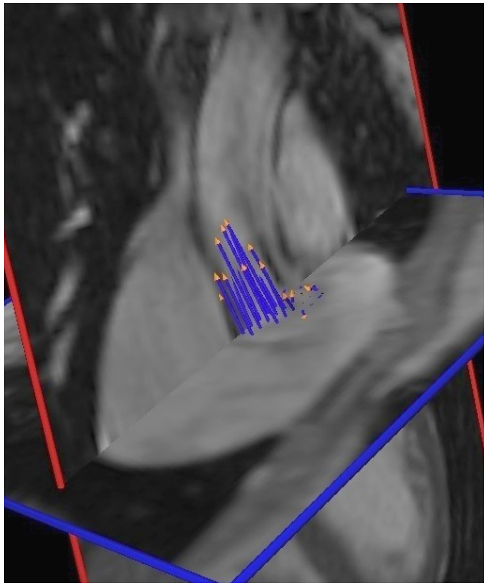


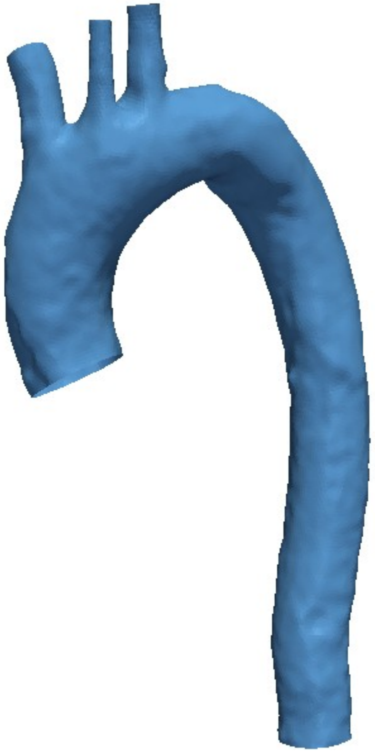
4. future work

real data

- perform simulations using measures from postprocessing of MRIs



- use 3D (real geometries)



improve computational performance

- implement more efficient preconditioning techniques
- use different optimization techniques for nonlinear problems (Newton-like methods)
- combine the formulation with model reduction methods
- move to parallel implementation

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- M. Benzi, Emory University, Atlanta, GA
- M. Gunzburger, Florida State University, Tallahassee, FL
- M. Perego, Florida State University, Tallahassee, FL

thank you for your attention

questions?

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