Discrete state, continuous time mean-field games Continuous state models N + 1 player games Convergence as  $N \to \infty$ More about continuous state models References

# An introduction to mean-field games: the finite state space case

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- Discrete state, continuous time mean-field games
- 2 Continuous state models
- 3 N + 1 player games
- 4 Convergence as  $N \to \infty$
- More about continuous state models





 $\begin{array}{c} \mbox{Continuous state models}\\ N+1 \mbox{player games}\\ \mbox{Convergence as } N\to\infty\\ \mbox{More about continuous state models}\\ \mbox{References}\\ \end{array}$ 

V+1 player symmetric games vlean field dynamics Single player point of view vlean field equations Jniqueness of solutions Frend to equilibrium A variational principle Jonnection with conservation laws

### Outline

- Discrete state, continuous time mean-field games
- 2 Continuous state models
- 3 N + 1 player games
- 4 Convergence as  $N o \infty$
- 5 More about continuous state models
- 6 References



Discrete state, continuous time mean-field games Continuous state models N+1 player games Convergence as  $N \to \infty$ 

More about continuous state models References N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation law:

### Problem set-up

### • N + 1 indistinguishable players;

- players can be in a finite number of states i = 1, ..., d;
- at any time each player knows only its state i(t) and the number of players n<sub>j</sub>(t) in state j;
- each player can only control its switching rate *α* from one state to another;
- players follow (independent) controlled Markov chains with transition rate  $\beta_{jk}$ .



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Discrete state, continuous time mean-field games Continuous state models N + 1 player games Convergence as  $N \to \infty$ More about continuous state models References Convergence as  $N \to \infty$ 

N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation laws

### **Optimization criterion**

- Each player chooses the switching rate in order to minimize an expected payoff;
- This payoff has a running cost c(i, <sup>n</sup>/<sub>N</sub>, α), where α is the switching rate
- and a terminal cost  $\psi^i(\frac{n}{N})$ ;

more precisely

$$\cot E = E \int_{t}^{T} c(\mathbf{i}(s), \frac{\mathbf{n}(s)}{N}, \alpha(s)) ds + \psi^{\mathbf{i}(T)} \left(\frac{\mathbf{n}(T)}{N}\right)$$



 Discrete state, continuous time mean-field games
 Mean field dynamics

 Continuous state models
 Single player point of view

 N+1 player games
 Mean field equations

 Convergence as N → ∞
 Uniqueness of solutions

 More about continuous state models
 Trend to equilibrium

 References
 A variational principle

 Connection with conservation la

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Discrete state, continuous time mean-field games Continuous state models N+1 player games Convergence as  $N \rightarrow \infty$ More about continuous state models References References

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Discrete state, continuous time mean-field games Continuous state models N + 1 player games Convergence as  $N \rightarrow \infty$ More about continuous state models References References

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Discrete state, continuous time mean-field games Continuous state models N + 1 player games Convergence as  $N \rightarrow \infty$ More about continuous state models References References

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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation law:

### **Technical hypothesis**

c(i, θ, α) is uniformly convex and superlinear in α
 c(i, θ, α) and ψ<sup>i</sup>(θ) are smooth in θ.



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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation law:

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Discrete state, continuous time mean-field games Continuous state models N + 1 player games Convergence as  $N \to \infty$ More about continuous state models References References Connection with conservation law

#### • Consider the case where N >> 1;

- We suppose the mean-field hypothesis holds, i.e. the fraction of players in each state *j* is given by a deterministic function θ<sup>j</sup>(t);
- if all players use the same Markovian control  $\beta = \beta_{jj}(t)$ , the evolution of  $\theta$  is determined by

$$\frac{d\theta^i}{dt} = \sum_j \theta^j \beta_{jj}.$$



N+1 player symmetric gamesDiscrete state, continuous time mean-field gamesMean field dynamicsContinuous state modelsSingle player point of viewN + 1 player gamesMean field equationsConvergence as  $N \to \infty$ Uniqueness of solutionsMore about continuous state modelsTrend to equilibriumReferencesA variational principleConnection with conservation laws

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N+1 player symmetric gamesDiscrete state, continuous time mean-field gamesMean field dynamicsContinuous state modelsSingle player point of viewN + 1 player gamesMean field equationsConvergence as  $N \to \infty$ Uniqueness of solutionsMore about continuous state modelsTrend to equilibriumReferencesA variational principleConnection with conservation laws

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N+ 1 player symmetric gamesDiscrete state, continuous time mean-field gamesMean field dynamicsContinuous state modelsSingle player point of viewN + 1 player gamesMean field equationsConvergence as  $N \to \infty$ Uniqueness of solutionsMore about continuous state modelsTrend to equilibriumReferencesA variational principle<br/>Connection with conservation laws

#### If $\theta$ is given, the objective of each player is to minimize

$$E\left[\int_t^T c(\mathbf{i}(s), \theta(s), \alpha(\mathbf{i}(s), s)) ds + \psi^{\mathbf{i}(T)}(\theta(T))\right],$$

where  $\alpha$  is the switching rate.



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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation laws

# Notation

• Let 
$$\Delta_i : \mathbb{R}^d \to \mathbb{R}^d$$
 be

$$\Delta_i z = \left(z^1 - z^i, ..., z^d - z^i\right).$$

 The infinitesimal generator for finite state continuous time Markov chain, with transition rate ν<sub>ij</sub>, is

$$\mathcal{A}_{i}^{
u}(arphi) = \sum_{j} 
u_{ij}(arphi^{j} - arphi^{j}) = 
u_{i\cdot} \cdot \Delta_{i} arphi \,.$$

• We define the generalized Legendre transform of *c* is

$$h(z,\theta,i) = \min_{\mu \in (\mathbb{R}^+_0)^d} c(i,\theta,\mu) + \sum_{\substack{i \in \mathcal{I}, i \in \mathcal{I}, i \in \mathcal{I}, i \in \mathcal{I}}} \mu_j \Delta_i z$$



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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation laws

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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation laws

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# Hamiltonian ODE

The value function is the unique solution to the following Hamilton-Jacobi ordinary differential equation:

$$\begin{cases} -\frac{du^{i}}{dt} = h(\Delta_{i}u, \theta, i), \\ u^{i}(T) = \psi^{i}(\theta(T)). \end{cases}$$

Furthermore, the optimal control is given by

$$\alpha_j^* = h_{Z_j}(\Delta_i u, \theta, i).$$

Discrete state, continuous time mean-field games Continuous state models N + 1 player games Convergence as  $N \to \infty$ More about continuous state models References N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation laws

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Discrete state, continuous time mean-field games Continuous state models N+1 player games Convergence as  $N \to \infty$ More about continuous state models References

N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation laws

### Mean-field equations

The mean-field equilibrium arises when all players use the same optimal switching rate.

This gives rise to the system

$$\begin{cases} \frac{d}{dt}\theta^{j} = \sum_{j} \theta^{j} \alpha_{i}^{*}(\Delta_{j}u, \theta, j) \\ -\frac{d}{dt}u^{j} = h(\Delta_{i}u, \theta, i). \end{cases}$$

together with the initial-terminal conditions

$$\theta(0) = \theta_0 \qquad u^i(T) = \psi^i(\theta(T))$$

Discrete state, continuous time mean-field games Continuous state models N + 1 player games Convergence as  $N \rightarrow \infty$ More about continuous state models Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation laws

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Discrete state, continuous time mean-field games<br/>Continuous state modelsMean field dynamicsN + 1 player games<br/>Convergence as  $N \to \infty$ Single player point of view<br/>Mean field equationsMore about continuous state models<br/>ReferencesTrend to equilibrium<br/>A variational principle<br/>Connection with conservation laws

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- from the ODE point of view these equations are non-standard as some of the variables have initial conditions whereas other variables have prescribed terminal data;
- existence of solution is by no means obvious;
- uniqueness (in general) does not hold.



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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation laws

### Example

Set

$$c(i, \theta, \alpha) = \sum_{j} \frac{\alpha_{j}^{2}}{2} + f^{i}(\theta)$$

Then

$$h(\Delta_i u, \theta, i) = f^i(\theta) - \frac{1}{2} \sum_j [(u^i - u^j)^+]^2,$$

and, for  $j \neq i$ ,

$$\alpha_j^* = (u^i - u^j)^+$$

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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation laws

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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation laws

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Discrete state, continuous time mean-field gamesMean field dynamicsContinuous state modelsSingle player point of viewN + 1 player gamesMean field dynamicsConvergence as  $N \to \infty$ Uniqueness of solutionsMore about continuous state modelsTrend to equilibriumReferencesA variational principle<br/>Connection with conservation laws

### Existence of mean-field equilibria

• Fix  $\theta$  and consider the map  $\mathcal{U}(\theta)$  to be the solution of

$$-\frac{d}{dt}u^i=h(\Delta_i u,\theta,i).$$

• given *u* consider the map  $\Theta(u)$  to be the solution to

$$\frac{d}{dt}\theta^{i} = \sum_{j} \theta^{j} \alpha_{j}^{*}(\Delta_{j} u, \theta, j)$$

 Existence of mean-field equilibria can be proved under very general conditions by showing the existence of a fixed point for Θ ∘ U.



	N+1 player symmetric games
Discrete state, continuous time mean-field games	Mean field dynamics
Continuous state models	Single player point of view
N + 1 player games	Mean field equations
Convergence as $N \to \infty$	Uniqueness of solutions
More about continuous state models	Trend to equilibrium
References	A variational principle
	Connection with conservation laws

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	N+1 player symmetric games
Discrete state, continuous time mean-field games	Mean field dynamics
Continuous state models	Single player point of view
N + 1 player games	Mean field equations
Convergence as $N \to \infty$	Uniqueness of solutions
More about continuous state models	Trend to equilibrium
References	A variational principle
	Connection with conservation laws

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Discrete state, continuous time mean-field games Continuous state models

N+1 player games Convergence as  $N \to \infty$ More about continuous state models References N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation laws

## Monotonicity hypothesis

We assume:

$$\sum_i ( heta^i - ilde{ heta}^i)(\psi^i( heta) - \psi^i( ilde{ heta})) \geq 0$$

and

$$heta \cdot (h(z, ilde{ heta}) - h(z, heta)) + ilde{ heta} \cdot (h( ilde{z}, heta) - h( ilde{z}, ilde{ heta})) \leq -\gamma \| heta - ilde{ heta}\|^2.$$

Furthermore define  $\|v\|_{\sharp} = \inf_{\lambda \in \mathbb{R}} \|v + \lambda \mathbf{1}\|$ . Then we suppose that uniformly on  $\|z\|_{\sharp} \leq M$  there exists  $\gamma_i > 0$  such that

$$h(z, \theta, i) - h(w, \theta, i) - \alpha^*(w, \theta, i) \cdot \Delta_i(z - w) \le -\gamma_i ||\Delta_i(z - w)|$$



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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation laws

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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation laws

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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation laws

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$$heta \cdot (h(z, ilde{ heta}) - h(z, heta)) + ilde{ heta} \cdot (h( ilde{z}, heta) - h( ilde{z}, ilde{ heta})) \leq -\gamma \| heta - ilde{ heta}\|^2.$$

Furthermore define  $\|v\|_{\sharp} = \inf_{\lambda \in \mathbb{R}} \|v + \lambda \mathbf{1}\|$ . Then we suppose that uniformly on  $\|z\|_{\sharp} \leq M$  there exists  $\gamma_i > 0$  such that

$$h(z, \theta, i) - h(w, \theta, i) - \alpha^*(w, \theta, i) \cdot \Delta_i(z - w) \le -\gamma_i \|\Delta_i(z - w)\|^2$$



	N+T player symmetric games
Discrete state, continuous time mean-field games	Mean field dynamics
Continuous state models	Single player point of view
N + 1 player games	Mean field equations
Convergence as $N  o \infty$	Uniqueness of solutions
More about continuous state models	Trend to equilibrium
References	A variational principle
	Connection with conservation laws

#### The last three hypothesis will be satisfied if h can be written as

$$h(\Delta_i z, \theta, i) = \tilde{h}(\Delta_i z, i) + f^i(\theta),$$

with  $\tilde{h}$  (locally) uniformly concave and f satisfying the monotonicity hypothesis

$$(f(\tilde{\theta}) - f(\theta)) \cdot (\theta - \tilde{\theta}) \le -\gamma |\theta - \tilde{\theta}|^2.$$

The previous property holds, for instance, if *f* is the gradient of a convex function  $f(\theta) = \nabla \Phi(\theta)$ .



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	N+T player symmetric games
Discrete state, continuous time mean-field games	Mean field dynamics
Continuous state models	Single player point of view
N + 1 player games	Mean field equations
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References	A variational principle
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Discrete state, continuous time mean-field games	Mean field dynamics
Continuous state models	Single player point of view
N + 1 player games	Mean field equations
Convergence as $N  ightarrow \infty$	Uniqueness of solutions
More about continuous state models	Trend to equilibrium
References	A variational principle
	Connection with conservation laws

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Discrete state, continuous time mean-field games

 $\begin{array}{c} \mbox{Continuous state models}\\ N+1 \mbox{player games}\\ \mbox{Convergence as } N \rightarrow \infty\\ \mbox{More about continuous state models}\\ \mbox{References}\\ \mbox{References}\\ \end{array}$ 

N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation law

#### Theorem

Under the monotonicity hypothesis, the mean-field equations have a unique solution  $(\theta, u)$ .



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Discrete state, continuous time mean-field games Continuous state models N + 1 player games

More about continuous state models References

N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation law

#### Lemma

Fix T > 0 and suppose that  $(\theta, u)$  and  $(\tilde{\theta}, \tilde{u})$  are solutions with  $\|u\|_{\sharp}, \|\tilde{u}\|_{\sharp} \leq C$  on [-T, T]Then there exists a constant C independent of T such that, for all  $0 < \tau < T$ , we have

$$egin{aligned} &\int_{- au}^{ au} \|( heta- ilde{ heta})(s)\|^2+\|(u- ilde{u})(s)\|_{\sharp}^2 ds \ &\leq C\int_{- au}^{ au}rac{d}{dt}iggl[( heta- ilde{ heta})\cdot(u- ilde{u})iggr] \ &\leq Ciggl(\|( heta- ilde{ heta})( au)\|^2+\|(u- ilde{u})( au)\|_{\sharp}^2iggr) \ &\leq Ciggl(\|( heta- ilde{ heta})( au)\|^2+\|(u- ilde{ heta})( au)\|_{\sharp}^2iggr) \end{aligned}$$



N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation law

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$$\int_{-\tau}^{\tau} \|(\theta - \tilde{\theta})(s)\|^{2} + \|(u - \tilde{u})(s)\|_{\sharp}^{2} ds$$

$$\leq C \int_{-\tau}^{\tau} \frac{d}{dt} \Big[ (\theta - \tilde{\theta}) \cdot (u - \tilde{u}) \Big]$$

$$\leq C \Big( \|(\theta - \tilde{\theta})(\tau)\|^{2} + \|(u - \tilde{u})(\tau)\|_{\sharp}^{2} \Big)$$

$$+ C \Big( \|(\theta - \tilde{\theta})(-\tau)\|^{2} + \|(u - \tilde{u})(-\tau)\|_{\pi}^{2} \Big)$$
Digo Gomes
An introduction to mean-field games



Discrete state, continuous time mean-field gamesContinuous state modelsMean field dynamicsN + 1 player gamesSingle player point of viewN + 1 player gamesMean field equationsConvergence as  $N \to \infty$ Uniqueness of solutionsMore about continuous state modelsTrend to equilibriumReferencesA variational principleConnection with conservation laws

 The proof of the lemma follows the Lions-Lasry monotonicity method. The inequality in the lemma is obtained by applying the monotonicity hypothesis to

$$\frac{d}{dt}\left[(\theta-\tilde{\theta})\cdot(u-\tilde{u})\right].$$

• Uniqueness follows trivially from the lemma.



Discrete state, continuous time mean-field games Continuous state models N + 1 player games Convergence as  $N \to \infty$ More about continuous state models References References N + 1 player games N + 1 player games N + 1 player games N + 2 player point of view Mean field dynamics Single player point of view Mean field dynamics Single player point of view Mean field dynamics N + 2 player games N + 2 player games N + 2 player point of view Mean field dynamics N + 2 player point of view Mean field dynamics N + 2 player point of view N + 1 player games N + 2 player point of view N + 1 player games N + 2 player point of view N + 1 player games N + 2 player point of view N + 1 player games N + 2 player point of view N + 1 player games N + 2 player point of view N + 1 player games N + 2 player point of view N + 1 player games N + 2 player point of view N + 1 player games N + 2 player point of view N + 1 player games N + 2 player point of view N + 1 player games N + 2 player point of view N + 1 player games N + 2 player point of view N + 1 player games N + 2 player point of view N + 1 player games N + 2 player point of view N + 1 player games N + 1 player games N + 2 player point of view N + 1 player games N + 2 player point of view N + 1 player games N + 1 player games N + 2 player point of view N + 1 player games N + 2 player point of view N + 1 player games N + 2 player

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Discrete state, continuous time mean-field gamesN+1 player symmetring gamesContinuous state modelsMean field dynamicsN + 1 player gamesSingle player point of viewN + 1 player gamesMean field equationsConvergence as  $N \to \infty$ Uniqueness of solutionsMore about continuous state modelsTrend to equilibriumReferencesA variational principleConnection with conservation laws

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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions **Trend to equilibrium** A variational principle Connection with conservation laws

## Contractive mean-field games

Let 
$$\langle u \rangle = \frac{1}{d} \sum_{i} u^{j}$$
.

We say that h is contractive if there exists M > 0 such that, if  $||u||_{\sharp} > M$ , then

$$(\Delta_i u)^j \leq 0 \ \forall \ j \text{ implies } h(\Delta_i u, \theta, i) - \langle h(u, \theta, \cdot) \rangle < 0 \,,$$

and

$$(\Delta_i u)^j \ge 0 \ \forall \ j \text{ implies } h(\Delta_i u, \theta, i) - \langle h(u, \theta, \cdot) \rangle > 0 \ .$$



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N+1 player symmetric gamesDiscrete state, continuous time mean-field gamesMean field dynamicsContinuous state modelsSingle player point of viewN + 1 player gamesMean field equationsConvergence as  $N \to \infty$ Uniqueness of solutionsMore about continuous state modelsTrend to equilibriumReferencesA variational principleConnection with conservation laws

#### These conditions are natural if one observes that

$$(\Delta_{i_1}u)^j \leq 0 \,\, orall j \,\,$$
 and  $\,\, (\Delta_{i_2}u)^j \geq 0 \,\, orall j$ 

implies

$$2\|u\|_{\sharp}=u^{i_1}-u^{i_2}.$$

So, if *u* is a smooth solution and  $||u(t)||_{\sharp}$  is differentiable with  $||u(t)||_{\sharp} > M$  then

$$\frac{d}{dt}\|u\|_{\sharp}>0\,.$$

This implies the flow is backwards contractive with respect to the  $\|\cdot\|_{\sharp}$  norm of the *u* component.



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N+1 player symmetric gamesDiscrete state, continuous time mean-field gamesContinuous state modelsMean field dynamicsN+1 player gamesMean field equationsConvergence as  $N \rightarrow \infty$ Uniqueness of solutionsMore about continuous state modelsTrend to equilibriumReferencesA variational principle<br/>Connection with conservation laws

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	N+1 player symmetric games
Discrete state, continuous time mean-field games	Mean field dynamics
Continuous state models	Single player point of view
N + 1 player games	Mean field equations
Convergence as $N  o \infty$	Uniqueness of solutions
More about continuous state models	Trend to equilibrium
References	A variational principle
	Connection with conservation laws

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	N+1 player symmetric games
Discrete state, continuous time mean-field games	Mean field dynamics
Continuous state models	Single player point of view
N + 1 player games	Mean field equations
Convergence as $N  o \infty$	Uniqueness of solutions
More about continuous state models	Trend to equilibrium
References	A variational principle
	Connection with conservation laws

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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions **Trend to equilibrium** A variational principle Connection with conservation laws

## Stationary solutions

### A triplet $(\bar{\theta}, \bar{u}, \kappa)$ is called a stationary solution if

$$\begin{cases} \sum_{j} \bar{\theta}^{j} \alpha_{i}^{*}(\Delta_{j} \bar{u}, \bar{\theta}, j) = 0, \\ h(\Delta_{i} \bar{u}, \bar{\theta}, i) = \kappa. \end{cases}$$

If  $(\bar{\theta}, \bar{u}, \kappa)$  is a stationary solution for the MFG equations, then  $(\bar{\theta}, \bar{u} - \kappa t)$  solves the time dependent problem with appropriate initial-terminal conditions.



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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions **Trend to equilibrium** A variational principle Connection with conservation laws

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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions **Trend to equilibrium** A variational principle Connection with conservation laws

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Discrete state, continuous time mean-field games Continuous state models N + 1 player games Convergence as  $N \to \infty$ More about continuous state models References References N+ 1 player games Single player point of view Mean field dynamics Single player solutions Trend to equilibrium A variational principle Connection with conservation laws

## Existence of stationary solutions

#### Theorem

### Suppose h is contractive. Then

- (a) For M large enough, the set  $\{(u, \theta), ||u||_{\sharp} < M\}$  is invariant backwards in time by the flow of the mean-field equations.
- (b) There exist a stationary solution.



Discrete state, continuous time mean-field games<br/>Continuous state models<br/>N + 1 player games<br/>Convergence as  $N \to \infty$ <br/>More about continuous state models<br/>ReferencesMean field dynamics<br/>Single player point of view<br/>Mean field equations<br/>Uniqueness of solutions<br/>Trend to equilibrium<br/>A variational principle<br/>Connection with conservation laws

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Discrete state, continuous time mean-field games<br/>Continuous state modelsMean field dynamics<br/>Single player point of view<br/>Mean field equations<br/>Uniqueness of solutions<br/>Trend to equilibrium<br/>A variational principle<br/>Connection with conservation laws

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Discrete state, continuous time mean-field games

 $\begin{array}{c} \mbox{Continuous state models}\\ N+1 \mbox{player games}\\ \mbox{Convergence as } N\to\infty\\ \mbox{More about continuous state models}\\ \mbox{References}\\ \end{array}$ 

N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions **Trend to equilibrium** A variational principle Connection with conservation law:

### Theorem

### Suppose that monotonicity and contractivity hold.

- (a) Suppose  $||u(T)||_{\sharp} \leq M$ , where u is a solution, and M is large enough. Then  $||u(t)||_{\sharp} \leq M \forall t \in [0, T]$ .
- (b) The stationary solution (θ
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(c) Given T > 0, a vector  $\theta_0$ , and a terminal condition  $\psi$ , let  $(\theta^T, u^T)$  be the solution with initial-terminal conditions  $\theta^T(-T) = \theta_0$  and  $u^{T,i}(T) = \psi^i(\theta^T(T))$ . As  $T \to \infty$ 

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Discrete state, continuous time mean-field games Continuous state models

N+1 player games Convergence as  $N \to \infty$ More about continuous state models References N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions **Trend to equilibrium** A variational principle Connection with conservation law

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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions **Trend to equilibrium** A variational principle Connection with conservation laws

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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions **Trend to equilibrium** A variational principle Connection with conservation laws

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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions **Trend to equilibrium** A variational principle Connection with conservation laws

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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions **Trend to equilibrium** A variational principle Connection with conservation laws

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Discrete state, continuous time mean-field games

 $\begin{array}{c} \mbox{Continuous state models} \\ N+1 \mbox{player games} \\ \mbox{Convergence as } N \rightarrow \infty \\ \mbox{More about continuous state models} \\ \mbox{References} \end{array}$ 

N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions **Trend to equilibrium** A variational principle Connection with conservation law

### We define

$$f_{\mathcal{T}}(\boldsymbol{s}) := \|( heta^{\mathcal{T}} - ilde{ heta}^{\mathcal{T}})(\boldsymbol{s})\|^2 + \|(u^{\mathcal{T}} - ilde{u}^{\mathcal{T}})(\boldsymbol{s})\|_{\sharp}^2,$$

and, for  $0 < \tau < T$ ,

 $F_T(\tau) := \int_{-\tau}^{\tau} f_T(s) ds.$ 

Then

$$F_T(\tau) \leq \frac{1}{\tilde{\gamma}}(f_T(\tau) + f_T(-\tau)).$$

Note that  $\dot{F}_T(\tau) = f_T(\tau) + f_T(-\tau)$ 

$$F_T(\tau) \leq \frac{1}{\tilde{\gamma}}\dot{F}_T(\tau).$$



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Discrete state, continuous time mean-field games Continuous state models N + 1 player games Convergence as  $N \rightarrow \infty$ 

More about continuous state models References N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions **Trend to equilibrium** A variational principle Connection with conservation law

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Discrete state, continuous time mean-field games Continuous state models N+1 player games Convergence as  $N \rightarrow \infty$ More about continuous state models References Nean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation

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$$f_T(\boldsymbol{s}) := \|(\theta^T - \tilde{\theta}^T)(\boldsymbol{s})\|^2 + \|(\boldsymbol{u}^T - \tilde{\boldsymbol{u}}^T)(\boldsymbol{s})\|_{\sharp}^2,$$

and, for  $0 < \tau < T$ ,

$$F_T(\tau) := \int_{-\tau}^{\tau} f_T(s) ds.$$

Then

$$F_{T}(\tau) \leq \frac{1}{\tilde{\gamma}}(f_{T}(\tau) + f_{T}(-\tau))$$
  
Note that  $\dot{F}_{T}(\tau) = f_{T}(\tau) + f_{T}(-\tau)$   
 $F_{T}(\tau) \leq \frac{1}{\tilde{\gamma}}\dot{F}_{T}(\tau).$ 



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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions **Trend to equilibrium** A variational principle Connection with conservation laws

#### From

$$F_T( au) \leq rac{1}{ ilde{\gamma}} \dot{F}_T( au),$$

it follows  $\frac{d}{dt} \ln F_T(\tau) \geq \tilde{\gamma}$ . Therefore

$$\ln F_T(\tau) - \ln F_T(1) \ge (\tau - 1)\tilde{\gamma},$$

for all  $0 < \tau < T$ . From this we get

$$\int_{-1}^{1} f_{T}(s) ds = F_{T}(1) \leq \frac{F_{T}(T)}{e^{(T-1)\tilde{\gamma}}} \to 0 \qquad \text{when } T \to \infty,$$

because F has sub-exponential growth in T.



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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions **Trend to equilibrium** A variational principle Connection with conservation law:

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N+1 player symmetry gamesDiscrete state, continuous time mean-field gamesMean field dynamicsContinuous state modelsSingle player point of viewN + 1 player gamesMean field equationsConvergence as  $N \to \infty$ Uniqueness of solutionsMore about continuous state modelsTrend to equilibriumReferencesA variational principle<br/>Connection with conservation laws

# Potential mean-field games

#### We say the mean-field game is potential if h has the form

$$h(z,\theta,i) = \tilde{h}(z,i) + f^{i}(\theta)$$
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where  $f(\cdot, \theta) = \nabla_{\theta} F(\theta)$  for some convex function  $F(\theta)$ .



N4 - I player symmetric gamesDiscrete state, continuous time mean-field gamesContinuous state modelsN + 1 player gamesSingle player point of viewN + 1 player gamesMean field equationsConvergence as  $N \to \infty$ Uniqueness of solutionsMore about continuous state modelsTrend to equilibriumReferencesA variational principleConnection with conservation laws

## Let $H : \mathbb{R}^{2d} \to \mathbb{R}$ be given by

$$egin{aligned} \mathcal{H}(u, heta) &= \sum_i heta^i ilde{h}(\Delta_i u, i) + \mathcal{F}( heta) \ &= heta \cdot ilde{h}(\Delta_i u, \cdot) + \mathcal{F}( heta) \end{aligned}$$

A direct computation shows the mean-field equations can be written as

$$\begin{pmatrix} \frac{\partial H}{\partial u^{j}} = \dot{\theta}^{j}, \\ \frac{\partial H}{\partial \theta^{j}} = -\dot{u}^{j}.
\end{cases}$$

This means the flow associated to the mean-field game is Hamiltonian.



N+1 player symmetric gamesDiscrete state, continuous time mean-field gamesMean field dynamicsContinuous state modelsSingle player point of viewN+1 player gamesMean field equationsN+1 player gamesUniqueness of solutionsConvergence as  $N \rightarrow \infty$ Trend to equilibriumMore about continuous state modelsA variational principleReferencesConnection with conservation laws

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N+1 player symmetric gamesDiscrete state, continuous time mean-field gamesMean field dynamicsContinuous state modelsSingle player point of viewN+1 player gamesMean field equationsN+1 player gamesUniqueness of solutionsConvergence as  $N \rightarrow \infty$ Trend to equilibriumMore about continuous state modelsA variational principleConvergence as not continuous state modelsConvergence as not continuous state models

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N+1 player symmetric gamesDiscrete state, continuous time mean-field gamesMean field dynamicsContinuous state modelsSingle player point of viewN+1 player gamesMean field equationsConvergence as  $N \to \infty$ Uniqueness of solutionsMore about continuous state modelsTrend to equilibriumReferencesA variational principle<br/>Connection with conservation laws

# Given a convex function G(p) we define the Legendre transform as

$$G^*(q) = \sup_p -q \cdot p - G(p).$$

If *G* is strictly convex and the previous supremum is achieved, then  $q = -\nabla G(p)$ , or equivalently  $p = -\nabla G^*(q)$ .



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N+ I player symmetric gamesDiscrete state, continuous time mean-field gamesMean field dynamicsContinuous state modelsSingle player point of viewN + 1 player gamesMean field equationsConvergence as  $N \to \infty$ Uniqueness of solutionsMore about continuous state modelsTrend to equilibriumReferencesA variational principle<br/>Connection with conservation laws

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	N+1 player symmetric games
Discrete state, continuous time mean-field games	Mean field dynamics
Continuous state models	Single player point of view
N + 1 player games	Mean field equations
Convergence as $N  o \infty$	Uniqueness of solutions
More about continuous state models	Trend to equilibrium
References	A variational principle
	Connection with conservation laws

If the function *F* is strictly convex in  $\theta$  then the Hamiltonian *H* is strictly convex in  $\theta$ . This allow us to consider the Legendre transform

$$L(u, \dot{u}) = \sup_{\theta} -\dot{u} \cdot \theta - H(u, \theta)$$
  
=  $\sup_{\theta} -(\dot{u} + \tilde{h}) \cdot \theta - F(\theta) = F^*(\dot{u} + \tilde{h}(\Delta, u, \cdot)).$ 

From this we conclude that any solution is a critical point of the functional

$$\int_0^1 F^*(\dot{u}+\tilde{h}(\Delta,u,\cdot))ds.$$



	N+1 player symmetric games
Discrete state, continuous time mean-field games	Mean field dynamics
Continuous state models	Single player point of view
N + 1 player games	Mean field equations
Convergence as $N  o \infty$	Uniqueness of solutions
More about continuous state models	Trend to equilibrium
References	A variational principle
	Connection with conservation laws

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	N+1 player symmetric games
Discrete state, continuous time mean-field games	Mean field dynamics
Continuous state models	Single player point of view
N + 1 player games	Mean field equations
Convergence as $N \to \infty$	Uniqueness of solutions
More about continuous state models	Trend to equilibrium
References	A variational principle
	Connection with conservation laws

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	N+1 player symmetric games
Discrete state, continuous time mean-field games	Mean field dynamics
Continuous state models	Single player point of view
N + 1 player games	Mean field equations
Convergence as $N  o \infty$	Uniqueness of solutions
More about continuous state models	Trend to equilibrium
References	A variational principle
	Connection with conservation laws

This variational problem has to be complemented by suitable boundary conditions. The initial-terminal value problem corresponds to

$$\theta_0 = -\nabla F^*(\dot{u}(0) + \tilde{h}(\Delta . u(0), \cdot)),$$
  
$$u(T) = \psi(\cdot, -\nabla F^*(\dot{u}(T) + \tilde{h}(\Delta . u(T), \cdot))).$$

Another important boundary condition arises in planning problems. In this case the objective is to find a terminal cost u(T) which steers a initial probability distribution  $\theta_0$  into a terminal probability distribution  $\theta^T$ . Hence we have the following

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	N+1 player symmetric games
Discrete state, continuous time mean-field games	Mean field dynamics
Continuous state models	Single player point of view
N + 1 player games	Mean field equations
Convergence as $N  o \infty$	Uniqueness of solutions
More about continuous state models	Trend to equilibrium
References	A variational principle
	Connection with conservation laws

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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation laws

## The master equation

Let

$$g(u, \theta, i) = \sum_{j} \theta^{j} \alpha_{i}^{*}(\Delta_{j} u, \theta, j).$$

Consider the PDE, called the master equation,

$$-\frac{\partial U^{i}}{\partial t}(\theta,t) = h(U,\theta,i) + \sum_{k} g(U,\theta,k) \frac{\partial U^{i}}{\partial \theta^{k}}(\theta,t),$$

together with the terminal condition

$$U^{i}(\theta, T) = \psi^{i}(\theta).$$



 $\begin{array}{c} \mbox{Continuous state models} \\ N+1 \mbox{player games} \\ \mbox{Convergence as } N \rightarrow \infty \\ \mbox{More about continuous state models} \\ \mbox{References} \end{array}$ 

N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation laws

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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation laws

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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation laws

#### Theorem

Suppose U is a solution. Let  $\theta$  and u be such that

- the first equation of the mean-field game is satisfied, i.e.  $\frac{d}{dt}\theta^{i} = g(U^{i}(\theta(t), t), \theta, i);$
- $e \theta(0) = \theta_0;$
- $u^i(t) = U^i(\theta(t), t) .$

Then u satisfies the second equation of the mean-field game, i.e.  $-\frac{d}{dt}u^i = h(\Delta_i u, \theta, i)$  as well as the terminal condition  $u^i(T) = \psi^i(\theta(T))$ . Therefore, u is the value function associated to  $\theta$ , and so it determines a Nash equilibria for the MFG.



N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation laws

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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation laws

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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation laws

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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation laws

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N+1 player symmetric games Mean field dynamics Single player point of view Mean field equations Uniqueness of solutions Trend to equilibrium A variational principle Connection with conservation laws

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Discrete state, continuous time mean-field games<br/>Continuous state modelsMean field dynamicsN + 1 player games<br/>Convergence as  $N \to \infty$ Single player point of viewMore about continuous state models<br/>ReferencesMean field equationsReferencesTrend to equilibriumA variational principle<br/>Connection with conservation laws

## A Hamilton Jacobi equation for potential MFG

For potential mean field games the master equation can be further simplified if we suppose that the terminal condition is given by a gradient

$$U^{i}(\theta, T) = \nabla_{\theta^{i}} \Psi_{T}(i, \theta).$$

In this case let  $\Psi$  be a solution of the PDE

$$\begin{cases} -\frac{\partial\Psi}{\partial t} = H(\nabla_{\theta}\Psi, \theta) \\ \Psi(\theta, T) = \Psi_{T}(\theta). \end{cases}$$

Then a direct calculation can show that  $U^{i}(\theta, t) = \nabla_{\theta^{i}} \Psi(\theta, t)$  i a solution of the master equation.



Discrete state, continuous time mean-field games<br/>Continuous state modelsMean field dynamicsN + 1 player games<br/>Convergence as  $N \to \infty$ Single player point of viewMore about continuous state models<br/>ReferencesMean field equationsReferencesTrend to equilibriumA variational principle<br/>Connection with conservation laws

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Continuous state mean-field models A new variational structure Evans-Aronsson's problem

# Outline

- Discrete state, continuous time mean-field games
- 2 Continuous state models
- 3 N + 1 player games
- 4 Convergence as  $N o \infty$
- 5 More about continuous state models

## 6 References



Continuous state mean-field models A new variational structure Evans-Aronsson's problem

## Mean field problems - continuous space and time

In continuous space and time a wide class of mean field equations has the form

$$N(u) = f(\theta)$$
$$L^*(\theta) = 0,$$

where *N* is a nonlinear operator and *L*<sup>\*</sup> is the adjoint of the linearization of *N*, and *f* is a monotone increasing function of  $\theta$ . The function  $u : \Omega \times [0, T] \to \mathbb{R}$  is supposed to be sufficiently regular, and  $\theta$  is a (probability) measure.



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#### An important example is

$$N(u) = -u_t + H(D_x u, x) - \frac{1}{2}\Delta u$$

to which corresponds

$$L^*(\theta) = \theta_t - \operatorname{div}(D_{\rho}H\theta) - \frac{1}{2}\Delta\theta,$$

and  $f(\theta) = \ln \theta$ .

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# **Controlled diffusions**

Suppose we know a the distribution of players in  $\mathbb{R}^n$  given by a probability measure  $\theta(t, \cdot)$ . The objective of an individual reference player is to minimize

$$V(x,t) = E \int_t^T \left( L(\mathbf{x}, \mathbf{v}, \theta) ds + \psi(\mathbf{x}(T)) \right).$$

among all diffusions

$$d\mathbf{x} = \mathbf{v}dt + dW_t$$
.

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Continuous state mean-field models A new variational structure Evans-Aronsson's problem

#### Then V(x, t) solves

$$-V_t + H(D_x V, x, \theta) = \frac{1}{2} \Delta V,$$

and the optimal drift v is

$$v = -D_{p}H(D_{x}V, x, \theta).$$



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#### If all the population acts according to the optimal strategy then

$$heta_t - \operatorname{div}(D_{
ho}H heta) = rac{1}{2}\Delta heta.$$



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- Suppose that  $f(z) = g'^{-1}(z)$ , for some convex increasing function g.
- We consider the variational problem

$$\int_0^T \int_\Omega g(N(u)).$$

• A simple computation shows that sufficiently smooth critical points of this functional are indeed solutions of the mean field equations, for

$$\theta = g'(N(u)).$$



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#### For instance, in the example we have the variational problem

$$\int_0^T \int_\Omega e^{-u_t + H(Du,x) - \frac{1}{2}\Delta u} dx dt.$$
 (2)



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- For convex nonlinear operators *N* this variational formulation yields in many instances uniqueness results for smooth solutions to the mean-field equations.
- Existence issues are more delicate as these functionals not coercive and thus delicate a-priori estimates or explicit formulas are required.



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#### The Evans-Aronsson variational problem is

$$\min\int_{\mathbb{T}^d} e^{H(Du,x)-\frac{1}{2}\Delta u} dx.$$

The lack of coercivity of this functional is the key technical problem.



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Continuous state mean-field models A new variational structure Evans-Aronsson's problem

# Key results

Existence and uniqueness of smooth solutions for:

•  $H(p, x) = \frac{|P+p|^2}{2} + V(x)$ , through explicit solutions;

• a wide class of Hamiltonians if *d* = 2, through a-priori bounds.



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# Hopf-Cole type transform

#### Theorem

Let u and v be periodic solutions to

$$\begin{cases} \frac{1}{2}\Delta u + \frac{|P+Du|^2}{2} + V(x) &= u - v \\ -\frac{1}{2}\Delta v + \frac{|P+Dv|^2}{2} + V(x) &= u - v. \end{cases}$$

Then u is a minimizer.



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# Note such solutions do exist and are smooth as we have the a-priori bound:

Theorem

#### $\sup |Du| + |Dv| \le C.$



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Continuous state mean-field models A new variational structure Evans-Aronsson's problem

## General case, dimension independent bounds

Set  $m = e^{\frac{1}{2}\Delta u + H(x,Du) - \lambda}$ . Here  $\lambda$  is such that m is a probability measure.

Theorem

$$\int_{\mathbb{T}^d} |D\ln m|^2 \le C,$$

and

$$\int_{\mathbb{T}^d} H(x, Du)m \leq C.$$



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#### Theorem

$$egin{aligned} &\int_{\mathbb{T}^d} |\Delta Du|^2 m \leq C \ &\|\sqrt{m}\|_{H^1} \leq C \ &\left(\int m^{rac{2^*}{2}}
ight)^{rac{2^*}{2^*}} \leq C \ &\int |D^2 u|^2 m \leq C, \end{aligned}$$

 $\int H^2 m \leq C.$ 

and

Theorem

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#### In dimension 2 the previous bounds yield

# which it is then enough to prove existence of smooth solutions by the continuation method.

 $\int_{\mathbb{T}^d} |D^2 u|^2 \leq C.$ 



N + 1 player dynamics Single player point of view Nash symmetric equilibrium

# Outline

- Discrete state, continuous time mean-field games
- 2 Continuous state models
- 3 N + 1 player games
  - 4 Convergence as  $N o \infty$
- 5 More about continuous state models

#### 6 References



N + 1 player dynamics Single player point of view Nash symmetric equilibrium

## Problem set-up - review

#### • N + 1 indistinguishable players;

- players can be in a finite number of states i = 1, ..., d;
- at any time each player knows only its state i(t) and the number of players n<sub>j</sub>(t) in state j;
- each player can only control its switching rate  $\alpha$  from one state to another;
- players follow (independent) controlled Markov chains with transition rate  $\beta_{jk}$ .



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# Optimization criterion - review

- Each player chooses the switching rate in order to minimize an expected payoff;
- This payoff has a running cost c(i, <sup>n</sup>/<sub>N</sub>, α), where α is the switching rate
- and a terminal cost  $\psi^i\left(\frac{n}{N}\right)$ ;

more precisely

$$\operatorname{cost} = E \int_{t}^{T} c(\mathbf{i}(s), \frac{\mathbf{n}(s)}{N}, \alpha(s)) ds + \psi^{\mathbf{i}(T)} \left(\frac{\mathbf{n}(T)}{N}\right)$$



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N + 1 player dynamics
 Single player point of view
 Nash symmetric equilibrium

- The reference player switches from state *i* to state *j* according to a switching Markovian rate α<sub>ij</sub>(n, t)
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Discrete state, continuous time mean-field games Continuous state models N + 1 player games Convergence as  $N \rightarrow \infty$ More about continuous state models References References N + 1 player dynamics Single player point of view Nash symmetric equilibrium

Let  $e_k$  be the k - th vector of the canonical basis of  $\mathbb{R}^d$ , and let  $e_{jk} = e_j - e_k$ . From the symmetry and independence of transitions assumption, for  $k \neq j$ , we have

$$\mathbb{P}\left(\mathbf{n}_{t+h} = n + e_{jk} \| \mathbf{n}_t = n, \mathbf{i}_t = i\right) = \gamma_{\beta,kj}^{n,i}(t).h + o(h)$$

where

$$\gamma_{\beta,kj}^{n,i}(t) = n_k \beta_{kj}(n+e_{ik},t).$$

The term  $n + e_{ik}$  instead of n, follows from the fact that from the point of view of a player which is in state k, and is distinct from the reference player, the number of other players in any state is given by  $\mathbf{n} + e_i - e_k = \mathbf{n} + e_{ik}$ .



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Discrete state, continuous time mean-field games  
Continuous state models  

$$N + 1$$
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Convergence as  $N \to \infty$   
More about continuous state models  
References  
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N + 1 player dynamics Single player point of view Nash symmetric equilibrium

## Control problem from a player's point of view

- Fix a reference player, and suppose the remaining N players use a transition rate β;
- Then the process  $\mathbf{n}(t)$  is a Markov process with rate  $\gamma_{\beta ki}^{n,i}(t)$

• The reference player wants to

$$u_{n}^{i}(t,\beta,\alpha) = \mathbb{E}^{\beta,\alpha}\left[\int_{t}^{T} c\left(\mathbf{i}_{s},\frac{\mathbf{n}_{s}}{N},\alpha(s)\right) ds + \psi^{\mathbf{i}_{T}}\left(\frac{\mathbf{n}_{T}}{N}\right)\right]$$

where the expectation is conditioned on  $\mathbf{i}_t = i$ ,  $\mathbf{n}_t = n$ .



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# Hamilton-Jacobi ODE

Fix an admissible control  $\beta$ . Consider the system of ODE's indexed by *i* and *n* given by

$$-\frac{d\varphi_n^i}{dt}(t) = \sum_{k,j} \gamma_{\beta,kj}^{n,i}(t) \left(\varphi_{n+e_{jk}}^i(t) - \varphi_n^i(t)\right) + h\left(\Delta_i \varphi_n(t), \frac{n}{N}, i\right) ,$$

where

$$h(p,\theta,i) = \min_{\alpha \ge 0} \left[ c(i,\theta,\alpha) + \alpha p \right],$$

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$$\varphi_n^i(T) = \psi^i\left(\frac{n}{N}\right)$$



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N + 1 player dynamics Single player point of view Nash symmetric equilibrium

# A verification theorem

#### Theorem

The previous terminal value problem for  $\varphi_n^i$  has a unique solution. This solution is the value function for the reference player, and

$$\tilde{\alpha}(\beta)(i,n,t) \equiv \alpha^* \left( \Delta_i \varphi_n(t), \frac{n}{N}, i \right)$$

is an optimal strategy. Furthermore  $\varphi_n^i$  is bounded uniformly in  $\beta$ .



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N + 1 player dynamics Single player point of view Nash symmetric equilibrium

Note that  $\alpha^*$  depends on  $\beta$ . We say that  $\beta$  is a symmetric Nash equilibrium if  $\alpha^*_{\beta} = \beta$ .

#### Theorem

There exists a unique Nash equilibrium  $\bar{\beta}$ .



N + 1 player dynamics Single player point of view Nash symmetric equilibrium

### A necessary condition for a control $\bar{\beta}$ to be a Nash equilibrium is

$$\bar{\beta}_{kj}(n,t) = \alpha_j^*\left(\Delta_k u_n(t;\bar{\beta}), \frac{n}{N}, k\right).$$



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Discrete state, continuous time mean-field games Continuous state models N + 1 player games Convergence as  $N \to \infty$ More about continuous state models References N + 1 player dynamics Single player point of view Nash symmetric equilibrium

Hence this gives rise to the system of nonlinear differential equations

$$-\frac{du_n^i}{dt} = \sum_{k,j} \gamma_{kj}^{n,i} (u_{n+e_{jk}}^i - u_n^i) + h\left(\Delta_i u_n, \frac{n}{N}, i\right) ,$$

with terminal condition

$$u_n^i(T)=\psi^i\left(\frac{n}{N}\right),$$

where  $\gamma_{kj}^{n,i}$  are given by

$$\gamma_{kj}^{n,i} = n_k \alpha_j^* \left( \Delta_k u_{n+e_{ik}}, \frac{n+e_{ik}}{N}, k \right)$$



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Discrete state, continuous time mean-field games Continuous state models N + 1 player games Convergence as  $N \rightarrow \infty$ More about continuous state models References N + 1 player dynamics Single player point of view Nash symmetric equilibrium

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Discrete state, continuous time mean-field games Continuous state models N + 1 player games Convergence as  $N \to \infty$ More about continuous state models References N + 1 player dynamics Single player point of view Nash symmetric equilibrium

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- Discrete state, continuous time mean-field games
- 2 Continuous state models
- 3 N + 1 player games
- 4 Convergence as  $N \to \infty$ 
  - 5 More about continuous state models

### 6 References



### Mean-field requations - review

$$\begin{cases} \frac{d}{dt}\theta^{i} = \sum_{j} \theta^{j} \alpha_{i}^{*}(\Delta_{j} u, \theta, j) \\ -\frac{d}{dt} u^{i} = h(\Delta_{i} u, \theta, i). \end{cases}$$



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### Master equation - review

Recall the master equation

$$-\frac{\partial U^{i}}{\partial t}(\theta,t) = h(U,\theta,i) + \sum_{k} g(U,\theta,k) \frac{\partial U^{i}}{\partial \theta^{k}}(\theta,t),$$

where

$$g(u, \theta, i) = \sum_{j} \theta^{j} \alpha_{i}^{*}(\Delta_{j} u, \theta, j).$$

together with the terminal condition

$$U^{i}(\theta, T) = \psi^{i}(\theta).$$



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### Master equation - review

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together with the terminal condition

$$U^{i}(\theta,T)=\psi^{i}(\theta).$$



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# Let U be a smooth solution to the mean-field master equation. Set $\tilde{u}_n^i = U^i(\frac{n}{N})$ . Then

$$-\frac{d\tilde{u}_n'}{dt} = \sum_{k,j} \gamma_{kj}^{n,i} (\tilde{u}_{n+e_{jk}}^i - \tilde{u}_n^i) + h\left(\Delta_i \tilde{u}_n, \frac{n}{N}, i\right) + E_N$$

where  $E_N \rightarrow 0$  as  $N \rightarrow \infty$ .





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## Consistency

Let *U* be a smooth solution to the mean-field master equation. Set  $\tilde{u}_n^i = U^i(\frac{n}{N})$ . Then

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## Consistency

Let *U* be a smooth solution to the mean-field master equation. Set  $\tilde{u}_n^i = U^i(\frac{n}{N})$ . Then

$$-\frac{d\tilde{u}_{n}^{\prime}}{dt}=\sum_{k,j}\gamma_{kj}^{n,i}(\tilde{u}_{n+e_{jk}}^{i}-\tilde{u}_{n}^{i})+h\left(\Delta_{i}\tilde{u}_{n},\frac{n}{N},i\right)+E_{N}$$

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### Stability of controls

### Lemma

We have

$$\left|\gamma_{kj}^{n+e_{rs},i}-\gamma_{kj}^{n,i}
ight|\leq \mathcal{C}+\mathcal{CN}\max_{rs}\|u_{n+e_{rs}}^{i}(t)-u_{n}^{i}(t)\|_{\infty}.$$



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### Gradient estimates

#### Lemma

Let  $u_n^i(t)$  be a solution. Then there exists C > 0 and  $T^* > 0$  such that, for  $0 < T < T^*$ , we have

$$\max_{rs} \|u_{n+e_{rs}}^{i}(t)-u_{n}^{i}(t)\|_{\infty} \leq \frac{2C}{N},$$

for all  $0 \le t \le T$ .

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### Theorem

There exists a constant *C*, independent of *N*, for which, if  $T < T^*$ , satisfies  $\rho = TC < 1$ , then

$$V_N(t) + W_N(t) \leq rac{C}{1-
ho}rac{1}{N}$$

for all  $t \in [0, T]$ , where

$$V_N(t) \equiv \mathbb{E}\left[\left\|\frac{\mathbf{n}_t}{N} - \theta(t)\right\|^2\right],$$

and

$$W_N(t) \equiv \mathbb{E}\left[\left\| u(t) - u_{\mathbf{n}_t}^N(t)\right\|^2\right]$$

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#### Lemma

Suppose  $T < T^*$ . There exists  $C_1 > 0$  such that

$$V_N(t) \leq \int_0^t C_1(V_N(s) + W_N(s))ds + rac{C_1}{N}.$$

#### \_emma

Suppose  $T < T^*$ . There exists  $C_2 > 0$  such that

$$W_N(t) \leq \int_t^T C_2(V_N(s) + W_N(s))ds + rac{C_2}{N}$$



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Adding both inequalities from previous Lemmas:

$$W_N(t)+V_N(t)\leq C\int_0^T(V_N(s)+W_N(s))ds+rac{C}{N}.$$

Suppose  $\rho = TC < 1$ . Set

$$W_N + V_N = \max_{0 \le t \le T} W_N(t) + V_N(t)$$

then

$$W_N + V_N \le \rho(W_N + V_N) + \frac{C}{N},$$

which proves the Theorem



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# Outline

- Discrete state, continuous time mean-field games
- 2 Continuous state models
- 3 N + 1 player games
- 4 Convergence as  $N \to \infty$
- More about continuous state models

### References



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- Discrete state, continuous time mean-field games
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### 6 References



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