Well-Posed Bayesian Geometric Inverse Problems Arising In Groundwater Flow

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Outline

Outline

Introduction

References

- D. A. White J. N. Carter.
  History matching on the Imperial College fault model using parallel tempering.
  Computational Geosciences, 17(2013) 43–65.

- M.Iglesias, K. Lin and A.M. Stuart
  Well-posed Bayesian geometric inverse problems arising in groundwater flow.
  In preparation.

Outline

Forward And Inverse Problems

Groundwater Flow Model

- pore pressure (head) : $p(x)$
- permeability (hydraulic conductivity) : $\kappa(x)$

Single-Phase Darcy Flow:

find $p \in H_0^1(D)$, given $\kappa \in L^\infty(D)$

$$\begin{aligned} -\nabla \cdot (\kappa(x)\nabla p) &= f, \quad x \in D, \\ p &= 0, \quad x \in \partial D, \end{aligned}$$

Inverse Problem: find $\kappa \in L^\infty(D)$, given noisy y_j

$$y_j = \ell_j(p) + \eta_j,$$
$$\eta_j \sim N(0, \gamma^2), \quad \ell_j \in H^{-1}(D).$$

Geometric Model

Piecewise Constant Permeability

$$\kappa(x) = \sum_{i=1}^{n} \kappa_i \, \chi_{D_i}(x),$$

where $\bigcup_i D_i = D$ and $D_i \cap D_j = \varnothing, \forall i \neq j$.

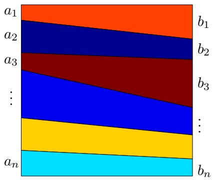


Figure : A multiple layers Model

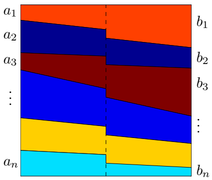
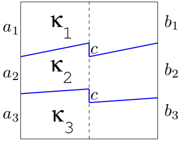


Figure : A fault Model

Forward and Observation Map

Geometry



parameterization

unknowns: $u = (\kappa^T, a^T, b^T, c)^T$
permeability : $u \to \kappa(x)$

Forward Operator

$$\begin{array}{rccccccc} \mathcal{G}: & u & \longmapsto & \kappa(x) & \longmapsto & p(x) & \longmapsto & y \\ & \mathbb{R}^d & \longmapsto & L^\infty(D) & \longmapsto & H_0^1(D) & \longmapsto & \mathbb{R}^J \end{array}$$

Find u from y

$$y = \mathcal{G}(u) + \eta.$$

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Bayesian Approach

$$y = \mathcal{G}(u) + \eta, \quad \eta \sim N(0, \Gamma).$$

- **Prior** distribution on u:

$$\mu_0(du) = \mathbb{P}(du).$$

- **Negative log likelihood** on $y|u$:

$$\Phi(u; y) = \frac{1}{2}|y - \mathcal{G}(u)|_\Gamma^2,$$

where $|\cdot|_\Gamma = |\Gamma^{-\frac{1}{2}} \cdot |$.

- **Posterior** distribution on $u|y$:

$$\mu^y(du) = \mathbb{P}(du|y).$$

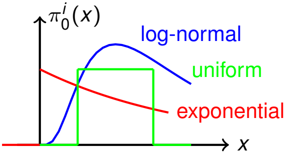
Prior Modeling

Geometric prior π_0^G and π_0^{slip}

- π_0^G: $a, b \sim$ uniformly in $A := \{\mathbf{x} \in \mathbb{R}^{n-1} | \sum_{i=1}^{n-1} x_i \leq 1, x_i \geq 0\}$.
- π_0^{slip}: $c \sim$ uniformly in $[-C, C]$.
- a, b, c are independent.

Permeability prior π_0^i on κ_i

- log-normal: $\kappa_i = \exp(\xi_i)$, ξ_i drawn from Gaussian.
- uniform: κ_i drawn from (> 0) uniform distribution.
- exponential: κ_i drawn from exponential distribution.



Prior on $u \in U := (0, \infty)^n \times A^2 \times [-C, C]$

$$\pi_0(u) = \prod_{i=1}^{n} \pi_0^i(\kappa_i) \times \pi_0^G(a)\pi_0^G(b)\pi_0^{slip}(c).$$

Well-Defined Posterior

Theorem (Bayes' Theorem [4])

Assume that $\Phi(u; y) : U \times Y \to \mathbb{R}$ *is measurable and,*

$$Z := \int_U \exp(-\Phi(u; y))\mu_0(du) > 0$$

Then the conditional distribution μ^y *of* $u|y$ *exists,* $\mu^y \ll \mu_0$ *and*

$$\frac{d\mu^y}{d\mu_0} = \frac{1}{Z} \exp\left(-\Phi(u; y)\right).$$

We establish this theorem for our problem by proving continuity of $\mathcal{G}$, (and hence of $\Phi(\cdot; y)$) on a set of full μ_0 measure.

Continuity of $\mathcal{G}$

Applying **Bayes' Theorem** (see [3])

$\mathcal{G}$ is continuous on U, $\mu_0(U) = 1$ and $Z > 0$ $\Longrightarrow$

$$\mu^y \ll \mu_0$$

$$\frac{d\mu^y}{d\mu_0} = \frac{1}{Z}\exp\left(-\Phi(u;y)\right).$$

- $u \to \kappa(x) \to p(x)$
- $u^\varepsilon \to \kappa^\varepsilon(x) \to p^\varepsilon(x)$

BUT

- $\kappa \in L^\infty(D) \to p \in H_0^1(D)$ is Lipschitz.
- $u \in \mathbb{R}^d \to \kappa \in L^r(D)$ continuous only for $r < \infty$.

$$\begin{aligned} -\nabla\cdot(\kappa^\varepsilon(x)\nabla(p^\varepsilon - p)) &= \nabla\cdot((\kappa^\varepsilon - \kappa)\nabla p), & x \in D \\ p^\varepsilon - p &= 0, & x \in \partial D. \end{aligned}$$

Continuity of $\mathcal{G}$

$$\|p^\varepsilon - p\|_V \le \frac{1}{\kappa_{min}} \left(\int_D |\kappa^\varepsilon - \kappa|^2 |\nabla p|^2 dx \right)^{\frac{1}{2}}$$

$$= \frac{1}{\kappa_{min}} \left(\sum_{i=1}^{n} |\kappa_i^\varepsilon - \kappa_i|^2 \int_{D_i^\varepsilon \cap D_i} |\nabla p|^2 dx + \sum_{i \neq j} |\kappa_i^\varepsilon - \kappa_j|^2 \int_{D_i^\varepsilon \cap D_j} |\nabla p|^2 dx \right)^{\frac{1}{2}}.$$

$$u \to u^\varepsilon$$

$$\kappa(x) \to \kappa^\varepsilon(x)$$

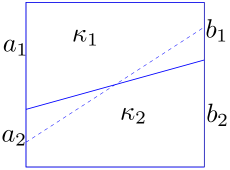


Figure : $\kappa(x)$, $\kappa^\varepsilon(x)$

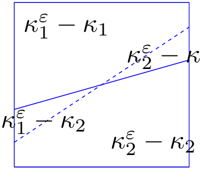


Figure : $\kappa^\varepsilon(x) - \kappa(x)$

Well-Poseness of Posterior

- The Hellinger distance:

$$d_{Hell}(\mu, \mu') = \left(\frac{1}{2} \int_U \left(\sqrt{\frac{d\mu}{d\nu}} - \sqrt{\frac{d\mu'}{d\nu}} \right)^2 d\nu \right)^{\frac{1}{2}}.$$

- The Total Variation distance:

$$d_{TV}(\mu, \mu') = \frac{1}{2} \int_U \left| \frac{d\mu}{d\nu} - \frac{d\mu'}{d\nu} \right| d\nu.$$

- Relationship:

$$\frac{1}{\sqrt{2}} d_{TV}(\mu, \mu') \le d_{Hell}(\mu, \mu') \le d_{TV}(\mu, \mu')^{\frac{1}{2}}.$$

Well-Poseness of Posterior

μ (resp μ') the posterior corresponding to data y (resp. y').

Theorem (Well-Posedness [3])

Assume that $\max\{|y|, |y'|\} < r$. *Then there are* $C_i = C_i(r)$ *such that:*
(I) $\kappa_i \sim$ *log-normal or* $\kappa_i \sim$ *uniform, then*

$$d_{TV}(\mu, \mu') \le C_1|y - y'|, \quad d_{Hell}(\mu, \mu') \le C_2|y - y'|.$$

(II) $\kappa_i \sim$ *exponential, then*

$$d_{TV}(\mu, \mu') \le C_1|y - y'|^{\alpha}, \quad d_{Hell}(\mu, \mu') \le C_2|y - y'|^{\frac{\alpha}{2}},$$

where $0 < \alpha < 1$.

Well-Poseness of Posterior

Sketch of proof:

$$d_{TV}(\mu,\mu') \le I_1 + I_2.$$

where

$$I_1 = \frac{1}{2Z}\int_U \left|\exp\left(-\Phi(u,y)\right) - \exp\left(-\Phi(u,y')\right)\right| d\mu_0(u),$$
$$I_2 = \frac{1}{2}|Z^{-1} - (Z')^{-1}|\int_U \exp\left(-\Phi(u,y')\right) d\mu_0(u).$$

And

$$I_2 \le CI_1.$$

Well-Poseness of Posterior

$$I_1 = \frac{1}{2Z}\int_U \left|\exp\left(-\Phi(u,y)\right) - \exp\left(-\Phi(u,y')\right)\right| d\mu_0(u)$$

$$\left|\exp\left(-\Phi(u,y)\right) - \exp\left(-\Phi(u,y')\right)\right| \le \left|\Phi(u,y) - \Phi(u,y')\right| \le M(r,u)|y-y'|.$$

(I): log-normal or uniform μ_0 on u

$$\int_U |\Phi(u;y) - \Phi(u,y')| d\mu_0(u) \le \int_U M(r,u) d\mu_0(u)\,|y-y'|.$$

(II): exponential μ_0 on u

- $\int_{\{|\Phi(u;y)-\Phi(u,y')|\le 1\}} |\Phi(u;y) - \Phi(u,y')|^{\alpha} d\mu_0 \le \int_U M^{\alpha} d\mu_0\,|y-y'|^{\alpha}.$
- $\mu_0\left(|\Phi(u;y) - \Phi(u,y')| > 1\right) \le \int_U M^{\alpha} d\mu_0\,|y-y'|^{\alpha}.$

Outline

Metropolis Hasting Method

General Metropolis-Hasting Algorithm: to sample $\pi(u)$:

1. $k = 0$, initialize $u^{(0)}$.
2. Propose $v \in \mathbb{R}^d$ from $q(u^{(k)}, v)$.
3. Set
$$u^{(k+1)} = \begin{cases} v & \text{with probability } a(u^{(k)}, v) \\ u^{(k)} & \text{otherwise} \end{cases},$$
4. $k \to k+1$ and repeat 2.

Here
$$a(u, v) = \frac{\pi(v)q(v, u)}{\pi(u)q(u, v)} \wedge 1.$$

Prior Reversible Proposal

To sample $\pi(u) = \exp\big(-\Phi(u)\big)\pi_0(u)$:

- Choose $q(u, v)$ s.t.

$$\pi_0(u)q(u, v) = \pi_0(v)q(v, u), \quad \forall u, v \in \mathbb{R}^d.$$

- Then

$$\begin{aligned} a(u, v) &= \frac{\pi(v)q(v, u)}{\pi(u)q(u, v)} \wedge 1, \\ &= \frac{\pi_0(v)\exp\big(-\Phi(v)\big)q(v, u)}{\pi_0(u)\exp\big(-\Phi(u)\big)q(u, v)} \wedge 1, \\ &= \exp\big(\Phi(u) - \Phi(v)\big) \wedge 1. \end{aligned}$$

- Propose using prior; accept/reject according to model-data misfit.

Two-Step Metropolis Hasting Method

Algorithm:

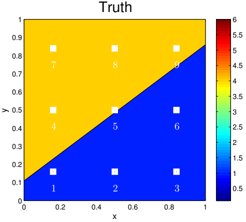
1. $k = 0$, initialize $u^{(0)} \in U$.
2. Draw $w \in \mathbb{R}^d$ from $p(u^{(k)}, w)$.
3. Propose $v \in U$ defined by
$$v = \begin{cases} w & w \in U \\ u^{(k)} & w \notin U \end{cases} .$$
4. Accept or reject v:
$$u^{(k+1)} = \begin{cases} v & \text{with probability } a(u^{(k)}, v) \\ u^{(k)} & \text{otherwise} \end{cases} .$$
5. $k \to k + 1$ and repeat 2.

Here
$$a(u, v) = \exp\big(\Phi(u; y) - \Phi(v; y)\big) \wedge 1.$$

Outline

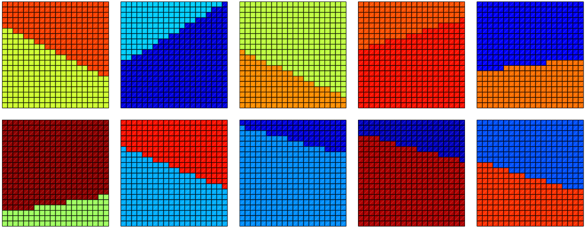
Two Layers Model

- Truth and Prior distribution

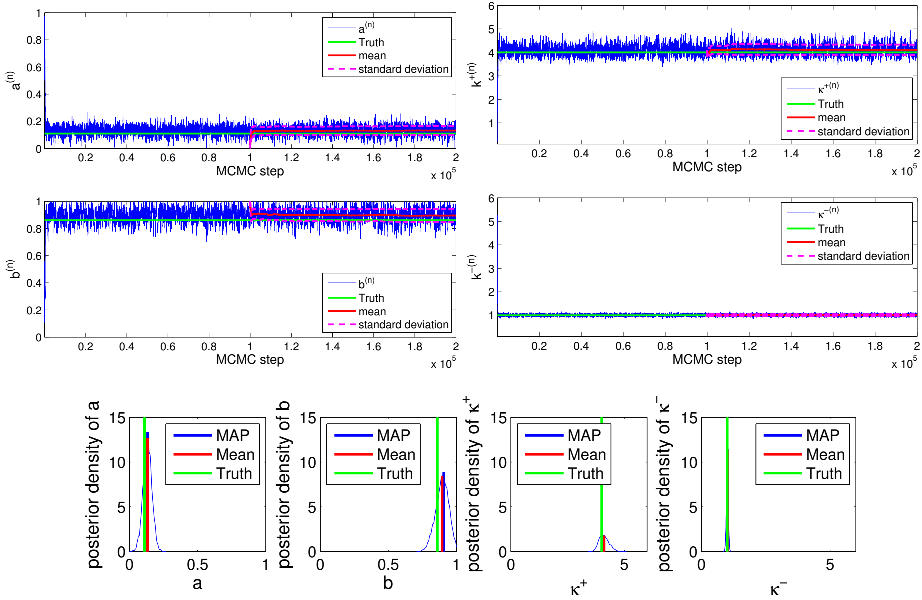


parameter	true value	prior distribution
a	0.11	$U[0, 1]$
b	0.86	$U[0, 1]$
κ^{+}	4	$U[0.1, 6]$ or $\exp(1)$
κ^{-}	1	$U[0.1, 6]$ or $\exp(1)$

- Random draws from prior

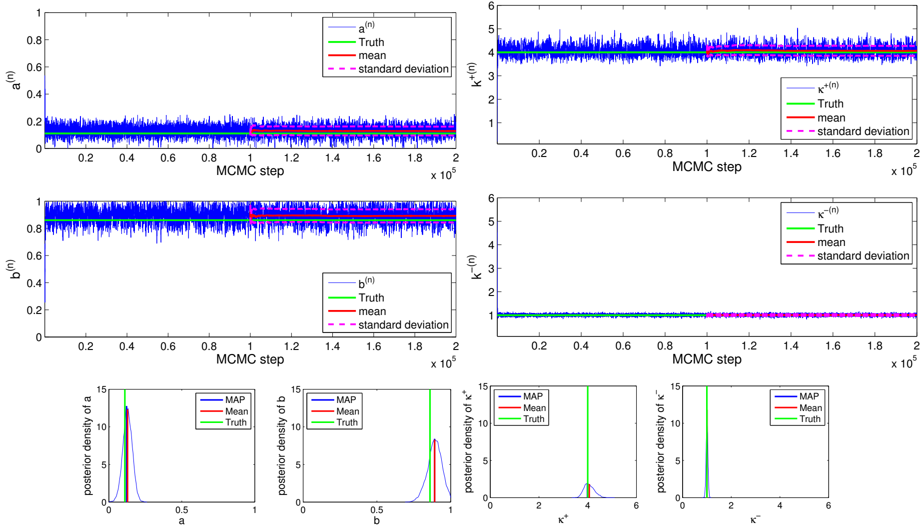


Two Layers Model: Uniform Prior Permeability



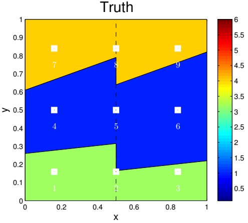
Two Layers Model: Exponential Prior Permeability

- $\kappa^+, \kappa^- \sim \exp(\lambda)$, where $\lambda = 1$.



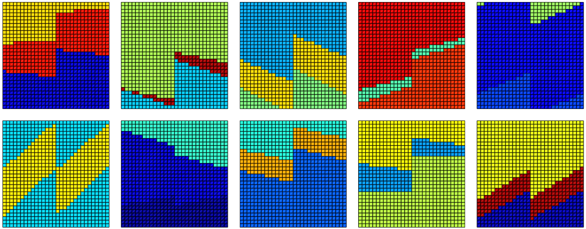
Fault Model

- Truth and Prior distribution

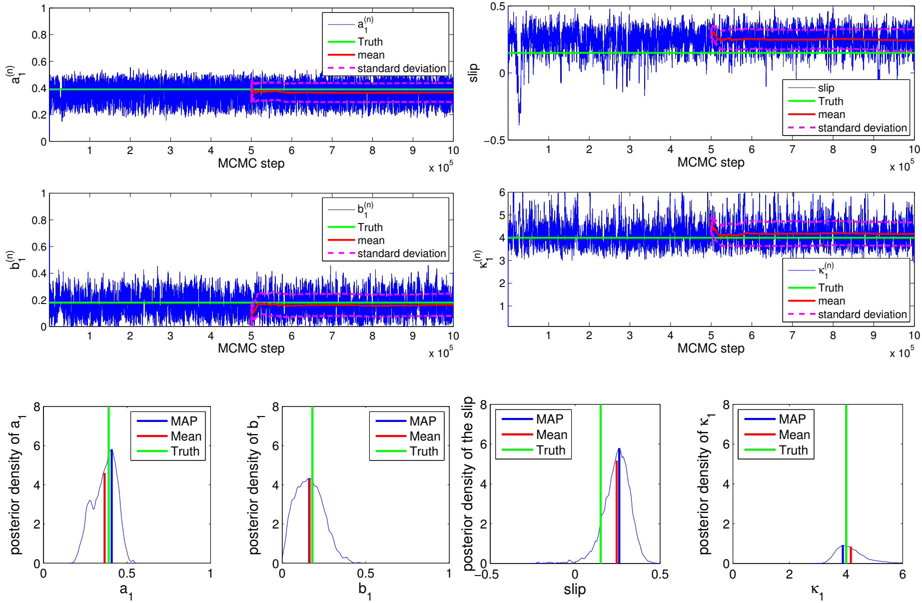


parameter	true value	prior distribution
a_1 a_2	0.39 0.35	$a \sim U(A)$
b_1 b_2	0.18 0.6	$b \sim U(A)$
c	0.15	$U[-0.5, 0.5]$
κ_1	4	$U[0.1, 6]$
κ_2	1	$U[0.1, 6]$
κ_3	1	$U[0.1, 6]$

- Random drawn from prior



Fault Model


$a_1^{(n)}$
Truth
mean
standard deviation
MCMC step
x 10^5
slip
Truth
mean
standard deviation
MCMC step
x 10^5
$b_1^{(n)}$
Truth
mean
standard deviation
MCMC step
x 10^5
$\kappa_1^{(n)}$
Truth
mean
standard deviation
MCMC step
x 10^5
posterior density of a_1
MAP
Mean
Truth
a_1
posterior density of b_1
MAP
Mean
Truth
b_1
posterior density of the slip
MAP
Mean
Truth
slip
posterior density of κ_1
MAP
Mean
Truth
κ_1

Outline

Conclusions

What we have shown:

- Geometric permeability prior modeling
- Well-definedness of posterior distribution (continuity argument)
- Well-posedness of posterior distribution (not Lipschitz)
- A tailored Metropolis-Hasting method

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