Examples of abelian surfaces with everywhere good reduction

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Motivation

Fontaine, Abrashkin: **There is no abelian variety over $\mathbb{Q}$ with everywhere good reduction.**

However, long before this result, there were few elliptic curves with unit conductor in the literature.

For example, the curve

$$E: \; y^2 + xy + \varepsilon^2 y = x^3,$$

where $\varepsilon = \frac{5+\sqrt{29}}{2}$ is the fundamental unit in $F = \mathbb{Q}(\sqrt{29})$, was known to Tate in the late 60s and has been extensively studied by Serre.

Motivation

There is now an abundance of elliptic curves with everywhere good reduction:

- Setzer, Stroeker, Comalada, Kida, Cremona, Pinch, Kagawa, etc.
- The database has been considerably expanded by Elkies.
- D-Voight: Assuming **modularity**, Elkies' database contains all the curves for all fundamental discriminants ≤ 1000.

In contrast, there is not a single example of an abelian surface with everywhere good reduction in the literature (except in the case of complex multiplication, or when the abelian surface is a product of elliptic curves).

It would be desirable to remedy that situation.

Motivation

A much more philosophical reason to study these objects:

Khare:

> "**The proof of the Serre conjecture in retrospect can be viewed as a method to exploit an accident which occurs in three different guises:**
>
> - *(Fontaine, Abrashkin) There are no non-zero abelian varieties over* $\mathbb{Z}$.
> - *(Serre, Tate) There are no irreducible representations*
>
> $$\bar{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}),$$
>
>   *where* $\overline{\mathbb{F}}$ *is the algebraic closure of* $\mathbb{F}_2$ *or* $\mathbb{F}_3$ *that are unramified outside of* 2 *and* 3 *respectively.*
> - $S_2(\mathrm{SL}_2(\mathbb{Z})) = 0$, *i.e., there are no cusp forms of level* $\mathrm{SL}_2(\mathbb{Z})$ *and weight* 2."

Our strategy

Let F be a number field of class number 1, and E an elliptic curve over F given by a (global minimal) Weierstrass equation

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

with $a_i \in \mathcal{O}_F$, the ring of integers of F.

The invariants c_4 and c_6 satisfy the equation $c_4^3 - c_6^2 = 1728\Delta$, where Δ is the discriminant of E. In other words, (c_4, c_6) is an $\mathcal{O}_F$-integral point on the curve

$$y^2 = x^3 - 1728\Delta. \quad (1)$$

E has **everywhere good reduction** $\iff$ Δ is a **unit** in $\mathcal{O}_F$.

Our strategy

So, to find all the elliptic curves over F with everywhere good reduction it is enough to solve (1) for all $\Delta \in \mathcal{O}_F^\times/(\mathcal{O}_F^\times)^{12}$ (finite).

Unfortunately, abelian varieties of higher dimension are not characterised by a nice diophantine equation as in (1).

For this reason, we need an additional input when looking for the ones with everywhere good reduction.

This extra input is provided by the **Eichler-Shimura conjecture.**

Eichler-Shimura conjecture

Conjecture (Eichler-Shimura)

Let F be a totally real number field and $\mathfrak{N}$ an intregal ideal of F. Let f be a Hilbert newform of parallel weight 2 and level $\mathfrak{N}$. Let $(a_{\mathfrak{m}}(f))_{\mathfrak{m}\subseteq\mathcal{O}_F}$ be the set of Fourier coefficients of f, and K_f the number field generated by them. There exists an abelian variety A_f/F of dimension $[K_f:\mathbb{Q}]$, with good reduction outside of $\mathfrak{N}$, such that

$$L(A_f, s) = \prod_{\tau\in\mathrm{Aut}(K_f)} L(f^\tau, s),$$

where

$$L(f^\tau, s) := \sum_{\mathfrak{m}\subseteq\mathcal{O}_F} \frac{a_{\mathfrak{m}}(f)^\tau}{\mathrm{N}\mathfrak{m}^s}.$$

Paramodularity conjecture

The following statement is a special case of the so-called **Paramodularity Conjecture** due to Brumer-Kramer, which encompasses Conjecture 1.

Conjecture (Brumer-Kramer)

Let g be a paramodular Siegel newform of genus 2, weight 2 and level N, with integer Hecke eigenvalues, which is not in the span of Gritsenko lifts. Then there exists an abelian surface B defined over $\mathbb{Q}$ of conductor N such that $\mathrm{End}_{\mathbb{Q}}(B) = \mathbb{Z}$ and $L(g, s) = L(B, s)$.

Eichler-Shimura construction for $F = \mathbb{Q}$

Let $N \in \mathbb{Z}_{>1}$, and let $X_1(N)$ be the modular curve of level $\Gamma_1(N)$.

This curve and its Jacobian $J_1(N)$ are defined over $\mathbb{Q}$.

The space $S_2(\Gamma_1(N))$ of cusp forms of weight 2 and level $\Gamma_1(N)$ is a $\mathbb{T}$-module, where $\mathbb{T}$ is the Hecke algebra.

Let $f \in S_2(\Gamma_1(N))$ be a newform, and let $I_f = \mathrm{Ann}_{\mathbb{T}}(f)$.

Shimura showed that the quotient

$$A_f := J_1(N)/I_f J_1(N)$$

is an abelian variety A_f of dimension $[K_f : \mathbb{Q}]$ defined over $\mathbb{Q}$ with endomorphisms by an order in K_f and that

$$L(A_f, s) = \prod_{g \in [f]} L(g, s),$$

where $[f]$ denotes the Galois orbit of f.

Eichler-Shimura construction: $F = \mathbb{Q}$

For **elliptic curves** defined over $\mathbb{Q}$, Cremona makes this construction explicit using the theory of **modular symbols**.

For **hyperelliptic curves of genus** 2, there is also the so-called **Jacobian nullwerte** approach used by the Barcelona school to compute equations for these curves.

Eichler-Shimura construction when $[F : \mathbb{Q}] > 1$

Case $[F : \mathbb{Q}]$ is odd:

Conjecture 1 is known and its proof exploits the **cohomology of Shimura curves.**

It is possible that one could use Voight's method to compute equations for abelian surfaces uniformised by Shimura curves.

Case: $[F : \mathbb{Q}]$ even:

Many cases of the conjecture are still unknown. For example, when $[F : \mathbb{Q}] = 2$ and f is a newform of level (1) and parallel weight 2 over F, it is not always possible to construct the associated surface A_f.

Examples

From now on, F is a real quadratic field of narrow class number one.

For a newform f of parallel weight 2 over F, recall that K_f denotes the coefficient field of f.

Notations:

$$\mathrm{Gal}(F/\mathbb{Q}) = \{1, \sigma\}, \ \mathrm{Gal}(K_f/\mathbb{Q}) = \{1, \tau\}.$$

Our examples can be subdivided in the following cases:

I: The form f is a base change from $\mathbb{Q}$.

II: The form f is not a base change from $\mathbb{Q}$:

- **(a)** The $\mathrm{Gal}(K_f/\mathbb{Q})$-orbit $\{f, f^\tau\}$ is $\mathrm{Gal}(F/\mathbb{Q})$-invariant.
- **(b)** The $\mathrm{Gal}(K_f/\mathbb{Q})$-orbit $\{f, f^\tau\}$ is not $\mathrm{Gal}(F/\mathbb{Q})$-invariant.

Examples: Case I

In this case, the Hecke eigenvalues of the Hilbert newform f satisfy

$$a_{\mathfrak{p}}(f) = a_{\mathfrak{p}^\sigma}(f).$$

Let g be a newform in $S_2(\Gamma_1(D))$ whose base change is f.

Since the level of f is (1), the form $g \in S_2(D, \chi_D)^{\mathrm{new}}$ by a result of Mazur-Wiles, where χ_D is the fundamental character of $F = \mathbb{Q}(\sqrt{D})$.

Furthermore, the abelian variety B_g attached to g is a fourfold such that

$$B_g \otimes_{\mathbb{Q}} F \simeq A_f \times A_f^\sigma,$$

where A_f and A_f^σ are isogenous.

So in this case, the existence of the surface A_f can be proved from the classical Eichler-Shimura construction.

Examples: Case I

The smallest discriminant for which we obtain such a surface is $D = 53$. The abelian surface A_f has real multiplication by $\mathbb{Z}[\sqrt{2}]$.

$N\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$	$(x^2 - a_{\mathfrak{p}}(f)x + N\mathfrak{p})(x^2 - a_{\mathfrak{p}}(f)^{\tau}x + N\mathfrak{p})$
4	2	$e+1$	$x^4 - 2x^3 + 7x^2 - 8x + 16$
7	$-w-2$	$-e-2$	$x^4 + 4x^3 + 16x^2 + 28x + 49$
7	$-w+3$	$-e-2$	$x^4 + 4x^3 + 16x^2 + 28x + 49$
9	3	$-3e+1$	$x^4 - 2x^3 + x^2 - 18x + 81$
11	$w-2$	$3e$	$x^4 + 4x^2 + 121$
11	$w+1$	$3e$	$x^4 + 4x^2 + 121$
13	$w-1$	$-2e+1$	$x^4 - 2x^3 + 19x^2 - 26x + 169$
13	$-w$	$-2e+1$	$x^4 - 2x^3 + 19x^2 - 26x + 169$
17	$-w-5$	-3	$x^4 + 6x^3 + 43x^2 + 102x + 289$
17	$w-6$	-3	$x^4 + 6x^3 + 43x^2 + 102x + 289$
25	5	$2e+4$	$x^4 - 8x^3 + 58x^2 - 200x + 625$
29	$-w-6$	$3e-3$	$x^4 + 6x^3 + 49x^2 + 174x + 841$

Table: Newform of level (1) and weight (2, 2) over $\mathbb{Q}(\sqrt{53})$.

Examples: Case I

Theorem

Let $C : y^2 + Q(x)y = P(x)$ be the curve over F, where

$$\begin{aligned} P(x) &:= (9w - 50)x^6 - (28w - 142)x^5 + (34w - 155)x^4 \\ &\quad - (2w - 12)x^3 + (21w - 83)x^2 - (105w - 439)x + 92w - 379 \\ Q(x) &:= -(w + 3)x^3 + (2w - 1)x^2 + (4w - 17)x - 7w + 29. \end{aligned}$$

Then

(a) *The discriminant of this curve is $\Delta_C = -\epsilon^{27}$, where ϵ is the fundamental unit. Thus C has everywhere good reduction.*

(b) *The surface $A := J(C)$ is modular and corresponds to the unique f in $S_2(1)$, the space of Hilbert newforms of weight $(2, 2)$ and level (1) over $F = \mathbb{Q}(\sqrt{53})$.*

Examples: Case II (a)

The following result explains the connection between Conjectures 1 and 2.

Lemma

Assume that Conjecture 2 is true. Let F be a real quadratic field. Let $\mathfrak{N}$ be an integral ideal of $\mathcal{O}_F$ such that $\mathfrak{N}^\sigma = \mathfrak{N}$. Let f be a Hilbert newform of weight $(2, 2)$ and level $\mathfrak{N}$ over F, which satisfies the hypotheses of Case II (a). Then f satisfies Conjecture 1.

Proof:

Since f is a non-base change, a result of Johnson-Leung-Roberts implies that there is a paramodular Siegel newform g of genus 2, level ND^2 and weight 2 attached to f, where $N = \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{N})$.

Moreover, since $\mathrm{Gal}(F/\mathbb{Q})$ preserves $\{f, f^\tau\}$, we must have the commutative diagram:

$$\begin{array}{ccc} \{a_\mathfrak{p}(f)\} & \xrightarrow{\sigma} & \{a_{\mathfrak{p}^\sigma}(f)\} \\ \| & & \| \\ \{a_\mathfrak{p}(f)\} & \xrightarrow[\tau]{} & \{a_\mathfrak{p}(f)^\tau\} \end{array}$$

In other words, we must have

$$a_{\mathfrak{p}^\sigma}(f) = a_\mathfrak{p}(f)^\tau, \ \forall\, \mathfrak{p} \subseteq \mathcal{O}_F.$$

Examples: Case II (a)

Proof (cont'd):

Therefore, the Hecke eigenvalues of the form g are integers.

So by Conjecture 2, there is an abelian surface B_g defined over $\mathbb{Q}$ with $\mathrm{End}_{\mathbb{Q}}(B_g) = \mathbb{Z}$ such that $L(B_g, s) = L(g, s)$.

Let A_f be the quadratic twist of B_g to F. Then A_f satisfies Conjecture 1, since it has the correct conductor N and L-factors.

Examples: Case II (a)

Remark

By Lemma 4, if A_f is an abelian surface attached to a Hilbert newform f satisfying Case II (a), then A_f is (essentially) the base change to F of some surface B defined over $\mathbb{Q}$, which acquires extra endomorphisms. Therefore, we know that the Igusa-Clebsch invariants of A_f are in $\mathbb{Q}$, and we can use this fact in looking for A_f.

Examples: Case II (a)

The first real quadratic field of narrow class number 1 where there is a form which satisfies Case II (a) is $F = \mathbb{Q}(\sqrt{193})$ (see Table 2).

The coefficients of f generate the ring of integers $\mathcal{O}_f := \mathbb{Z}[\frac{1+\sqrt{17}}{2}]$ of the field $K_f = \mathbb{Q}(\sqrt{17})$.

By Conjecture 1 (Eichler-Shimura conjecture), this form corresponds to a surface A over F with everywhere good reduction.

Notations:

$$w = \frac{1+\sqrt{193}}{2}, \quad e = \frac{1+\sqrt{17}}{2}.$$

$$\mathrm{Gal}(F/\mathbb{Q}) = \{1, \sigma\}, \ \mathrm{Gal}(K_f/\mathbb{Q}) = \{1, \tau\}.$$

Examples: Case II (a)

$\mathrm{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$	$(x^2 - a_{\mathfrak{p}}(f)x + \mathrm{N}\mathfrak{p})(x^2 - a_{\mathfrak{p}}(f)^{\tau}x + \mathrm{N}\mathfrak{p})$
2	$9w - 67$	e	$x^4 - x^3 - 2x + 4$
2	$9w + 58$	$-e + 1$	$x^4 - x^3 - 2x + 4$
3	$-2w + 15$	e	$x^4 - x^3 + 2x^2 - 3x + 9$
3	$2w + 13$	$-e + 1$	$x^4 - x^3 + 2x^2 - 3x + 9$
7	$-186w - 1199$	$-e + 2$	$x^4 - 3x^3 + 12x^2 - 21x + 49$
7	$186w - 1385$	$e + 1$	$x^4 - 3x^3 + 12x^2 - 21x + 49$
23	$38w - 283$	$-e - 6$	$x^4 + 13x^3 + 84x^2 + 299x + 529$
23	$-38w - 245$	$e - 7$	$x^4 + 13x^3 + 84x^2 + 299x + 529$
25	5	1	$x^4 - 2x^3 + 51x^2 - 50x + 625$
31	$-16w - 103$	$e - 3$	$x^4 + 5x^3 + 64x^2 + 155x + 961$
31	$-16w + 119$	$-e - 2$	$x^4 + 5x^3 + 64x^2 + 155x + 961$

Table: The first few Hecke eigenvalue attached to a non-base change for of conductor (1) over $\mathbb{Q}(\sqrt{193})$

Examples: Case II (a)

Theorem

Let $C : y^2 + Q(x)y = P(x)$ *be the curve over* F, *where*

$$P(x) := 2x^6 + (-2w+7)x^5 + (-5w+47)x^4 + (-12w+85)x^3 + (-13w+97)x^2 + (-8w+56)x - 2w + 1,$$
$$Q(x) := -x - w.$$

Then

(a) *The discriminant* $\Delta_C = -1$, *hence* C *has everywhere good reduction.*

(b) *The surface* $J(C)$ *is modular and corresponds to the form* f *listed in Table 2.*

Examples: Case II (a)

Remark

(1) *Both C and $J(C)$ have everywhere good reduction. However, this is not true in general. Indeed, it can happen that the curve C has bad reduction at a prime $\mathfrak{p}$ while $J(C)$ does not.*

(2) *A result of Stroeker states that if E is an elliptic curve defined over a real quadratic field F having good reduction everywhere, then $\Delta_E \notin \{-1, 1\}$.*

Theorem 6 shows that this is not true for genus 2 curves.

Examples: Case II (a)

Sketch of proof:

By direct calculation, one sees that $\Delta_C = -1$. So C and $J(C)$ have everywhere good reduction.

However, it is important to know that we found the curve C based on our heuristics which rely on Conjectures 1 and 2.

Indeed, let $S_2(1)$ be the space of Hilbert cuspforms of level (1) and parallel weight 2 over $F = \mathbb{Q}(\sqrt{193})$.

- $S_2(1)$ has dimension 9, and decomposes into two Hecke constituents of dimension 2 and 7 respectively.
- The Hecke constituent of the form f in Table 2 is 2-dimensional. It is a **non-base change** and is stable under $\mathrm{Gal}(K_f/\mathbb{Q})$.

So we can look for our surface A_f with the help of Lemma 4.

Examples: Case II (a)

For a fundamental discriminant D, let $Y_-(D)$ be the Hilbert modular surface which parametrises principally polarised abelian surfaces with real multiplication by $\mathcal{O}_D$, the ring of integers of $\mathbb{Q}(\sqrt{D})$.

The surface $Y_-(D)$ has a model over $\mathbb{Q}$.

In the 1970s, Hirzebruch et al. computed the geometric invariants of many of these surfaces and determined their Enriques-Kodaira classification.

Recently, Elkies and Kumar computed explicit equations for the birational models of these surfaces for all the fundamental discriminants ≤ 100.

Examples: Case II (a)

They showed that $Y_{-}(17)$ is given as a double-cover of $\mathbf{P}^2_{g,h}/\mathbb{Q}$ by

$$z^2 = (256h^3 - 192g^2h^2 - 464gh^2 - 185h^2 + 48g^4h - 236g^3h - 346g^2h - 144gh - 18h - 4g^6 - 20g^5 - 41g^4 - 44g^3 - 26g^2 - 8g - 1).$$

With this parametrisation, the Igusa-Clebsch invariants are given by:

$$I_2 := -\frac{24B_1}{A_1},$$
$$I_4 := -12A,$$
$$I_6 := \frac{96AB_1 - 36A_1B}{A_1},$$
$$I_{10} := -4A_1B_2,$$

Examples: Case II (a)

where

$$
\begin{aligned}
A_1 &:= (2g+1)^2,\\
A &:= -\frac{16h^2 - 8g^2h + 20gh + 64h + g^4 + 4g^3 + 6g^2 + 4g + 1}{3},\\
B_1 &:= -\frac{48h^2 - 16g^2h - 40gh - 16h + 4g^4 - 12g^3 - 23g^2 - 12g - 2}{3},\\
B &:= -\frac{2}{27}\Big(64h^3 - 48g^2h^2 - 312gh^2 + 978h^2 + 12g^4h - 6g^3h\\
&\qquad + 72g^2h + 210gh + 120h - g^6 - 6g^5 - 15g^4 - 20g^3\\
&\qquad -15g^2 - 6g - 1\Big),\\
B_2 &:= -64h^3.
\end{aligned}
$$

Examples: Case II (a)

A search for points of **low height** on this surface yields:

$$g = 0, h = -1/4,$$
$$I_2 = 40,\ I_4 = -56,\ I_6 = -669,\ I_{10} = -4,$$
$$j_1 = -3200000,\ j_2 = -208000,\ j_3 = -16400.$$

Over $\mathbb{Q}$, this gives the curve

$$C' : \ y^2 = -8x^6 + 220x^5 - 44x^4 - 14828x^3 - 4661x^2 - 21016x + 10028.$$

Twisting C' to F and then reducing yields curve C in Theorem 6.

Examples: Case II (a)

To prove modularity, we note that 3 is inert in $K_f = \mathbb{Q}(\sqrt{17})$, and consider the 3-adic representation attached to A,

$$\rho_{A,3} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_2(K_{f,(3)}) \simeq \mathrm{GL}_2(\mathbb{Q}_9).$$

By computing the orders of Frobenius for the first few primes, we see that the mod 3 representation

$$\bar{\rho}_{A,3} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_2(\mathbb{F}_9)$$

is surjective, and absolutely irreducible.

- We prove that $\bar{\rho}_{A,3}$ is modular using a result of Ellenberg.
- The modularity of $\rho_{A,3}$ the follows from work of Gee.

Examples: Case II (a)

Corollary

Let B be the Jacobian of the curve $C'/\mathbb{Q}$ in the proof of Theorem 6. Then B is paramodular of level 193^2.

Remark

In their paper, Brumer and Kramer remarked that Conjecture 2 should be verifiable by current technology for paramodular abelian surfaces B over $\mathbb{Q}$ with $\mathrm{End}_{\overline{\mathbb{Q}}}(B) \supsetneq \mathbb{Z}$. The majority of the surfaces we found fall in Case II (a), and provide such evidence.

Data

D	RM
53	$8^{(c)}$
61	12
73	$5^{(c)}$
193	17
233	17
277	29
349	21

D	RM
353	$5^{(*)}$
373	93
389	8
397	24
409	13
433	12
461	$5^{(c)}, 5^{(*)}, 29$

D	RM
613	21
677	$13, 29, 85$
709	5
797	$8, 29$
809	5
821	44
853	21

D	RM
929	13
997	13

Conventions:

Case I: $(^{c})$ means that the form f is a base change from $\mathbb{Q}$.

Case II (b): (*) means that both the form f and its theta lift are not base change.

An incomplete example over $\mathbb{Q}(\sqrt{-223})$

Consider the polynomials

$$P := -8x^6 + (54w - 27)x^5 + 9103x^4 + (-14200w + 7100)x^3 \\ - 697185x^2 + (326468w - 163234)x + 3539399$$

$$Q := x^3 + (2w - 1)x^2 - x,$$

where $w = \frac{1+\sqrt{-223}}{2}$.

The curve $C : y^2 + Q(x)y = P(x)$ has discriminant 1, and its Jacobian $J(C)$ has real multiplication by an $\mathbb{Z}[\sqrt{2}]$.

Question: Show that $J(C)$ is modular and corresponds to a Bianchi modular form.