Dimension formulas for vector-valued Hilbert modular forms

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- Jacobi forms over number fields
  - Same type of correspondence as over $\mathbb{Q}$ (between scalar and vector-valued)
  - Liftings between Hilbert modular forms and Jacobi forms (Shimura lift)

- $K/\mathbb{Q}$ number field of degree n
- Embeddings: $\sigma_i : K \to \mathbb{R}$, $1 \leq i \leq n$,
- Trace and norm:

$$\mathrm{Tr}\alpha = \sum \sigma_i \alpha, \quad \mathrm{N}\alpha = \prod \sigma_i \alpha.$$

- If $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(K)$ we write $A_{\sigma_i} = \begin{pmatrix} \sigma_i(\alpha) & \sigma_i(\beta) \\ \sigma_i(\gamma) & \sigma_i(\delta) \end{pmatrix}$.

There are two important lattices related to K:

- $\mathcal{O}_K$ the ring of integers with integral basis $1 = \alpha_1, \alpha_2, \ldots \alpha_n$

$$\mathcal{O}_K \simeq \alpha_1 \mathbb{Z} \oplus \cdots \oplus \alpha_n \mathbb{Z},$$

- $\mathcal{O}_K^\times$ the unit group with generators $\pm 1, \varepsilon_1, \ldots, \varepsilon_{n-1}$

$$\mathcal{O}_K^\times \quad \simeq \quad \langle \pm 1 \rangle \times \langle \varepsilon_1 \rangle \times \cdots \langle \varepsilon_{n-1} \rangle$$

- Λ the logarithmic unit lattice: $v_i = (\ln|\sigma_1 \varepsilon_i|, \ldots, \ln|\sigma_{n-1} \varepsilon_i|)$

$$\Lambda = v_1 \mathbb{Z} \oplus \cdots \oplus v_{n-1} \mathbb{Z}.$$

  The "volume" of Λ is called the regulator $\mathrm{Reg}(K)$.

- The volume of $\mathcal{O}_K$ is $|d_K|^{\frac{1}{2}}$, d_K is the discriminant of K.

More preliminary notation

- Define the ring $\mathbb{C}_K := \mathbb{C} \otimes_{\mathbb{Q}} K$
  - Multiplication:
    $$(z \otimes a, w \otimes b) \mapsto (zw \otimes ab)$$
    Algebra structure over $\mathbb{C}$ and K by identifications $K = 1 \otimes_{\mathbb{Q}} K$ and $\mathbb{C} = \mathbb{C} \otimes_{\mathbb{Q}} 1$
- Also $\mathbb{R}_K := \mathbb{R} \otimes_{\mathbb{Q}} K$ as a subring of $\mathbb{C}_K$.
- Imaginary part (similarly for real part):
  $$\Im(z \otimes a) = \Im(z) \otimes a,$$
- Extend embeddings:
  $$\sigma(z \otimes a) = z\sigma(a)$$
- For $x \in \mathbb{R}$ we say that $x \otimes a$ is totally positive, $x \otimes a \gg 0$ if
  $$\sigma_i(x \otimes a) > 0,\ i = 1, 2$$

In $\mathbb{Q}(\sqrt{5})$ we have the fundamental unit ε and its conjugate ε^*:

$$\varepsilon_0 = \frac{1}{2}\left(1+\sqrt{5}\right), \quad \varepsilon^* = -\varepsilon_0^{-1} = \frac{1}{2}\left(1-\sqrt{5}\right).$$

And

$$\begin{aligned} \mathcal{O}_K &\simeq \mathbb{Z} + \varepsilon_0 \mathbb{Z}, \\ \Lambda &\simeq \mathbb{Z} \ln \left| \frac{1+\sqrt{5}}{2} \right| \end{aligned}$$

with the volume given by

$$\begin{aligned} |\mathcal{O}_K| &= \left| \det \begin{pmatrix} \frac{1}{2}\left(1+\sqrt{5}\right) & \frac{1}{2}\left(1-\sqrt{5}\right) \\ 1 & 1 \end{pmatrix} \right| = \sqrt{5} \\ |\Lambda| &= \left| \ln \frac{1}{2}\left(1+\sqrt{5}\right) \right| \simeq 0.4812\ldots \end{aligned}$$

- For $r \in \mathbb{R}_K$ and $z \in \mathbb{C}_K$ we define $z^r \in \mathbb{C}_K$ by

$$\sigma(z^r) = \exp\left(i\sigma(r)\operatorname{Arg}\sigma(z) + \sigma(r)\log|\sigma(z)|\right), \quad \forall\sigma$$

- Subgroups

$$\mathrm{SL}_2(K) \subseteq \mathrm{SL}(2, \mathbb{R}_K) \subseteq \mathrm{SL}(2, \mathbb{C}_K)$$

- Generalized upper half-plane

$$\mathbb{H}_K = \{z \in \mathbb{C}_K \,:\, \Im(z) \gg 0\}.$$

- Action by $\mathrm{SL}(2, \mathbb{R}_K)$ on $\mathbb{H}_K$:

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = (az+b)(cz+d)^{-1}.$$

The Hilbert modular group:

$$\Gamma_K = \mathrm{SL}_2(\mathcal{O}_K) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), a, b, c, d \in \mathcal{O}_K, ad - bc = 1 \right\}$$

If $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_K$ and $\tau \in \mathbb{H}_K$ then

$$A\tau := (a\tau + b)(c\tau + d)^{-1} \in \mathbb{H}_K.$$

Cusps of $\mathrm{SL}_2(\mathcal{O}_K)$

- Cusp: $\lambda = (\rho : \sigma) \in \mathbb{P}_1(K)$
- Fractional ideal $\mathfrak{a}_\lambda = (\rho, \sigma)$
- Known: $\lambda \sim \mu \pmod{\mathrm{SL}_2(\mathcal{O}_K)} \Leftrightarrow \mathfrak{a}_\lambda = (\alpha)\,\mathfrak{a}_\mu$
- The number of cusp classes equals the class number of K.
- Cusp-normalizing map: $\exists \xi, \eta \in \mathfrak{a}_\lambda^{-1}$ s.t.

$$\begin{aligned} A_\lambda &= \begin{pmatrix} \rho & \xi \\ \sigma & \eta \end{pmatrix} \in \mathrm{SL}_2(K), \\ A_\lambda^{-1}\mathrm{SL}_2(\mathcal{O}_K)A_\lambda &= \mathrm{SL}_2\left(\mathfrak{a}^2 \oplus \mathcal{O}_K\right) \end{aligned}$$

- Let V be a complex $\mathrm{SL}_2(\mathcal{O}_K)$-module of rank $d < \infty$ s.t.
  - the kernel of V is a finite index normal subgroup Γ.
  - $\alpha \in Z(\mathrm{SL}_2(\mathcal{O}_K))$ acts with multiplication by $1|_k\alpha$.
- Denote the action by $(\gamma, v) \mapsto \gamma.v$
- For $f \in \mathcal{O}(\mathbb{H}_K, V)$ and $A \in \mathrm{SL}_2(\mathcal{O}_K)$ we define $(A.f)(z) = A.(f(z))$

- Define

$$M_k(V) = \{f \in \mathcal{O}(\mathbb{H}_K, V), A.f = f|_k A, \forall A \in \mathrm{SL}_2(\mathcal{O}_K)\}$$

- If $f \in M_k(V)$ and $f = \sum f_i v_i$ then $f_i \in M_k(\Gamma)$ (scalar-valued)

$$S_k(V) = \left\{ f = \sum f_i v_i \in M_k(V), : f_i \in S_k(\Gamma) \right\}$$

If $k \in \mathbb{Z}^n$ with $k \gg 2$ then:

$$\begin{aligned} \dim S_k(V) &= \frac{1}{2}\dim V \cdot \zeta_K(-1) \cdot \mathrm{N}(k-1) \\ &+\text{"elliptic order terms"} \\ &+\text{"parabolic terms} \end{aligned}$$

- Identity (main) term: $\zeta_K(-1)$ (a rational number)
  - Example: $\zeta_{\mathbb{Q}(\sqrt{5})} = \frac{1}{30}$, $\zeta_{\mathbb{Q}(\sqrt{193})}(-1) = 16 + \frac{1}{3}$, $\zeta_{\mathbb{Q}(\sqrt{1009})}(-1) = 211$.
- Finite order (“elliptic”) terms
- Parabolic (“cuspidal”) term

$$\text{"elliptic terms"} = \sum_{\mathfrak{U}} \frac{1}{|\mathfrak{U}|} \sum_{\pm 1 \neq A \in \mathfrak{U}} \chi_V(A) \cdot E(A)$$

here $\mathfrak{U}$ runs through elliptic conjugacy classes and

$$\begin{aligned} \chi_V(A) &= \mathrm{Tr}(A, V), \\ E(A) &= \prod_{\sigma} \frac{\rho(A_\sigma)^{1-k_\sigma}}{\rho(A_\sigma) - \rho(A_\sigma)^{-1}}, \\ \rho(A) &= \frac{1}{2}\left(t + \mathrm{sgn}(c)\sqrt{t^2-1}\right), \; t = \mathrm{Tr}A \end{aligned}$$

The cuspidal contribution is the value at $s=1$ of the twisted Shimizu L-series

$$L(s;\mathcal{O}_K,V)=\frac{\sqrt{|d_K|}}{(-2\pi i)^2}\sum_{0\neq a\in\mathcal{O}_K/U^2}\chi_{\overline{V}}\left(\left(\begin{smallmatrix}1&a\\0&1\end{smallmatrix}\right)\right)\frac{\operatorname{sgn}(\mathrm{N}(a))}{|\mathrm{N}(a)|^s}.$$

- The “untwisted” L-series ($V=1$) is known to have analytic cont. and functional equation

$$\Lambda(s)=\Gamma\left(\frac{s+1}{2}\right)^n\left(\frac{\operatorname{vol}(\mathcal{O}_K)}{\pi^{n+1}}\right)^s L(s;\mathcal{O}_K,1)=\Lambda(1-s)$$

- It is easy to see that the L-function for $V\neq 1$ also has AC. FE is more complicated (cf. Hurwitz-Lerch).
- If K has a unit of norm -1 then $L(s;\mathcal{O}_K,1)=0$ (conditions on V in general)

- Note that $L(s; O_K, 1)$ is proportional to

$$L(s,\chi) = \sum_{0 \neq \mathfrak{a} \subseteq O_K} \frac{\chi(\mathfrak{a})}{|\mathrm{N}(\mathfrak{a})|^s}$$

  where the sum is over all integral ideals of O_K and $\chi(\mathfrak{a}) = \mathrm{sgn}(\mathrm{N}(\mathfrak{a}))$.
- Studied by Hecke, Siegel, Meyer, Hirzebruch and others.
- Can be expressed in terms of Dedekind sums (Siegel)
- Proof uses Kronecker's limit formula.

- The proof goes in essentially the same way as the "usual" Eichler-Selberg trace formula.

- Scalar if $A = \pm 1$
- Elliptic: A has finite order.
- Parabolic: If A is not scalar but $\mathrm{Tr} A = \pm 2$.
- Mixed (these do not contribute to the dimension formula).

Let $A \in \mathrm{SL}_2(K) \setminus \{\pm 1\}$ have trace t. Then TFAE

- A is of finite order m
- $\sigma(A)$ is elliptic in $\mathrm{SL}_2(\mathbb{R})$ for every embedding σ.
- $t = z + z^{-1}$ for an m-th root of unity z

In this case $\mathbb{Q}(t)$ is the totally real subfield of $\mathbb{Q}(z)$ and

$$2\,[\mathbb{Q}(t) : \mathbb{Q}] = \varphi(m)$$

where $[\mathbb{Q}(t) : \mathbb{Q}]$ divides the degree of K since $t \in K$.

If $K = \mathbb{Q}\left(\sqrt{D}\right)$ then the possible orders are:

- $3, 4, 6$ (solutions of $\varphi(l) = 2$), and
- $5, 8, 10, 12$ (solutions of $\varphi(l) = 4$)

Lemma

Let $\mathfrak{a}$ be a fractional ideal and $t \in K$ be such that $K\left(\sqrt{t^2-4}\right)$ is a cyclotomic field. Then

$$A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \lambda(A) = \frac{a-d+\sqrt{t^2-4}}{2c}$$

defines a bijection between the set of elements of $SL_2(\mathfrak{a} \oplus \mathcal{O}_K)$ with trace t and

$$\left\{ z = \frac{x+\sqrt{t^2-4}}{2y} \in \mathbb{H}_K \,:\, x \in \mathcal{O}_K, \, y \in \mathfrak{a}, \; x^2 - t^2 + 4 \in 4\mathcal{O}_K \right\}.$$

Key:

- Can compute set of representatives for elliptic fixed points
- Explicit bound on the x, y which can appear.

Distance to a cusp

- Distance to infinity

$$\Delta(z,\infty) = \mathrm{N}(y)^{-\frac{1}{2}}$$

- Distance to other cusps

$$\Delta(z,\lambda) = \Delta\left(A_{\lambda}^{-1} z, \infty\right).$$

- λ is a closest cusp to z if

$$\Delta(z,\lambda) \leq \Delta(z,\mu), \quad \forall \mu \in \mathbb{P}^1(K).$$

Reduction algorithm for $z \in \mathbb{H}_K$

- Find closest cusp λ and set $z^* = x^* + iy^* = A_\lambda^{-1} z$.
- z^* is $\mathrm{SL}_2(\mathcal{O}_K)$-reduced if it is Γ_∞-reduced, where

$$\Gamma_\infty = \left\{ \begin{pmatrix} \varepsilon & \mu \\ 0 & \varepsilon^{-1} \end{pmatrix}, \varepsilon \in \mathcal{O}_K^\times, \mu \in \mathcal{O}_K \right\}.$$

- Local coordinate (wrt. lattices Λ and $\mathcal{O}_K$):

$$\begin{aligned} \Lambda Y &= \tilde{y} \\ B_{\mathcal{O}_K} X &= x^* \end{aligned}$$

where $\tilde{y}_i = \ln \frac{y_i^*}{\sqrt[n]{\mathrm{N} y^*}}$.

- Then z is $\mathrm{SL}_2(\mathcal{O}_K)$-reduced iff

$$|Y_i| \leq \frac{1}{2},\ 1 \leq i \leq n-1, \quad |X_i| \leq \frac{1}{2},\ 1 \leq i \leq n.$$

- If z not reduced we can reduce:
  - Y_i by acting with $\varepsilon = \varepsilon_i^k \in \mathcal{O}_K^\times$:

$$U(\varepsilon) = A_\lambda^{-1} \left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{smallmatrix}\right) A_\lambda : z^* \mapsto \varepsilon_i^{2k} z^*,\ Y_i \mapsto Y_i + k.$$

  - X by acting with $\zeta = \sum a_i \alpha_i \in \mathcal{O}_K$:

$$T(\zeta) = A_\lambda^{-1} \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix} A_\lambda : z^* \mapsto z^* + \zeta,\ X_i \mapsto X_i + a_i.$$

Remarks

- Once in a cuspidal neighbourhood reduce in constant time.
- The hard part is to find the closest cusp.
- Elliptic points are on the boundary, i.e. can have more than one “closest” cusp.

- Let $z \in \mathbb{H}_K$ and $\lambda = \frac{a}{c} \in \mathbb{P}^1(K)$.
- Then
$$\Delta(z,\lambda)^2 = \mathrm{N}(y)^{-1}\,\mathrm{N}\left((-cx+a)^2 + c^2y^2\right).$$
- For each $r > 0$ there is only a finite (explicit!) number of pairs $(a', c') \in \mathcal{O}_K^2 / \mathcal{O}_K^\times$ s.t.
$$\Delta(z,\lambda') \le r.$$
- In fact, for $i = 1, \ldots, n$ we have bounds on each embedding:
$$\begin{aligned} |\sigma_i(c)| &\le c_K r^{\frac{1}{2}} \sigma_i\left(y^{-\frac{1}{2}}\right), \\ |\sigma_i(a - cx)|^2 &\le \sigma_i\left(r c_K^2 y - c^2 y^2\right) \end{aligned}$$
- Here c_K is an explicit constant.

Lemma

If $K/\mathbb{Q}$ is a number field and $\alpha \in K$ with $\mathrm{N}\alpha = 1$ then there exists $\varepsilon \in \mathcal{O}_K^\times$ such that

$$|\sigma_i(\alpha\varepsilon)| \leq r_K^{\frac{n-1}{2}}$$

where

$$r_K = \max_k \left\{ \frac{\max(|\sigma_1(\varepsilon_k)|, \ldots, |\sigma_n(\varepsilon_k)|, 1)}{\min(|\sigma_1(\varepsilon_k)|, \ldots, |\sigma_n(\varepsilon_k)|, 1)} \right\}.$$

Remark

$r_K \geq 1$ always. If $K = \mathbb{Q}\left(\sqrt{D}\right)$ has a f.u. ε_0 with $\sigma_1(\varepsilon_0) > 1 > \sigma_2(\varepsilon_0)$ then $r_K = |\sigma_1(\varepsilon_0)|^2$.

Example $\mathbb{Q}(\sqrt{5})$

- The orders which can appear are: 3, 4, 5, 6, 8, 10, 12
- The possible traces are:

m	t	
3	-1	
4	0	
5	$\frac{1}{2}\left(\sqrt{5}-1\right)$	$\frac{1}{2}\left(-\sqrt{5}-1\right)$
6	1	
8	-	
10	$\varepsilon_0 = \frac{1}{2}\left(\sqrt{5}+1\right)$	$\varepsilon_0^* = \frac{1}{2}\left(-\sqrt{5}+1\right)$
12	-	

Example (contd.)

A set of reduced fixed points is:

order	trace	fixed pt	ell. matrix
4	0	i	$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
4	0	$i\varepsilon_0^*$	$SE(\varepsilon^*) = \begin{pmatrix} 0 & \varepsilon_0^* \\ -\varepsilon_0^* & 0 \end{pmatrix}$
6	1	ρ	$TS = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$
6	1	$\rho\varepsilon_0^*$	$SE(\varepsilon_0)T^{\varepsilon^3} = \begin{pmatrix} 0 & \varepsilon_0^* \\ \varepsilon_0 & 1 \end{pmatrix}$
10	ε	$-\frac{1}{2}\varepsilon_0 + \frac{i}{2}\sqrt{3-\varepsilon_0}$	$ST^{\varepsilon_0} = \begin{pmatrix} 0 & -1 \\ 1 & \varepsilon_0 \end{pmatrix}$
10	ε^*	$\frac{1}{2}\varepsilon_0 + \frac{i}{2}\varepsilon_0^*\sqrt{3-\varepsilon_0^*}$	$T^{\varepsilon_0^*}S = \begin{pmatrix} \varepsilon_0^* & -1 \\ 1 & 0 \end{pmatrix}$

Here $\rho^3 = 1$ and we always choose “correct” Galois conjugates to get points in $\mathbb{H}^n$.

Example $\mathbb{Q}(\sqrt{3})$

	t	z_t	$\frac{1}{\sqrt{\mathrm{N}y}}$	Y	X_1	X_2		
$4a$	0	$\frac{-1+\sqrt{3}}{2} - i\frac{1+\sqrt{3}}{2}$	$\sqrt{2}$	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	
4	0	$\frac{-1+\sqrt{3}}{2} + i\frac{1-\sqrt{3}}{2}$	$\sqrt{2}$	$\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\sim 4a$
$4b$	0	$\varepsilon_0 i$	1	$-\frac{1}{2}$	0	0	0	
$4c$	0	i	1	0	0	0	0	
6	1	$\frac{1}{2} - i\left(1+\frac{\sqrt{3}}{2}\right)$	2	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$\sim 12a$
$6a$	1	$\frac{1}{2} + \frac{1}{2}i\sqrt{3}$	$\sqrt{\frac{4}{3}}$	0	$\frac{1}{2}$	0	0	
$6b$	1	$\frac{\sqrt{3}}{2} - i\left(\frac{1}{\sqrt{3}}+\frac{1}{2}\right)$	$\sqrt{\frac{4}{3}}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	-1	
$12a$	$-\sqrt{3}$	$\frac{1}{2}\sqrt{3} + \frac{1}{2}i$	2	0	0	$-\frac{1}{2}$	0	

We have two cusp classes: $c_0 = \infty = [1:0]$ and $c_1 = \left[3 : 1+\sqrt{10}\right]$

Orders: 4 (trace 0) and 6 (trace 1).

order	label	fixed pt	close to
4	$4a$	$\left(\frac{1}{2}\sqrt{10}+\frac{3}{2}\right)\sqrt{-4}^{\pm}$	∞
4	$4b$	$\frac{1}{2}\sqrt{-4} = i$	∞
4	$4c$	$\left(\frac{1}{4}\sqrt{10}-\frac{3}{4}\right)\sqrt{-4}^{\pm}+\frac{1}{2}$	∞
4	$4d$	$\frac{1}{2}\sqrt{10}-\frac{1}{2}+\frac{1}{4}\sqrt{-4}$	∞
4	$4e$	$\frac{5}{13}\sqrt{10}-\frac{1}{2}+\frac{1}{52}\sqrt{-4}$	c_1
4	$4f$	$\frac{129}{370}\sqrt{10}-\frac{86}{185}+\left(-\frac{3}{740}\sqrt{10}+\frac{1}{185}\right)\sqrt{-4}^{\pm}$	c_1

Here $\sqrt{-4}^{\pm} = \pm 2i$ with sign choosen depending on the embedding of $\sqrt{10}$.

Example $\mathbb{Q}(\sqrt{-10})$ order 4

label	x	$N(x)$	y	$N(y)$
$4a$	0	0	$\sqrt{10}-3$	-1
$4b$	0	0	-1	1
$4c$	$2\sqrt{10}+6$	-4	$2\sqrt{10}+6$	-4
$4d$	$-2\sqrt{10}+2$	-36	-2	4
$4e$	$-20\sqrt{10}+26$	-3324	-26	676
$4f$	-86	7396	$-15\sqrt{10}-20$	-1850

Example $\mathbb{Q}\left(\sqrt{-10}\right)$

Note that if A is the cuspnormalizing map of c_1 then

label	$A^{-1}z$	x	y
$4e$	$\left(-\frac{1}{9}\sqrt{10}-\frac{7}{18}\right)\sqrt{-4}$	0	7
$4f$	$\left(\frac{-1}{36}\sqrt{10}+\frac{1}{36}\right)\sqrt{-4}^{\pm}+\frac{1}{2}$	$-2\sqrt{10}-2$	$-2\sqrt{10}-2$

Given elliptic element A:

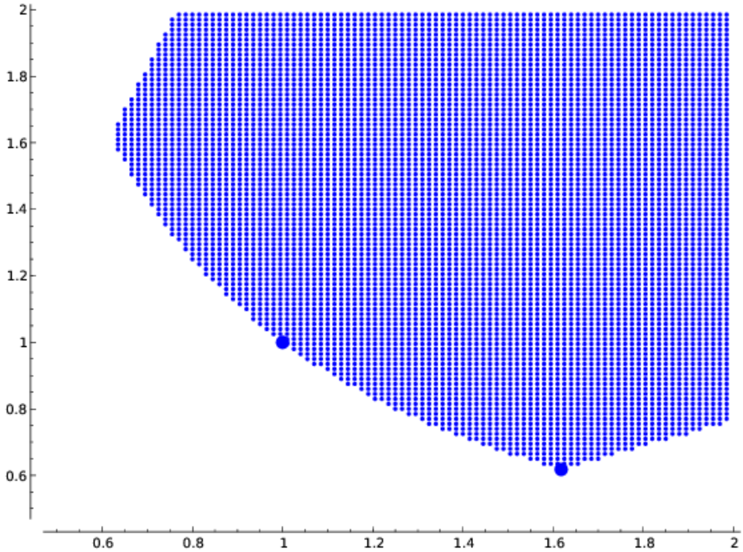
- Find fixed point z
- Set $z_0 = z + \varepsilon$ s.t. $z_0 \in \mathcal{F}_\Gamma$ (well into the interior).
- $w_0 = Az_0$
- Find pullback of w_0 in to $\mathcal{F}_\Gamma$ (make sure $w_0^* = z_0$).
- Keep track of matrices used in pullback.

$K = \mathbb{Q}\left(\sqrt{3}\right)$, $z = \frac{-1+\sqrt{3}}{2} - i\frac{1+\sqrt{3}}{2}$ $A = \begin{pmatrix} -1 & -\sqrt{3}+1 \\ \sqrt{3}+1 & 1 \end{pmatrix}$

- $w_0 = Az_0 \sim$ (close to 0)
- $w_1 = Sw_0 \sim$ (close to $a-1$)
- $w_2 = ST^{1-a}w_1$
- $w_3 = T^{1+a}w_2$ – reduced
- $A = T^{1+a}ST^{a-1}S$ (as a map)
- $A = S^2T^{1+a}ST^{a-1}S$ (in $\mathrm{SL}_2(\mathcal{O}_K)$)

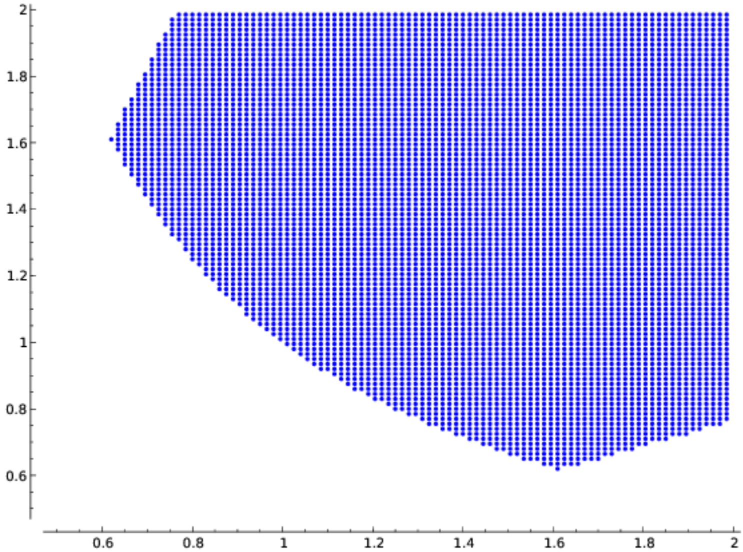
Section of a fundamental domain

$x = (0.00, 0.00)$



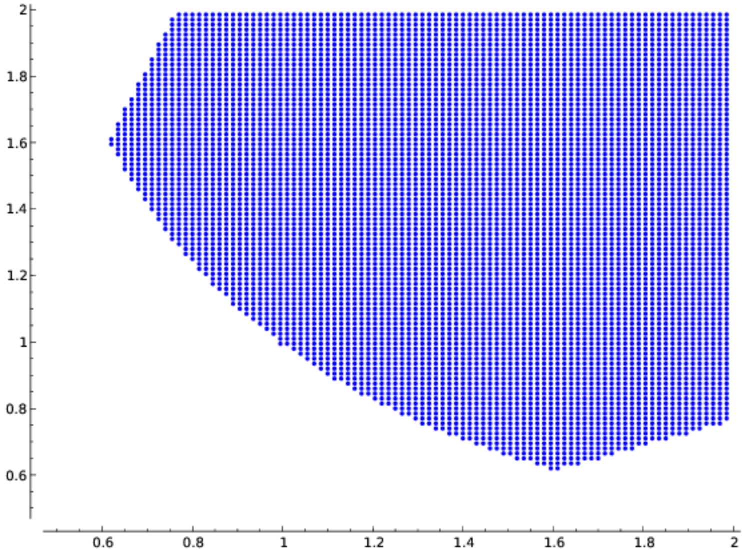
Section of a fundamental domain

$$x = (0.05, 0.05)$$



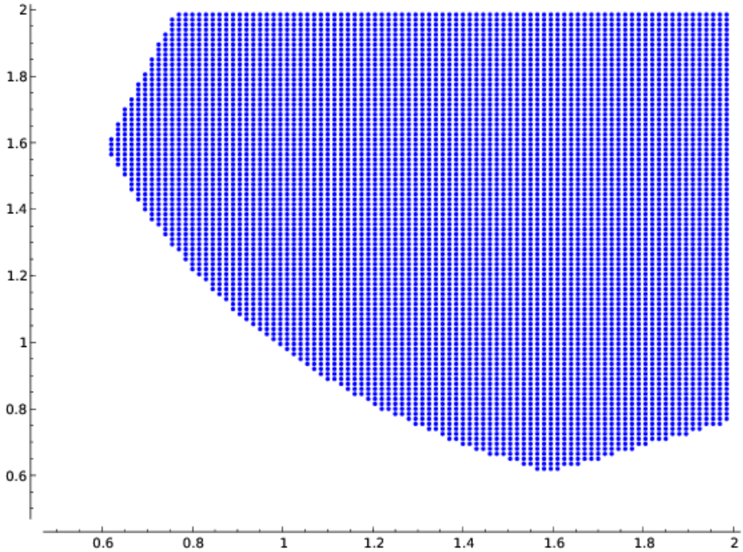
Section of a fundamental domain

$x = (0.10, 0.10)$



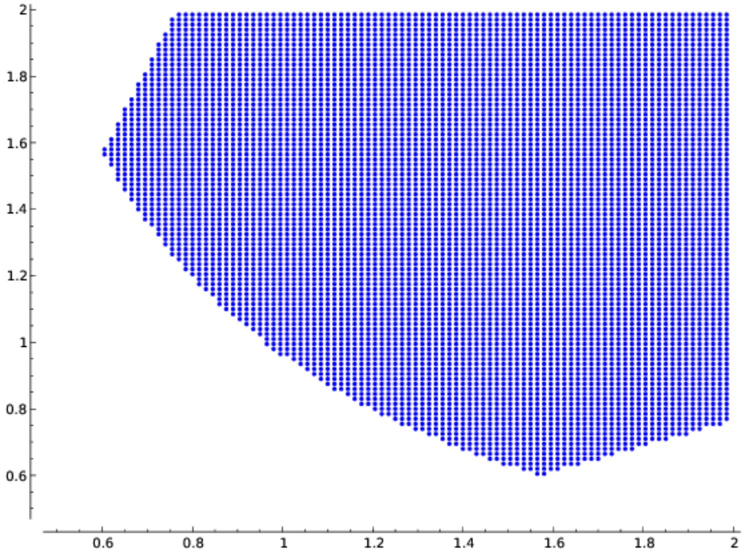
Section of a fundamental domain

$x = (0.15, 0.15)$



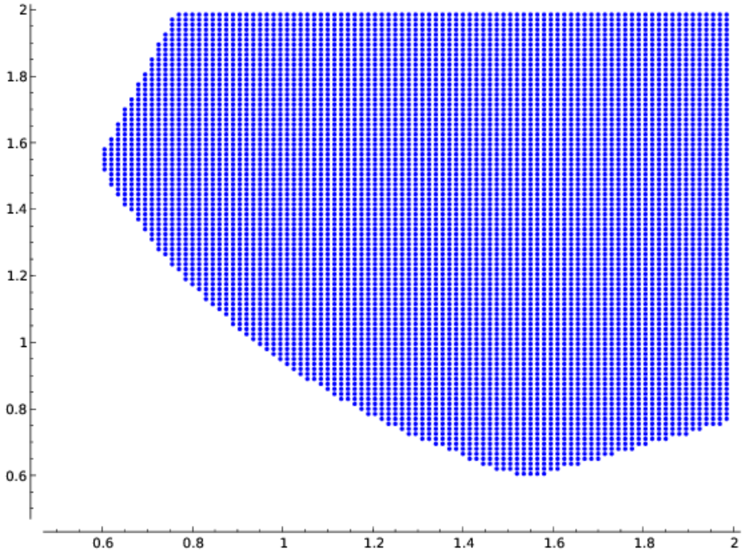
Section of a fundamental domain

$x = (0.20, 0.20)$



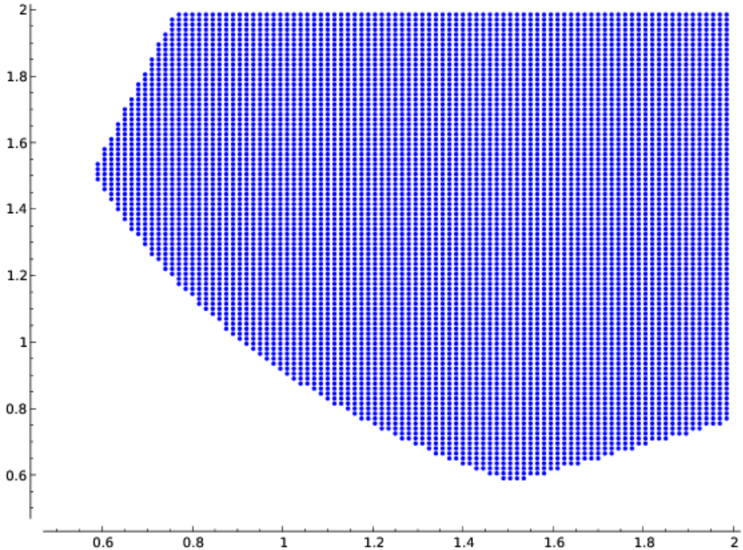
Section of a fundamental domain

$$x = (0.25, 0.25)$$



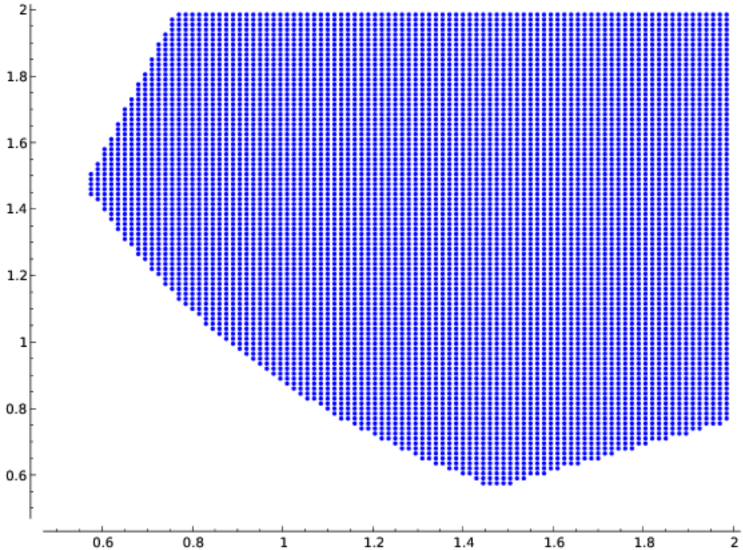
Section of a fundamental domain

$$x = (0.30, 0.30)$$



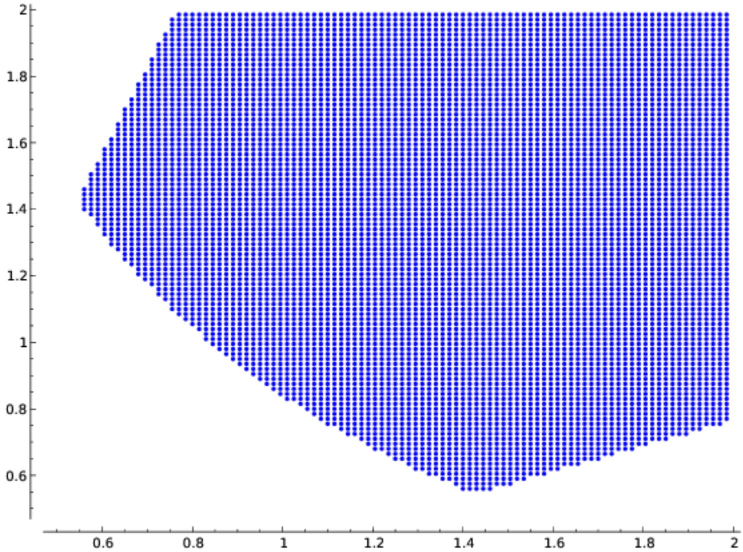
Section of a fundamental domain

$$x = (0.35, 0.35)$$



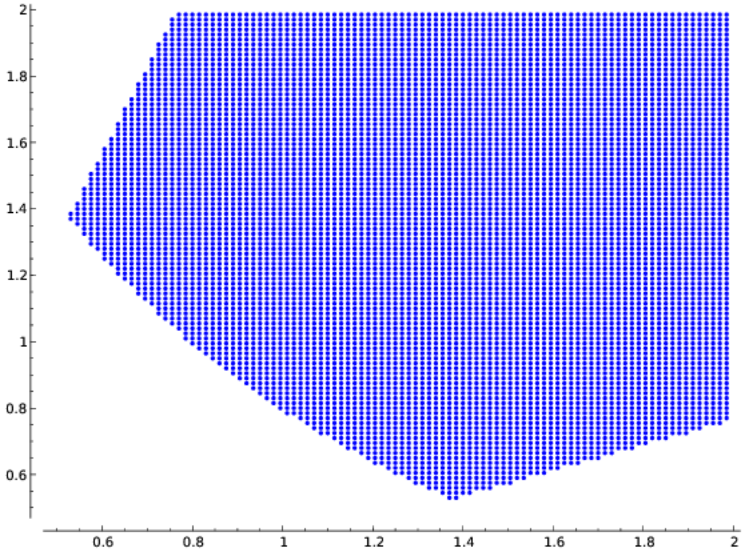
Section of a fundamental domain

$$x = (0.40, 0.40)$$



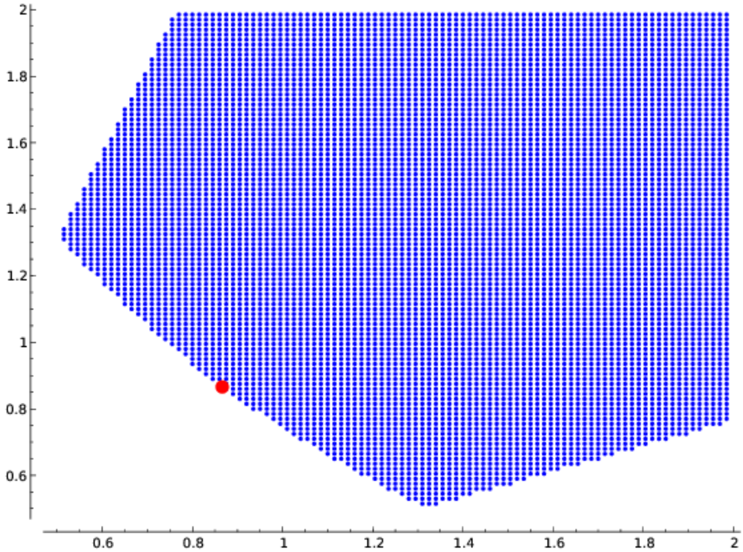
Section of a fundamental domain

$x = (0.45, 0.45)$



Section of a fundamental domain

$x = (0.50, 0.50)$



Elliptic points of order 4 and 10

$$x = (-0.3090\ldots, 0.8090\ldots)$$

er 10

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