Power series expansions for modular forms

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joint work with
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q-expansions

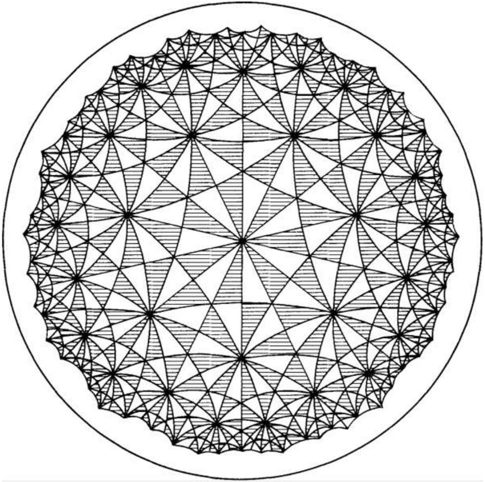
Let $f : \mathcal{H} \to \mathbb{C}$ be a classical modular form of weight $k \in 2\mathbb{Z}_{\geq 0}$ for $\Gamma_0(N)$. Then f satisfies the translation invariance $f(z+1) = f(z)$ for $z \in \mathcal{H}$, so f admits a Fourier expansion (or *q-expansion*)

$$f(z) = \sum_{n=0}^{\infty} a_n q^n$$

at the cusp ∞, where $q = e^{2\pi i z}$. If further f is a normalized eigenform for the Hecke operators T_n, then the coefficients a_n are the eigenvalues of T_n for n relatively prime to N.

Cocompact groups

Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ be a cocompact Fuchsian group.



Modular forms, no cusps!

A *modular form* f of weight $k \in 2\mathbb{Z}_{\geq 0}$ for a cocompact Fuchsian group Γ is a holomorphic map $f : \mathcal{H} \to \mathbb{C}$ satisfying

$$f(gz) = (cz + d)^k f(z)$$

for all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. We also write $j(g, z) = cz + d$.

As Γ is cocompact, the quotient $X = \Gamma \backslash \mathcal{H}$ has no cusps, so there are no q-expansions!

However, not all is lost: such a modular form f still admits a power series expansion in the neighborhood of a point $p \in \mathcal{H}$.

Power series expansions in unit disc

A q-expansion is really just a power series expansion at ∞ in the parameter q, convergent for $|q| < 1$. So it is natural to consider a neighborhood of p normalized so the expansion also converges in the unit disc $\mathcal{D}$ for a parameter w. So we map

$$\begin{aligned} w : \mathcal{H} &\to \mathcal{D} \\ z &\mapsto w(z) = \frac{z - p}{z - \overline{p}}. \end{aligned}$$

We then consider series expansions of the form

$$f(z) = (1 - w)^k \sum_{n=0}^{\infty} b_n w^n$$

where $w = w(z)$. The term

$$(1 - w(z))^k = \left(\frac{p - \overline{p}}{z - \overline{p}} \right)^k$$

is the automorphy factor arising by slashing by linear fractional transformation $w(z)$.

Example

Let $f \in S_2(\Gamma_0(11))$ be defined by

$$f(z) = q \prod_{n=1}^{\infty} (1-q^n)^2(1-q^{11n})^2 = q - 2q^2 - q^3 + 2q^4 + \ldots.$$

Consider expansions about the point $p = (-9+\sqrt{-7})/22 \in \mathcal{H}$, a CM point on $X_0(11)$ for $K = \mathbb{Q}(\sqrt{-7})$. From the q-expansion:

$$\begin{aligned} f(z) &= (1-w)^2 \sum_{n=0}^{\infty} b_n w^n = f(p)(1-w)^2 \sum_{n=0}^{\infty} \frac{c_n}{n!}(\Theta w)^n \\ &= -\sqrt{3+4\sqrt{-7}}\Omega^2(1-w)^2 \cdot \\ &\quad \cdot \left(1 + \Theta\omega + \frac{5}{2!}(\Theta w)^2 - \frac{123}{3!}(\Theta w)^3 - \frac{59}{4!}(\Theta w)^4 - \ldots\right) \end{aligned}$$

where

$$\Theta = \frac{-4+2\sqrt{-7}}{11}\pi\Omega^2$$

and $\Omega = 0.500491\ldots$ is the Chowla-Selberg period for K.

Arithmetic groups

Let F be a totally real number field with ring of integers $\mathbb{Z}_F$. Let B be a quaternion algebra over F with a unique split real place $\iota_\infty : B \hookrightarrow \mathrm{M}_2(\mathbb{R})$. Let $\mathcal{O} \subset B$ be an Eichler order of level $\mathfrak{N}$, let $\mathcal{O}_1^*$ be the group of units of reduced norm 1 in $\mathcal{O}$. Then the group

$$\Gamma = \Gamma_0^B(\mathfrak{N}) = \iota_\infty(\mathcal{O}_1^*/\{\pm 1\}) \subset \mathrm{PSL}_2(\mathbb{R})$$

is a Fuchsian group with $X = \Gamma\backslash\mathcal{H}$ of finite area.

If F has narrow class number 1, the space $M_k(\Gamma)$ of modular forms of weight k for Γ has an action of Hecke operators $T_\mathfrak{p}$ indexed by the prime ideals $\mathfrak{p} \nmid \mathfrak{D}\mathfrak{N}$.

Algebraicity

Let K be a totally imaginary quadratic extension of F that embeds in B, and let $\nu \in B$ be such that $F(\nu) \cong K$. Let $p \in \mathcal{H}$ be a fixed point of $\iota_\infty(\nu)$. Then we say p is a *CM point* for K.

Theorem (Shimura)

There exists $\Theta \in \mathbb{C}^\times$ such that for every CM point p for K, every congruence subgroup Γ commensurable with $\Gamma^B(1)$, and every $f \in M_k(\Gamma)$ with $f(p) \in (K^{\mathrm{ab}})^\times$, we have for all $n \in \mathbb{Z}_{\geq 0}$ that

$$\frac{b_n(f)}{\Theta^n} \in K^{\mathrm{ab}}.$$

Rodriguez-Villegas and Zagier link these coefficients to square roots of central values of the Rankin-Selberg L-function $L(s, f \times \theta^n)$, where θ is associated to a Hecke character for K. Many authors have pursued this further, including O'Sullivan-Risager, Bertolini-Darmon-Prasanna, Mori, ...

Our result

We exhibit a general method for numerically computing power series expansions of modular forms for cocompact Fuchsian groups. Our method has generalizations to a wide variety of settings (noncongruence groups, real analytic modular forms, higher dimensional groups) and applies equally well for arithmetic Fuchsian groups over any totally real field F.

(There is another recent method, due to Nelson, which directly computes the Shimizu lift of a modular form on a Shimura curve over $\mathbb{Q}$ to a classical modular curve!)

Our method is inspired by the method of Stark and Hejhal, who used the same basic principle to compute Fourier expansions for Maass forms on $\mathrm{SL}_2(\mathbb{Z})$ and the Hecke triangle groups.

Basic idea

Let Γ be a cocompact Fuchsian group. Let $D \subset \mathcal{D}$ be a fundamental domain for Γ contained in a circle of radius $\rho > 0$. Let $f \in S_k(\Gamma)$. We consider an approximation

$$f(z) \approx f_N(z) = (1-w)^k \sum_{n=0}^{N} b_n w^n$$

valid for all $|w| \leq \rho$ to some precision $\epsilon > 0$.

For a point $w = w(z) \notin D$, there exists $g \in \Gamma$ such that $z' = gz \in D$; by the modularity of f we have

$$f_N(z') \approx f(z') = j(g,z)^k f(z)$$

$$(1-w')^k \sum_{n=0}^{N} b_n (w')^n \approx j(g,z)^k (1-w)^k \sum_{n=0}^{N} b_n w^n,$$

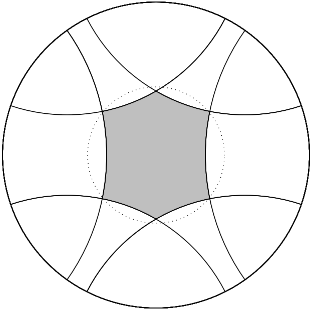
imposing a (nontrivial) linear relation on the unknowns b_n.

Better idea

Use the Cauchy integral formula:

$$b_n = \frac{1}{2\pi i} \oint \frac{f(z)}{w^{n+1}(1-w)^k}\, dw.$$

We take the contour to be a circle of radius ρ, apply automorphy, and again obtain linear relations among the coefficients b_n.



Computing a fundamental domain

For a point $p \in \mathcal{H}$, we denote by $\Gamma_p = \{g \in \Gamma : g(p) = p\}$ the stabilizer of p in Γ.

Theorem (V)

There exists an algorithm that, given as input a cocompact Fuchsian group Γ and a point $p \in \mathcal{H}$ with $\Gamma_p = \{1\}$, computes as output a fundamental domain $D(p) \subset \mathcal{H}$ for Γ and an algorithm that, given $z \in \mathcal{H}$ returns a point $z' \in D(p)$ and $g \in \Gamma$ such that $z' = gz$.

The fundamental domain $D(p)$ is the *Dirichlet domain*

$$D(p) = \{z \in \mathcal{H} : d(z,p) \leq d(gz,p) \text{ for all } g \in \Gamma\}$$

where d is the hyperbolic distance. The set $D(p)$ is a closed, connected, and hyperbolically convex domain whose boundary consists of finitely many geodesic segments.

Confirming the output

Although (at the moment) our results are not provably correct, there are several tests that allow one to be quite convinced that they are correct. (See also Booker-Strömbergsson-Venkatesh.)

First we simply decrease the error ϵ and see if the coefficients b_n converge. The second is to look at the singular values to see that the approximately nonzero eigenvalues are sufficiently large.

More seriously, we can also verify that f is modular at point $w \notin D$ with $|w| \leq \rho$. This shows that the computed expansion transforms like a modular form of weight k for Γ.

Finally, when f is an eigenform for a congruence group Γ, we can check that f is indeed numerically an eigenform (with the right eigenvalues) and that the normalized coefficients appear to be algebraic using the LLL-algorithm.

Cubic example

Let $F = \mathbb{Q}(a)$ where $h(a) = a^3 - a^2 - 4a + 1 = 0$. Then F is a totally real cubic field of discriminant $321 = 3 \cdot 107$ and ring of integers $\mathbb{Z}_F = \mathbb{Z}[a]$. F is not Galois; the Galois group of h is S_3. The narrow class number of F is $\#\operatorname{Cl}^+ \mathbb{Z}_F = 1$.

Let $B = \left(\dfrac{-1, b}{F}\right)$ where $b = a^2 - a - 4 \in \mathbb{Z}_F^\times$, so that B is generated by i, j subject to

$$i^2 = -1, \quad j^2 = b, \quad \text{and} \quad ji = -ij.$$

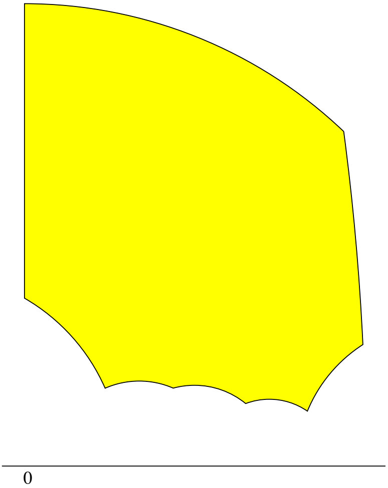
Then the quaternion algebra B is ramified only at two of three real places and no finite place.

We compute a maximal order

$$\mathcal{O} = \mathbb{Z}_F \oplus \mathbb{Z}_F i \oplus \mathbb{Z}_F \left(\frac{1 + a^2 i + j}{2}\right) \oplus \mathbb{Z}_F \left(\frac{a^2 + i + ij}{2}\right).$$

Fundamental domain

The group of units Γ with respect to an embedding $\iota : B \hookrightarrow \mathrm{M}_2(\mathbb{R})$ is a group of signature $(0; 2, 3, 3, 3)$: in particular, the quotient $X = \mathcal{H}/\Gamma$ has genus 0.



Power series expansions

We choose the center p to be CM point on X of class number 1, a fixed point of an element μ satisfying

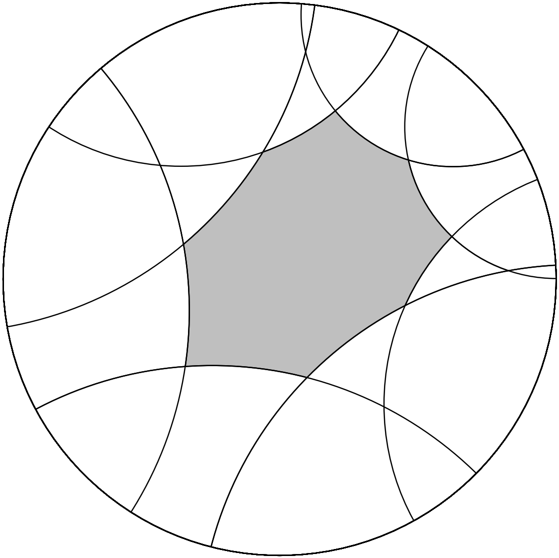
$$\mu^2 + (-a^2 - a)\mu + (2a^2 + 3a) = 0$$

with discriminant $-(a+3)$ and norm discriminant -23:

$$\mu = \frac{a^2 + a}{2} + \frac{-a+1}{2} i + \frac{a-1}{2} j - \frac{a}{2} ij$$

and so $p = 0.517256587\ldots + (0.28484776\ldots)\sqrt{-1}$.

Fundamental domain II



Uniformizing function

For $k \in 2\mathbb{Z}_{\geq 0}$, define

$M_k(\Gamma) = S_k(\Gamma) = \{f : \mathcal{H} \to \mathbb{C} \mid f(\gamma z) = (cz+d)^k f(z) \text{ for all } \gamma \in \Gamma\}$.

Then the canonical ring of X is

$$S(\Gamma) = \bigoplus_k S_k(\Gamma) = \frac{F[f_4, g_6, h_6]}{(r(f_4, g_6, h_6))}$$

where

$$r(f_4, g_6, h_6) = f_4^3 g_6 - (g_6^3 + (7a^2 + 4a - 20)g_6^2 h_6 + (-179a^2 + 196a + 871)g_6 h_6^2 \\ + (-2048a^2 + 1600a + 8640)h_6^3)$$

Anyway, $S(\Gamma) \cong F[g, h]$ so $X \cong \mathbb{P}^1$ by the map $j = h/g$.

Uniformizing function

$$
\begin{aligned}
j(z) &= \sum_{n=0}^{\infty} \frac{c_n}{n!}\left(\Theta\frac{z-p}{z-\overline{p}}\right)^n = \sum_{n=0}^{\infty}\frac{c_n}{n!}(\Theta w)^n \\
&= (a-1)(\Theta w) + \frac{2a^2+14a-10}{2!}(\Theta w)^2 \\
&\quad + \frac{154a^2+256a-578}{3!}(\Theta w)^3 \\
&\quad + \frac{1544a^2+7688+728}{4!}(\Theta w)^4 \\
&\quad + \frac{142960a^2+157120a-301400}{5!}(\Theta w)^5 + \dots
\end{aligned}
$$

where

$$\Theta = 0.3613682530143011\ldots - (0.334606902934795\ldots)\sqrt{-1}.$$

The method to compute these power series expansions is joint work with John Willis and the computation of the ring of modular forms is thesis work of Michael Klug.

Jugendtraum for CM extensions of F

Consider the CM extension $K = F(\sqrt{d})$ where $d = -3a^2 + 10a - 7$; then $\mathbb{Z}_K$ has discriminant d and $N(d) = -87$. We have $\mathrm{Cl}\,\mathbb{Z}_K \cong \mathbb{Z}/2\mathbb{Z}$. We find an element $\nu \in \mathcal{O}$ such that ν has minimal polynomial of discriminant d. Its fixed point w_ν has $j(w_\nu)$ satisfying the equation

$$j(w_\nu)^2 + (5a+5)j(w_\nu) + (-19a^2 + 18a + 89) = 0$$

and $H = K(j(w_\nu)) = K(\sqrt{13a^2 - 9a - 53})$ is the Hilbert class field of K.

Example

Let $F = \mathbb{Q}(a) = \mathbb{Q}(\sqrt{5})$ where $a^2 + a - 1 = 0$, and let $\mathbb{Z}_F$ be its ring of integers. Let $\mathfrak{p} = (5a + 2)$, so $N\mathfrak{p} = 31$. Let B be the quaternion algebra ramified at $\mathfrak{p}$ and the real place sending $\sqrt{5}$ to its positive real root: we take $B = \left(\dfrac{a, 5a+2}{F}\right)$.

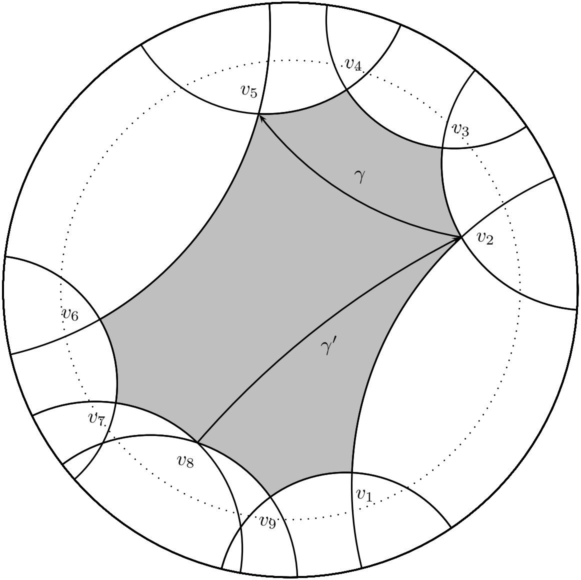
As before, we compute a maximal order $\mathcal{O} \subset B$. Then $\Gamma = \Gamma_0^B(1)$ has signature $(1; 2^2)$, so $X = \Gamma\backslash\mathcal{H}$ can be given the structure of a compact Riemann surface of genus 1. The space $M_2(\Gamma)$ of modular forms on Γ of weight 2 is 1-dimensional.

The field $K = F(\sqrt{-7})$ embeds in $\mathcal{O}$ with

$$\mu = -\frac{1}{2} - \frac{5a+10}{2}i - \frac{a+2}{2}j + \frac{3a-5}{2}ij \in \mathcal{O}$$

and $\mathbb{Z}_F[\mu] = \mathbb{Z}_K$ the maximal order with class number 1. We take $p = -3.1653\ldots + 1.41783\ldots \in \mathcal{H}$ to be the fixed point of μ.

Fundamental domain



An expansion for the form

$$f(z) = (1-w)^2 \left(1 + (\Theta w) - \frac{70a+114}{2!}(\Theta w)^2 - \frac{8064a+13038}{3!}(\Theta w)^3 + \frac{174888a+282972}{4!}(\Theta w)^4 - \frac{13266960a+21466440}{5!}(\Theta w)^5 - \frac{1826784288a+2955799224}{6!}(\Theta w)^6 - \frac{2388004416a+3863871648}{7!}(\Theta w)^7 + \dots\right)$$

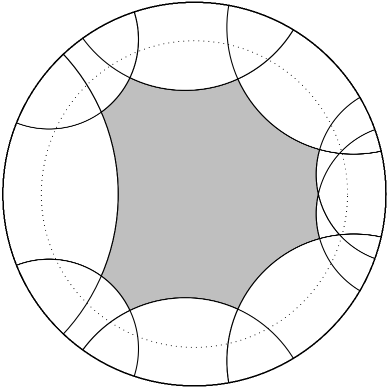
where

$$\Theta = 0.0462185795292084999 18\ldots - 0.0759873175318325683 51\ldots i$$

is a period related to the CM abelian variety given by the point p.

The conjugate curve

We further compute the other embedding of this form by repeating the above with an algebra ramified at $\mathfrak{p}$ and the other real place.



The coefficients agree with the conjugates under the nontrivial element of $\mathrm{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q})$.

Finding an equation

We can identify the equation of the Jacobian J of the curve X by computing the associated periods. We first identify the group Γ using the sidepairing relations coming from the computation of $D(p)$:

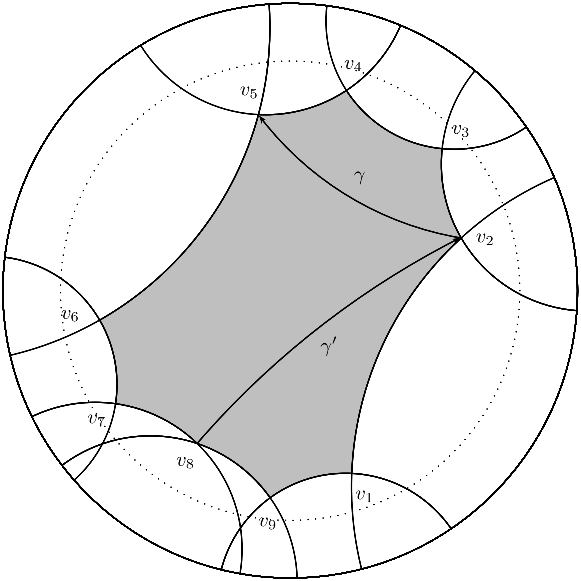
$$\Gamma \cong \langle \gamma, \gamma', \delta_1, \delta_2 \mid \delta_1^2 = \delta_2^2 = \gamma^{-1}\gamma'^{-1}\delta_1\gamma\gamma'\delta_2 = 1 \rangle$$

where

$$\begin{aligned} \gamma &= \frac{a+2}{2} - \frac{2a+3}{2}\alpha + \frac{a+1}{2}\alpha\beta \\ \gamma' &= \frac{2a+3}{2} + \frac{7a+10}{2}\alpha + \frac{a+2}{2}\beta - (3a+5)\alpha\beta \end{aligned}$$

generate the free part of the maximal abelian quotient of Γ.

Fundamental domain again



Finding an equation

Therefore, we compute two independent periods ω_1, ω_2

$$\omega_1 = \int_{v_2}^{v_5} f(z)\frac{dw}{(1-w)^2} \approx \left(\sum_{n=0}^{N} \frac{b_n}{n+1} w^{n+1}\right)\Bigg|_{v_2}^{v_5}$$

$$= -0.654017\ldots + 0.397799\ldots i$$

$$\omega_2 = \int_{v_8}^{v_2} f(z)\frac{dw}{(1-w)^2} = 0.952307\ldots + 0.829145\ldots i.$$

We then compute the j-invariant

$$j(\omega_1/\omega_2) = -18733.423\ldots$$

$$= -\frac{118896117223833394a + 8629385062119691}{31^8}.$$

We identify the elliptic curve J as

$$y^2 + xy - ay = x^3 - (a-1)x^2 - (31a+75)x - (141a+303).$$

Heegner point

Finally, we compute the image on J of a degree zero divisor on X.

The fixed points w_1, w_2 of the two elliptic generators δ_1 and δ_2 are CM points of discriminant -4. Let $K = F(i)$ and consider the image of $[w_1] - [w_2]$ on J given by the Abel-Jacobi map as

$$\int_{w_1}^{w_2} f(z) \frac{dw}{(1-w)^2} \equiv -0.177051\ldots - 0.291088\ldots i \pmod{\Lambda}$$

where $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is the period lattice of J. Evaluating the elliptic exponential, we find the point

$$(-10.503797\ldots, 5.560915\ldots - 44.133005\ldots i) \in J(\mathbb{C})$$

which matches to the precision computed $\epsilon = 10^{-20}$ the point

$$Y = \left(\frac{-81a - 118}{16}, \frac{(358a + 1191)i + (194a + 236)}{64} \right) \in J(K).$$

We have $J(K) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}$ and Y generates the free quotient.

Higher genus

One can just as easily compute equations for Shimura curves in higher genus.

For example, let $F = \mathbb{Q}(a) = \mathbb{Q}(\sqrt{13})$ and let $\mathfrak{p} = (2w-1) = (\sqrt{13})$. The group $\Gamma^B(1)$ has signature $(2; -)$. Computing a basis for the space $S_2(\Gamma)$ of holomorphic differentials, we obtain the equation

$$X^B(1) : y^2 + y = x^5 + 12x^4 + 17x^3 - 107x^2 + 134x - 56$$

And explicit power series x, y realizing the uniformization of the curve defined by this equation.

This method has really produced the canonical model of X over the reflex field F, in the sense of Shimura and Deligne.

Conclusion

The coefficients of a power series expansion of a modular form f encode interesting information about f that is of independent interest.

We have exhibited a general method for numerically computing power series expansions of modular forms for cocompact Fuchsian groups with good results in practice.

The potential for this algorithm to transport familiar algorithms for modular curves to the more general setting of Shimura curves (even quaternionic Shimura varieties) is promising.