Front progression in the East model

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LPMA – Paris 7; ENS Paris

Glassy Systems and Constrained Stochastic Dynamics
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- Markov process on $\{0,1\}^{\mathbb{Z}}$ with dynamics of creation/destruction of particles. $p \in (0,1)$ density parameter.
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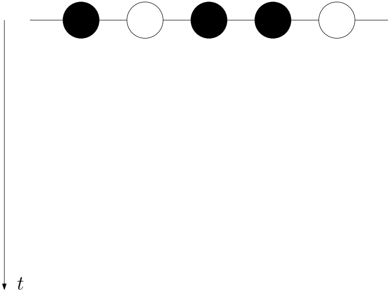

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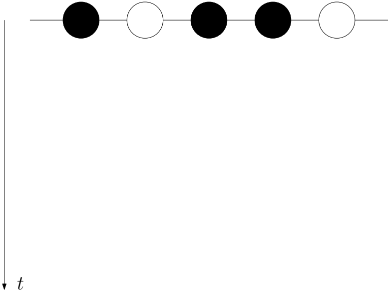

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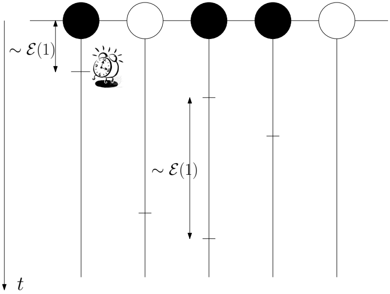

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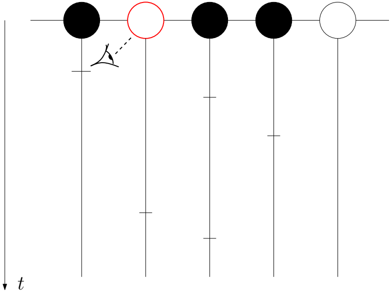

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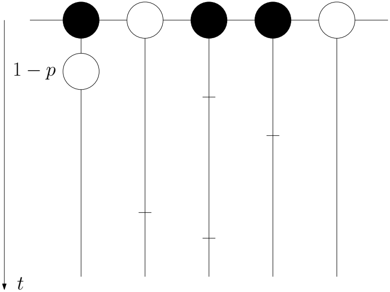

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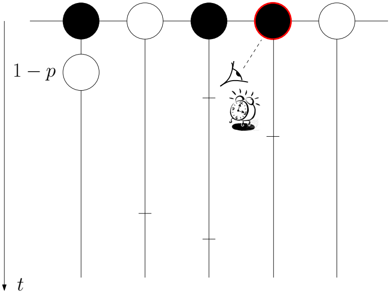

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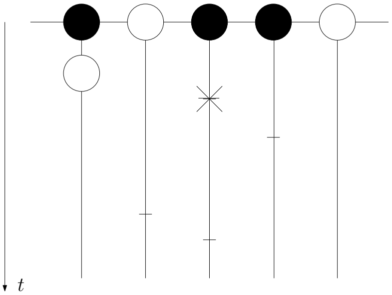

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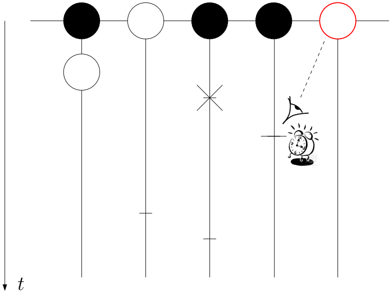

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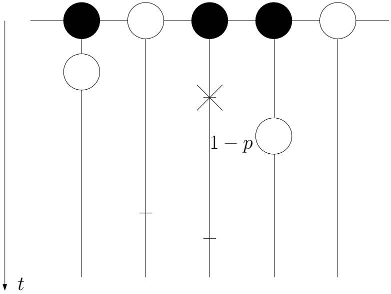

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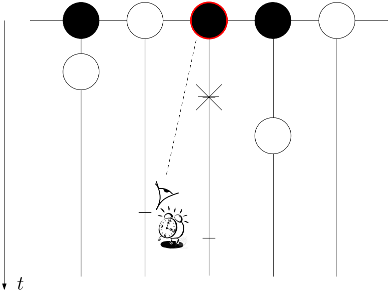

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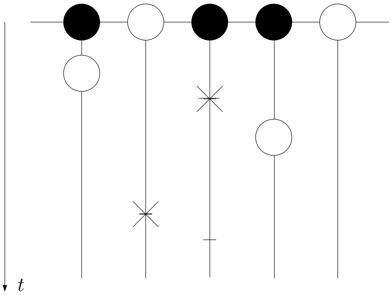

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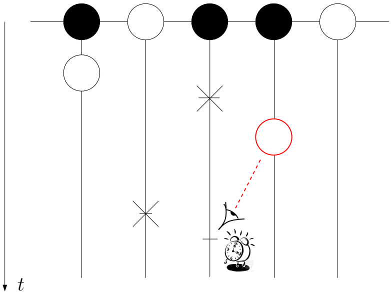

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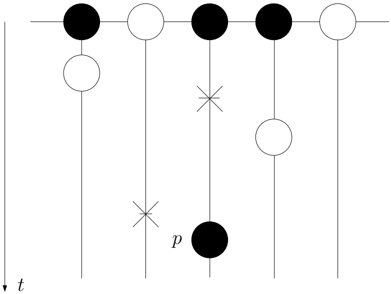

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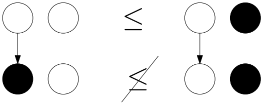

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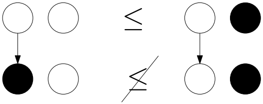

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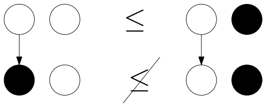
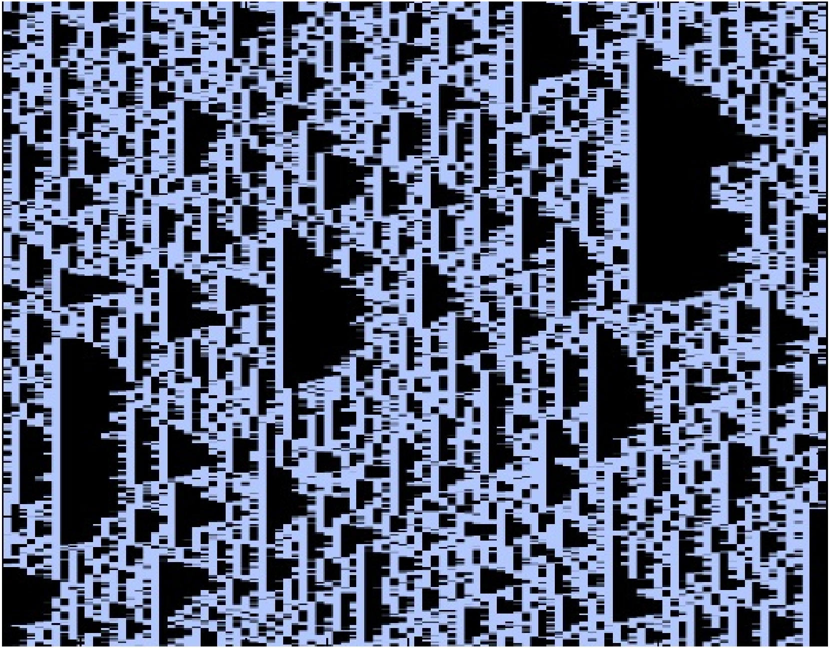
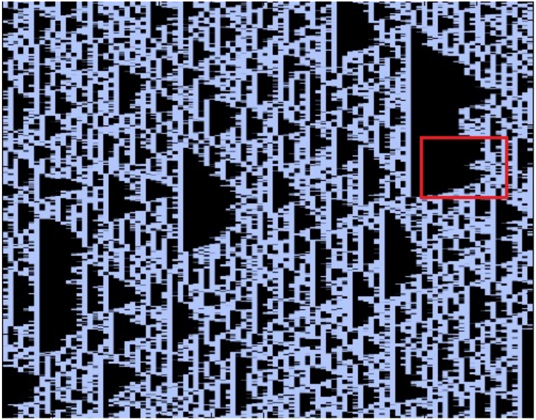
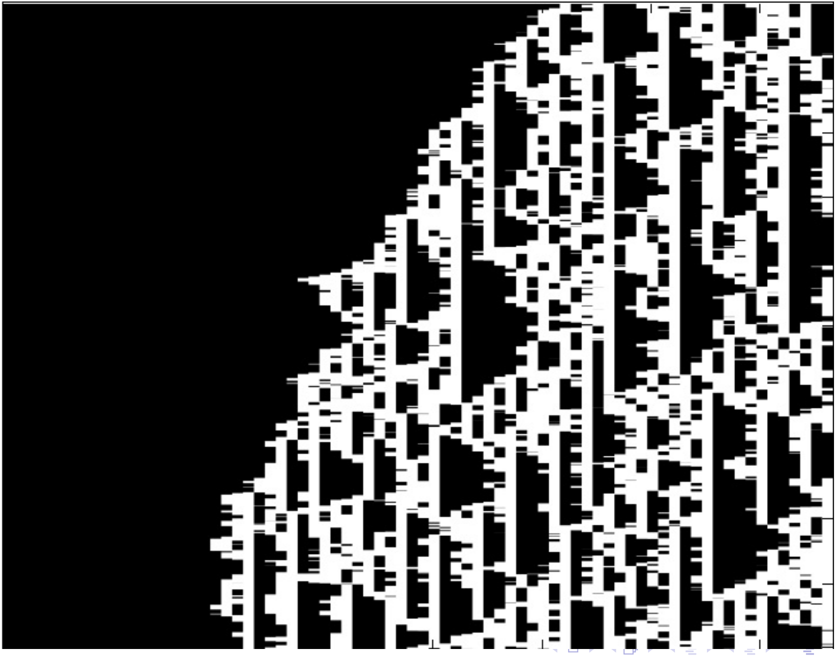


- Equilibrium measure $\mu = \mathcal{B}(p)^{\otimes \mathbb{Z}}$ (reversible).
- Positive spectral gap [AD '02] (exponential return to equilibrium, but not uniform).
  Recall: $\tau = \mathrm{gap}^{-1}$ is the smallest quantity s.t. for all $f \in L^2(\mu)$

$$\mu\left(\left(\mathbb{E}_\eta\left[f(\eta(t))\right] - \mu(f)\right)^2\right) \leq e^{-2t/\tau} Var_\mu(f).$$







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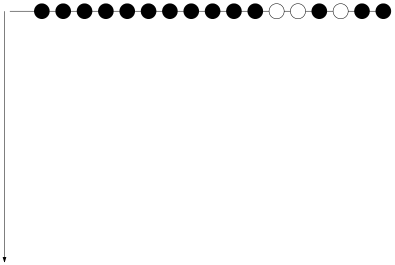

- Start from any configuration with left-most active site at 0.

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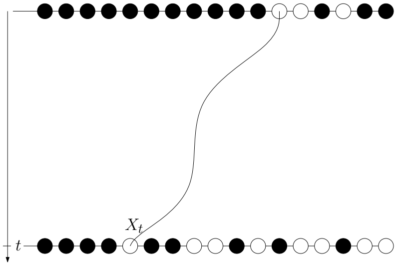

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- Let the East dynamics run for time t.

X_t: position of the front (*i.e.* the left-most active site) at time t.
$\theta\eta(t)$: configuration seen from the front at time t.

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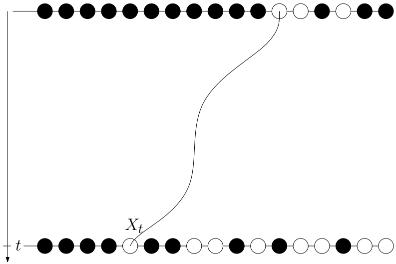

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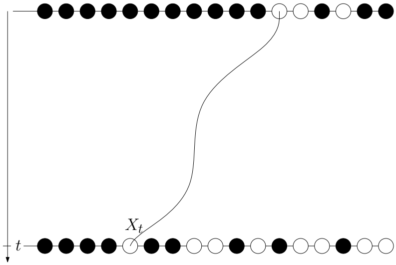

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- Long-time behaviour of X_t?
- What does the front see? Invariant measure for $(\theta\eta(t))_{t\geq 0}$? Convergence of $(\theta\eta(t))_{t\geq 0}$?

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N.B.: Only transitions on the front or on its left change its position.

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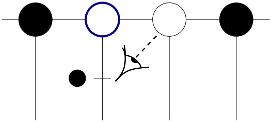

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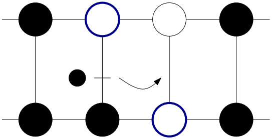

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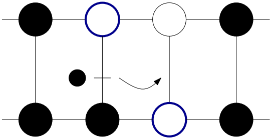

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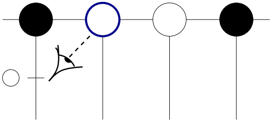

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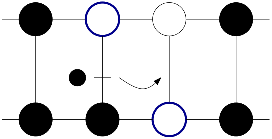

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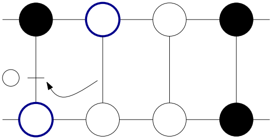

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Results

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- [B. '12] *There exists* $v < 0$ *such that for every initial* η *as above*

$$\frac{X_t}{t} \underset{t\to\infty}{\longrightarrow} v \qquad \text{in probability.}$$

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N.B.: $v = -(1-p) + p\nu(1-\omega_1)$.

Invariant measure

Facts:

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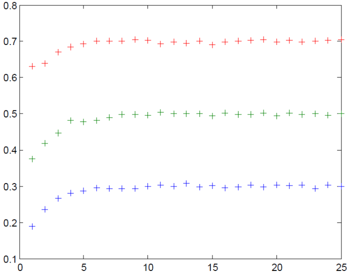

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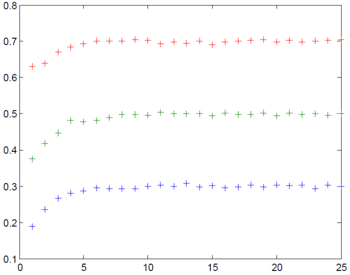

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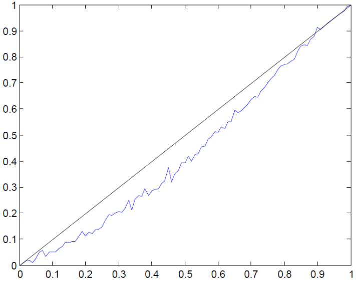

$\nu(\omega_1) = p^2$? Other nice expression?

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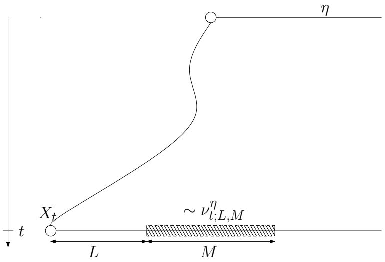

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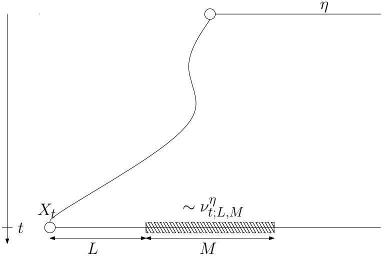
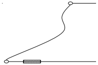


Theorem (B., 2012)

- *If $L + M \leq Ct$* 

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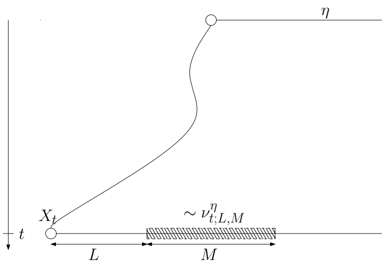

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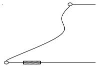
- *If* $L + M \leq Ct$ 

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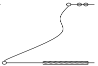
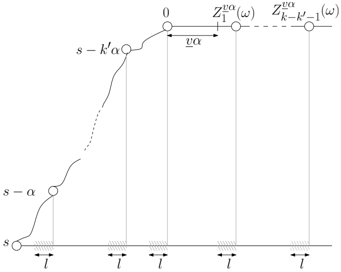
- *If* $L + M > Ct$ *and* η *has "enough zeros"* 

$$\|\nu^{\eta}_{t;L,M} - \mu\|_{TV} \leq e^{-\epsilon(L \wedge t)}$$

Return to equilibrium behind the front

Two steps:

- Guarantee that at all times there are "many" active sites behind the front.



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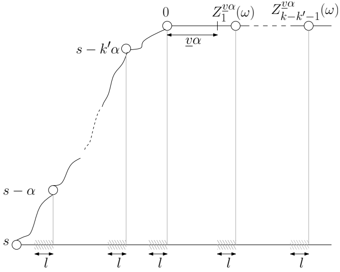
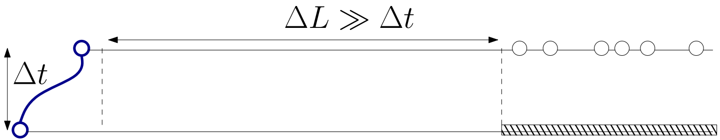
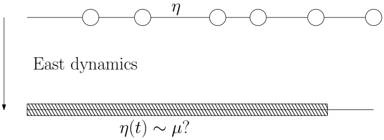


- Use those active sites to relax to equilibrium.



Relaxation to equilibrium

New problem:



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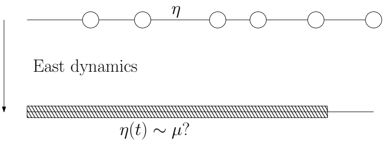

- [Cancrini-Martinelli-Schonmann-Toninelli '10] If f has support in $[0, x_+]$ and $\eta_z = 0$ with $z > x_+$

$$|\mathbb{E}_\eta\left[f(\eta(t))\right] - \mu(f)| \le \sqrt{Var(f)} K(p)^z e^{-t/\tau},$$

where $K(p) = 1/(p \wedge (1-p)) > 1$.

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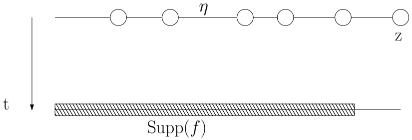

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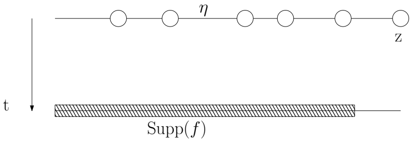

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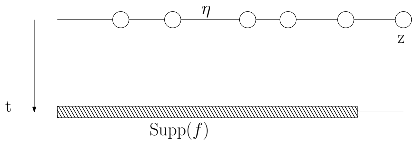

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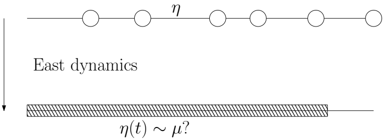
Works well as long as $t \gg z$. Does not take advantage of the *many* zeros.

Relaxation on large sets

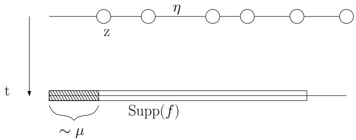


Proposition (B. '12)

f bounded with support in $\mathbb{N}$, $\eta_z = 0$.

$$\left|\mathbb{E}_\eta\left[f(\eta(t)) - \mu_{\{0,\dots,z-1\}}(f)(\eta(t))\right]\right| \leq \sqrt{2}\|f\|_\infty K(p)^z e^{-t/\tau}.$$

Relaxation on large sets



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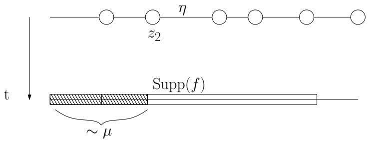

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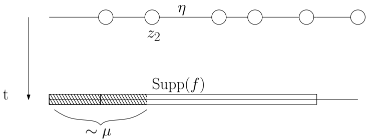

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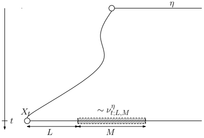
Works well if max distance between active sites $\ll t$.

Conclusion: coupling

Recall: The configuration seen from the front at time t starting from η, $\theta\eta(t)$, is exponentially close to μ in total variation distance.



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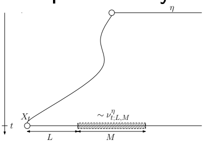

Consequence: let η, σ with left-most active site in 0. We can find (η_t, σ_t) random configurations such that:

- $\eta_t \overset{(d)}{=} \theta\eta(t)$,
- $\sigma_t \overset{(d)}{=} \theta\sigma(t)$,
- with probability exponentially close to 1, $\eta_t = \sigma_t$ far from the front.

Conclusion: coupling

Goal: starting from η, σ with left-most active site at 0, build a coupling (η_t, σ_t) between $\theta\eta(t)$ and $\theta\sigma(t)$ such that for all $L \in \mathbb{N}$, $P(\eta_t = \sigma_t \text{ on } [0, L]) \underset{t\to\infty}{\longrightarrow} 1$.

Ingredients:

- With the previous theorem, coupling far from the front OK.
- The part close to the front is finite.

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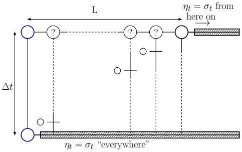

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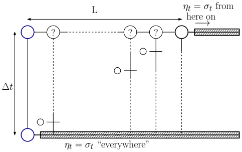

- Try many times. "When one tries continuously, one ends up succeeding..."

Thank you for your attention!

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