Scaling Limits for a Weakly Pinned Gaussian Random Field at a Critical Parameter

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Standard Gaussian lattice free field, in dimension d: $D \subset\subset \mathbb{Z}^d$. Outer boundary ∂D. Hamiltonian:

$$H_N(\phi) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{\langle i,j \rangle \in D \cup \partial D} \left(\phi_i - \phi_j\right)^2,$$

with boundary condition $\mathbf{a} \in \mathbb{R}^{\partial D}$ given by $\phi_i = a_i,\ i \in \partial D$. Gibbs measure on $\mathbb{R}^D$

$$\mu_D^{\mathbf{a}}(d\phi) \stackrel{\text{def}}{=} \frac{1}{Z_D^{\mathbf{a}}} \exp\left[-H_N(\phi)\right] \prod_i d\phi_i.$$

Local pinning at the origin, so-called δ-pinning with pinning parameter $\varepsilon > 0$:

$$\mu_D^{\mathbf{a},\varepsilon}(d\phi) \stackrel{\text{def}}{=} \frac{1}{\zeta_D^{\mathbf{a},\varepsilon}} \exp\left[-H_N(\phi)\right] \prod_i \left(d\phi_i + \varepsilon \delta_0 (d\phi_i)\right),$$

where ζ is always used for pinning partition functions

$$\zeta_D^{\mathbf{a},\varepsilon} \stackrel{\mathsf{def}}{=} \int_{\mathbb{R}^D} \exp\left[-H_N(\phi)\right] \prod_i \left(d\phi_i + \varepsilon\delta_0\left(d\phi_i\right)\right).$$

For 0-boundary conditions, this strongly localizes the field: If $D_N \uparrow \mathbb{Z}^d$, then

$$\lim_{N\to\infty} \mu_{D_N}^{\mathbf{0},\varepsilon}$$

exists, and has exponentially decaying correlations, and a postive density of zeros. $d \geq 3$ by Brydges, Fröhlich, and Spencer 1982. $d = 2$: B-Brydges 2000, Deuschel-Velenik, Ioffe-Velenik 2000, B-Velenik 2001.

Positivity of the **surface tension:**

$$\xi^\varepsilon \stackrel{\mathsf{def}}{=} \lim_{N\to\infty} \frac{1}{D_N} \log \frac{\zeta_{D_N}^{\mathbf{0},\varepsilon}}{Z_{D_N}^{\mathbf{0}}} > 0, \ \forall \varepsilon > 0.$$

What about non-zero (constant) boundary conditions? Interesting only, if the boundary condition increases with N: Put $aN, \ a > 0$.

Plausible: If a is large: The local attraction is not strong enough to disturb the field $\Longrightarrow$ detached. If a is small: $\exists$ region in the center where the field is attached to the wall.

Variational formula (heuristics):

Assume $D_N = ND, \ D \subset\subset \mathbb{R}^d$ nice. Let $A \subset D$ be a nice subset.

The part of the partition function coming from surfaces attached to the wall on NA and detached on $N(D \backslash A)$:

$$Z'_{N(D\backslash A)} \zeta^{0,\varepsilon}_{NA}$$

$Z'_{N(D\backslash A)}$: partition function of the (unpinned) free field with boundary conditions Na on ∂D_N and 0 on $N(\partial A)$.

$$Z'_{N(D\backslash A)} \approx \exp\left[N^d\left(\eta\,|D\backslash A| - q_{D,A}\right)\right],$$

where

$$q_{D,A} \stackrel{\text{def}}{=} \inf\left\{\frac{1}{2}\int_{D\backslash A} |\nabla f|^2\,dx : f|_{\partial D} = a,\ f|_{\partial A} = 0\right\},$$

$$\eta \stackrel{\text{def}}{=} \lim_{N\to\infty}\frac{1}{|D_N|}\log Z^{\mathbf{0}}_{D_N}.$$

$$\zeta^{0,\varepsilon}_{NA} \approx \exp\left[N^d\,|A|\,(\xi^{\varepsilon}+\eta)\right],$$

so

$$Z'_{N(D\backslash A)}\zeta^{0,\varepsilon}_{NA} \approx \mathrm{e}^{N^d|D|\eta}\exp\left[N^d\left(|A|\,\xi^{\varepsilon} - q_{D,A}\right)\right].$$

Finally, one has to "sum" over the possible A's:

$$\zeta_{D_N}^{aN,\varepsilon} \approx \sum_A Z'_{N(D\setminus A)} \zeta_{NA}^{0,\varepsilon} \approx \mathrm{e}^{N^d |D| \eta} \exp\left[N^d \sup_{A : A \subset D} \left(|A|\, \xi^{\varepsilon} - q_{D,A} \right) \right].$$

Problematic steps:

- Splitting of the partition functions

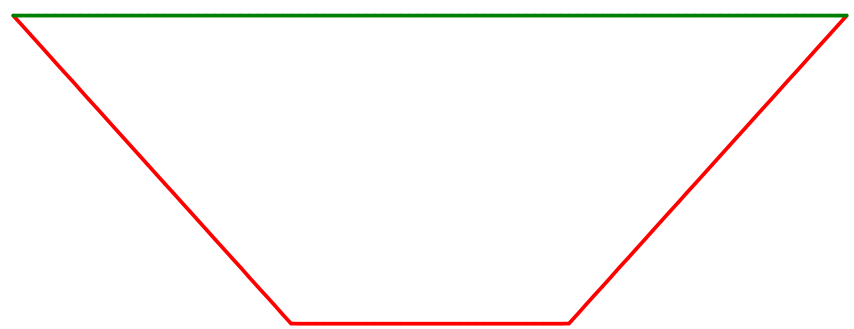
- Replacing the summation by the supremum.

If the steps are justified, then: If $\sup$ is attained at $A = \emptyset$: no attachment, otherwise partial attachment on NA for the optimal A.

One dimensional case: $D=[0,1]$, $A=[x,1-x]$, $x\leq 1/2$. $q_{D,A}=a^2x^{-2}$, and we have to maximize

$$-\frac{a^2}{x}+\xi_\varepsilon(1-2x),\ x\leq 1/2.$$

For $a<\sqrt{\xi_\varepsilon/8}$, the optimal choice is $x=a/\sqrt{2\xi_\varepsilon}$, and for $a>\sqrt{\xi_\varepsilon/8}$, the optimal choice is to take $A=\emptyset$. The critical value is $a_{\text{crit}}\overset{\text{def}}{=}\sqrt{\xi_\varepsilon/8}$. First order transition: The optimal profile jumps from partly attached to completely detached. At a_{crit}: two optimal profiles $\bar{h}(t)\overset{\text{def}}{=}a,\ \forall t,$ or $\hat{h}$



For $d = 1$: B.-Funaki-Otobe 2008. Define $h^N : [0, 1] \to \mathbb{R}$ by

$$h^N \left(\frac{k}{N} \right) \stackrel{\text{def}}{=} \frac{1}{N} \phi_k,$$

and interpolated between, then at a_{crit}: For some $\alpha > 0$

$$\lim_{N \to \infty} \mu_N^{N a_{\text{crit}}, \varepsilon} \left(\left\| h^N - \hat{h} \right\|_\infty \leq N^{-\alpha} \right) = 1,$$

i.e. the surface prefers the attached solution.

Remark: With free boundary condition at one end of the interval, $\bar{h}$ and $\hat{h}$ get positive weight.

Conjecture: For $d \geq 2$, the surface stays attached, too.

Precise description of our results: D_N a cylinder over a torus: Let $\mathbb{T}_N \stackrel{\text{def}}{=} \mathbb{Z}/N\mathbb{Z}$, and

$$D_N \stackrel{\text{def}}{=} \{0, 1, \ldots, N\} \times \mathbb{T}_N^{d-1},\ \partial D_N \stackrel{\text{def}}{=} \{0, N\} \times \mathbb{T}_N^{d-1},\ D_N^0 \stackrel{\text{def}}{=} D_N \backslash \partial D_N.$$

$\phi = \{\phi_i\}_{i \in D_N^0}$ with boundary condition aN on ∂D_N, Hamiltonian as above, and local pinning with strength $\varepsilon > 0$. The law is $\mu_N^{aN,\varepsilon}$.

The **macroscopic profile** $h^N : D \stackrel{\text{def}}{=} [0, 1] \times \mathbb{T}^{d-1} \to \mathbb{R}$, where $\mathbb{T} \stackrel{\text{def}}{=} \mathbb{R}/\mathbb{Z}$ is defined by

$$h^N\left(\frac{i}{N}\right) \stackrel{\text{def}}{=} \frac{\phi_i}{N},\ i \in D_N,$$

with some interpolation.

Torus symmetry$\Longrightarrow$variational problem is one-dimensional on D. The variation formula for the macroscopic profile reduces to 1-dim: Two possible maximizers:

$$\bar{h}(\mathbf{t}) \equiv a,\ \hat{h}(\mathbf{t}) = \hat{h}_1(t_1),\ \mathbf{t} = (t_1, \ldots, t_d),$$

where

$$\hat{h}_1(t) = \begin{cases} a\left(1 - \frac{t}{x}\right) & \text{for } t \leq x \\ 0 & \text{for } x \leq t \leq 1 - x \\ (t - 1 + x)\frac{a}{x} & \text{for } 1 - x \leq t \leq 1 \end{cases},$$

where

$$x = x(a, \varepsilon) \stackrel{\text{def}}{=} \frac{a}{\sqrt{2\xi_\varepsilon}},$$

but ξ_ε is the d-dimensional surface tension. Critical value for $a : a_{\text{crit}}(\varepsilon) = \sqrt{\xi_\varepsilon / 8}$, where from rough LDP, $\bar{h}$ and $\hat{h}$ are equally favorable.

Theorem Assume $d \geq 3$, and ε large enough. Then there exists $\alpha > 0$ such that

$$\lim_{N\to\infty} \mu_N^{Na_{\text{crit}},\varepsilon}\left(\left\|h^N - \hat{h}\right\|_1 \leq N^{-\alpha}\right) = 1.$$

Three crucial steps:

(I) **LDP**: There exists $\alpha > 0$ such that

$$\lim_{N\to\infty} \mu_N^{Na,\varepsilon}\left(\text{dist}_{L^1}\left(h^N, \left\{\hat{h}, \bar{h}\right\}\right) \geq N^{-\alpha}\right) = 0.$$

(II) **Lower bound at $\hat{h}$**: For $\alpha'' < 1$ there $\exists c > 0$ s.th. for large enough N

$$\frac{\zeta_N^{aN,\varepsilon}}{Z_N}\mu_N^{Na,\varepsilon}\left(\left\|h^N - \hat{h}\right\|_1 \geq N^{-\alpha''}\right) \geq \exp\left[cN^{d-1}\right].$$

(Remark here that $Z_N = Z_N^{aN}$).

(III) **Upper bound at $\bar{h}$**: If $\alpha' > d$ one has for large enough N :

$$\frac{\zeta_N^{aN,\varepsilon}}{Z_N}\mu_N^{Na,\varepsilon}\left(\left\|h^N - \bar{h}\right\|_1 \leq (\log N)^{-\alpha'}\right) \leq 2.$$

The **LDP (I)** uses heavily a technique introduce in B-Ioffe (CMP, 1997) on a Winter-bottom construction. Expansion of the pinned measure:

$$\begin{aligned}
\mu_N^{Na,\varepsilon}(d\phi) &= \frac{1}{\zeta_N^{aN,\varepsilon}} \exp\left[-H_N(\phi)\right] \prod_{i\in D_N^0} \left(d\phi_i + \varepsilon\delta_0(d\phi_i)\right) \\
&= \sum_{A\subset D_N^0} \frac{Z_A^{aN,0}}{\zeta_N^{aN,\varepsilon}} \varepsilon^{|A^c|} \mu_A^{Na,0}(d\phi) :
\end{aligned}$$

$\mu_A^{Na,0}$: the free field on A with boundary conditions 0 on $D_N^0 \setminus A$ and aN on ∂D_N.

Difficulty: The summation over A's is too big. On the other hand: the summation over all A's produces the surface tension ξ_ε.

Way out: Reduce the combinatorial complexity in the detached region, but not in the attached one.

Key idea: Introduce a **mesoscopic** scale N^β, $0 < \beta < 1$, and devide D_N^0 into subboxes B of sidelength N^β. **Mesoscopically smoothed surface**: average over the mesoscopic subboxes $\Longrightarrow$Mesoscopic profile $h^{N,\text{meso},\beta}$

$$\mu_N^{Na,\varepsilon}\left(\left\|h^N - h^{N,\text{meso},\beta}\right\|_1 \geq N^{-\alpha}\right) \leq \exp\left[-N^{d+\delta}\right],\ \text{some } \delta > 0,$$

This idea stems from the **Donsker-Varadhan** treatment of the LDP for the Wiener sausage.

"Mesoscopically wetted" region

$$W_{\text{meso}} \stackrel{\text{def}}{=} \left\{\mathbf{t} \in D : h^{N,\text{meso},\beta}(\mathbf{t}) \geq N^{-\gamma}\right\}.$$

Fixing a mesoscopic region, the sum over microsopic A's is essentially only over subsets of $(W_{\text{meso}})^c$. This requires to splits A in the original summation over $A \cap W_{\text{meso}}$ and $A \cap \left(D_N^0 \backslash W_{\text{meso}}\right)$, the latter producing the surface tension on $D_N^0 \backslash W_{\text{meso}}$, and $A \cap W_{\text{meso}}$ has to be small.

The number of possible mesoscopic regions is subexponential $\Longrightarrow$ summation can be replaced by maximum $\Longrightarrow$ the mesoscopic profile is with high probability the optimal one.

The argument needs an analytic rigidity property.

The proof of the **lower bound (II)** unfortunately heavily uses that ε is large. As one cannot hope to get anything better than $\exp\left[cN^{d-1}\right]$, the coarse graining technique from (I) cannot be used. Let $K \in \mathbb{N}$, and split D_N^0 into five parts: Two layers of width K in the first coordinate near the optimal position $x_{\mathrm{crit}}N$, call them γ_L, γ_R, then two regions F_L, F_R left of γ_L and right of γ_R, and finally the rest B in the middle. Then define

$$\Gamma \stackrel{\mathrm{def}}{=} \{\phi : \phi_i \neq 0,\ i \in F_L \cup F_R,\ \phi_i = 0,\ i \in \gamma_L \cup \gamma_R\},$$

and estimate

$$\begin{aligned}\frac{\zeta_N^{\alpha N,\varepsilon}}{Z_N}\mu_N^{aN,\varepsilon}\left(\left\|h^N - \hat{h}\right\|_1 \leq \delta\right) \quad \geq \quad & \frac{Z_{F_L}^{aN,0}\zeta_B^{0,\varepsilon}Z_{F_R}^{0,aN}}{Z_N}\varepsilon^{2|\gamma_L|}\mu_{F_L}^{aN,0}\left(\left\|h^N - \hat{h}\right\|_1 \leq \delta\right) \\ & \times\mu_B^{aN,\varepsilon}\left(\left\|h^N - \hat{h}\right\|_1 \leq \delta\right)\mu_{F_R}^{0,aN}\left(\left\|h^N - \hat{h}\right\|_1 \leq \delta\right).\end{aligned}$$

The μ-probabilities are all ≈ 1, even with $\delta = N^{-\alpha'}$, and then a somewhat messy computation gives that for ε large enough

$$\frac{Z_{F_L}^{aN,0}\zeta_B^{0,\varepsilon}Z_{F_R}^{0,aN}}{Z_N}\varepsilon^{2|\gamma_L|}\gtrapprox \exp\left[cN^{d-1}\log\varepsilon\right].$$

In the **upper bound** (III), we use $d\geq 3$. If one would replace $\|\cdot\|_1$ by $\|\cdot\|_\infty$, then the statement is trivial (with bound 1). This is not possible because of (I) does not work with $\|\cdot\|_\infty$. However, restricted to $\left\|h^N-\bar{h}\right\|_1\leq(\log N)^{-\alpha'}$, $\alpha'>d$, in the expansion over $A\stackrel{\text{def}}{=}$zero set, one can restrict to $|A|\leq(N/\log N)^d$, and it then suffices to prove

$$\sum_{A\subset D_N^0,\ |A|\leq(N/\log N)^d}\varepsilon^{|A^c|}\frac{Z_A^{aN,0}}{Z_N}\leq 2.$$

where $Z_A^{aN,0}$ refers to the ($\varepsilon=0$) partition function with aN boundary on ∂D_N and

0 boundary on $A^c \overset{\mathsf{def}}{=} D_N^0 \setminus A$. For that, one uses an estimate

$$\frac{Z_A^{aN,0}}{Z_N} \leq \mathsf{e}^{\mathsf{const} \times |A^c|} \,\mathsf{exp}\left[-N^2 a \,\mathsf{cap}_{D_N}(A^c)\right],$$

where cap_{D_N} refers to the capacity with respect to the transient random walk on D_N^0 with killing at ∂D_N. From that, one gets the estimate if N is large enough.