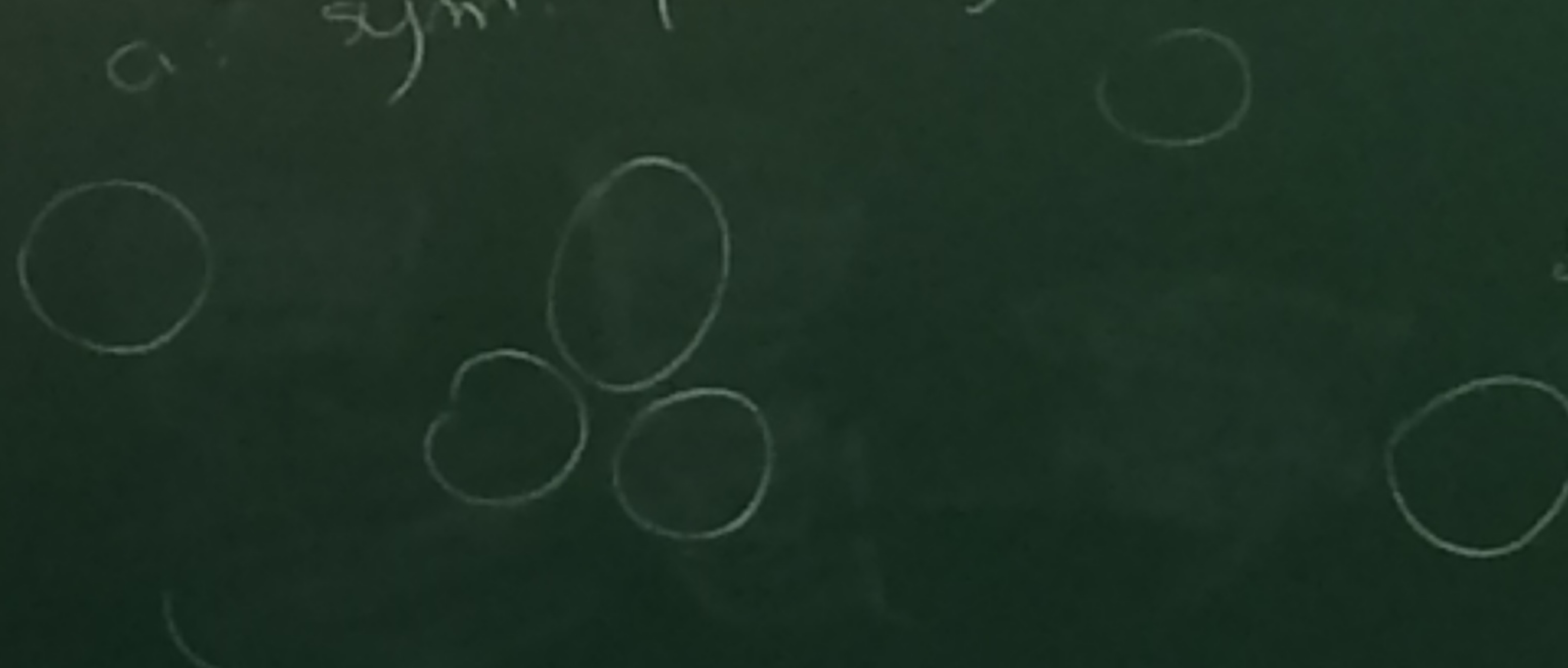
Fluctuations in homogenization

Heat equation: $\partial_t u = \nabla \cdot a \nabla u$

$u : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$

a : sym. pos. def. matrix   function of space



a. random

law translation invariant

ergodic

$$\begin{cases} \partial_t u_\varepsilon = \nabla \cdot a(\cdot/\varepsilon) \nabla u_\varepsilon \\ u_\varepsilon(0, \ ) = f \end{cases}$$

$u_\varepsilon \xrightarrow[\varepsilon \to 0]{} \bar{u}$ solving $\begin{cases} \partial_t \bar{u} = \nabla \cdot \bar{a} \nabla \bar{u} \\ \bar{u}(0, \ ) = f \end{cases}$

$\bar{a}$ constant in space, deterministic

(1) Probabilistic

$$u_\varepsilon(t,x) = E_{x/\varepsilon}\left[f\left(\varepsilon X_{\varepsilon^{-2}t}\right)\right]$$

Prove a CLT for (X_t) $\qquad \xi \in \mathbb{R}^d$

We would like $\xi \cdot X_t$ = martingale + small rest

$f(X_t)$ mart. $\quad \nabla \cdot a \nabla f = 0$ $\qquad$ sub-linear

$$f(x) = \xi \cdot x + \phi_\xi(x)$$

$$\nabla \cdot a \left(\xi + \nabla \phi_\xi\right) = 0$$

$$\int (\xi + \nabla \phi_\xi) \cdot a \, (\xi + \nabla \phi_\xi)$$

ϕ stationary

Gloria - Otto 10. iid & $d \geq 3 \to$ $\underline{\phi_\xi \text{ stationary}}$

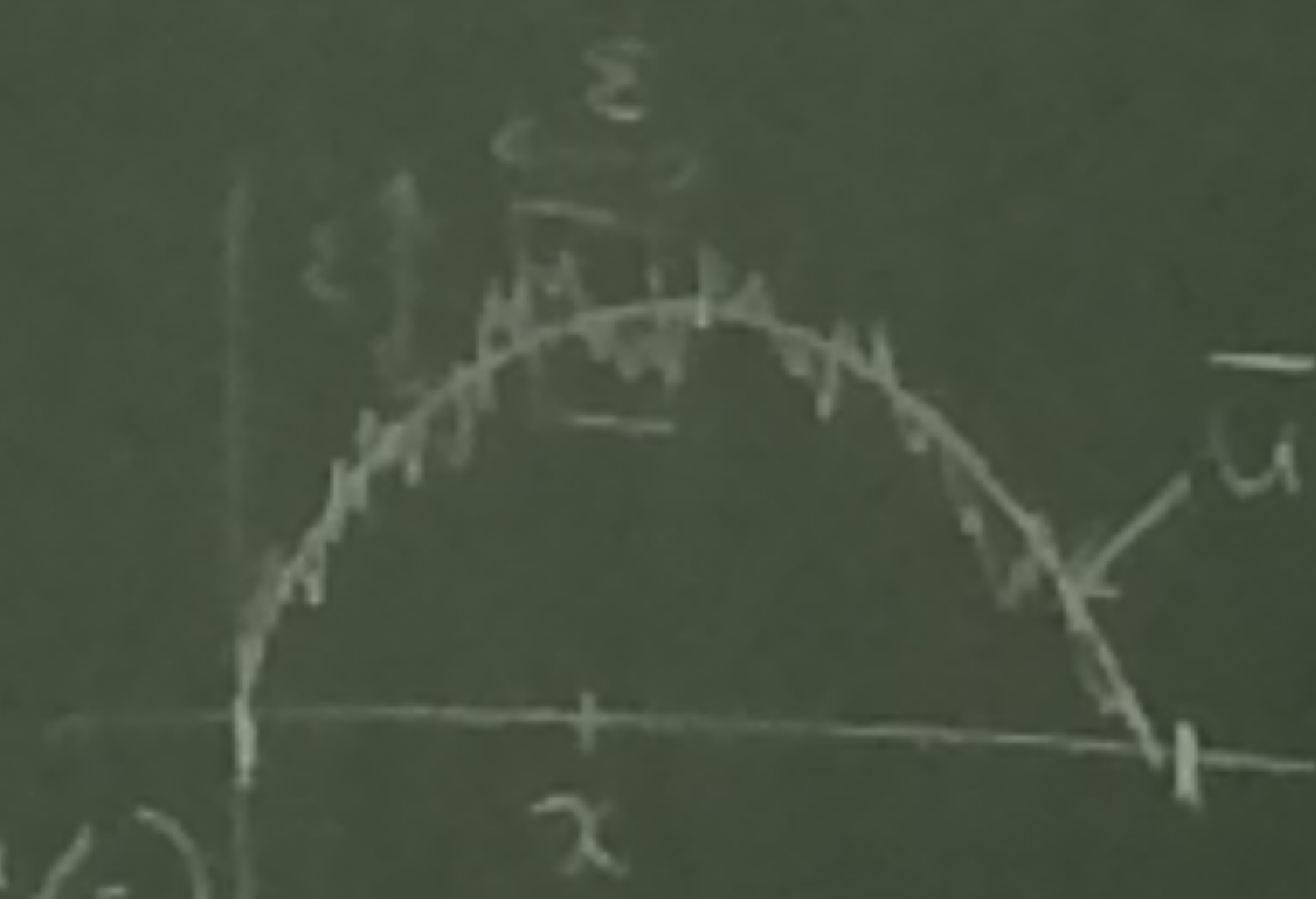
$$\xi \cdot X_t = \underbrace{\left(\xi \cdot X_t + \phi_\xi(X_t)\right)}_{\pi_t} - \phi_\xi(X_t)$$

$$\langle \pi \rangle_t \sim ct \qquad (t \to +\infty)$$

$$\underline{\xi \cdot \bar{a}\, \xi} = \langle (\xi + \nabla\phi_\xi) \cdot \bar{a}\, (\xi + \nabla\phi_\xi) \rangle$$

(2) PDE: formal two-scale expansion

$$-\nabla\cdot a(\cdot/\varepsilon)\nabla u_\varepsilon = f$$



$$u_\varepsilon(x) = \bar{u}(x) + \varepsilon v(x, x/\varepsilon) + \varepsilon^2 w(x, x/\varepsilon) + \dots$$

$$v(x,\cdot) = \phi_{\nabla\bar{u}(x)}$$

$$u_\varepsilon(x) \simeq \bar{u}(x) + \varepsilon \sum_{i=1}^{d} \partial_{x_i}\bar{u}(x)\, \phi_{e_i}(x/\varepsilon)$$

To sum up: want to find the scaling limit of ϕ. $\varepsilon^{-\frac{d-2}{2}} \phi(\cdot/\varepsilon)$?

(1) $\langle \phi(0)\, \phi(x) \rangle \simeq |x|^{-(d-2)}$?

(2) $\varepsilon^{-\frac{(d-2)}{2}} \int \phi(x/\varepsilon) f(x)\, dx \Big/ \sqrt{\mathrm{Var}(\cdot)} \to \mathcal{N}(0,1)$?

$(\mathbb{Z}^d, \mathbb{B})$ $\quad (a_e)_{e \in \mathbb{B}}$ iid

$$\nabla \cdot a \nabla f(x) = \sum_{y \sim x} a_{xy} (f(y) - f(x))$$

(ζ_e) iid $\mathcal{N}(0,1)$

Assume $\quad a_e = a(\zeta_e)$

<u>Thm</u> (w. Otto). let $\mathcal{G}$ = Green function of homog. $\nabla \cdot \bar{a} \nabla$

There exists a matrix $Q^{(\zeta)}$ s.t. if

$$K_\zeta(x) = \int \nabla \mathcal{G}(y) \cdot Q^{(\zeta)} \nabla \mathcal{G}(x-y)\, dy \text{ , Then}$$

$$\left| \langle \phi(0) \phi(x) \rangle - K_\zeta(x) \right| \lesssim \frac{\log^2 |x|}{|x|^{d-1}}$$

$$\left| \langle \Phi(0)\, \Phi(x) \rangle - K_\xi(x) \right| \sim \frac{}{|x|^{d-1}}$$

Rem Fourier transform of K_ξ is $\dfrac{p \cdot Q^{(\xi)} p}{(p \cdot \bar{a}\, p)^2}$

is not like $\dfrac{1}{p \cdot A p}$ for some A.

Ideas of proof. $\sum V(\nabla \phi_i)$ Naddaf Spencer 97

Helffer-Sjöstrand for iid $\mathcal{N}(0,1)$ r.v.

$\xi = (\xi_e)$ $f(\xi), g(\xi)$ centred.

$$\partial_e f = \frac{\partial f}{\partial \xi_e} \quad ; \quad \partial_e^* f = -\partial_e f + \xi_e f$$

$\partial f = (\partial_e f)_e$, $\partial^* F = \sum_e \partial_e^* F_e$; $\mathcal{L} = \partial^* \partial$

$\langle f \; g \rangle = \sum_e \langle \partial_e f \; (\mathcal{L}+1)^{-1} \partial_e g \rangle$.

$\phi(0)$ $\phi(x)$

Say $g = \mathcal{L} u$

$\langle f \; g \rangle = \langle f \; \mathcal{L} u \rangle$

$= \sum_e \langle \partial_e f \;\; \partial_e u \rangle$

$\partial_e u = (\mathcal{L}+1)^{-1} \partial_e g$?

$\partial_e g = \sum_{e'} \partial_e \partial_{e'}^* \partial_{e'} u$

$[\partial_e, \partial_{e'}^*] = \partial_e(-\partial_{e'} + \zeta_{e'})$

$- (-\partial_{e'} + \zeta_{e'}) \partial_e$

$= \mathbb{1}_{\{e = e'\}}$

$$\partial_e g = \sum_{e'} \partial_e \partial_{e'} \partial_{e'} u$$

$$-(-\partial_{e'} + \xi_{e'})\partial_e$$

$$= \mathbb{1}_{\{e=e'\}}$$

$$\partial_e g = \sum_{e'} \left(\partial_{e'}^* \partial_e + \mathbb{1}_{\{e=e'\}}\right) \partial_{e'} u$$

$$= \left(\sum_{e'} \partial_{e'}^* \partial_{e'} \partial_e u\right) + \partial_e u = (\mathcal{L}+1)\partial_e u \quad \square$$

2^{d} step: compute $\partial_e \phi$

$$-\nabla \cdot a(\xi + \nabla\phi) = 0$$

$$-\nabla \cdot \mathbb{1}_e (\xi + \nabla\phi) = 0$$

$$-\nabla \cdot a \nabla \partial_e \phi - \nabla \cdot \mathbb{1}_e(\xi + \nabla\phi)$$

$$\partial_e \phi(x) = \sum_y G(x,y) \nabla \cdot \mathbb{1}_e (\xi + \nabla\phi)(y)$$

$$= \sum_b \nabla G(x,b) \cdot \mathbb{1}_e (\xi + \nabla\phi)(b)$$

$$= -\nabla G(x,e)(\xi + \nabla\phi)(e)$$

$$\langle \phi(0)\,\phi(x)\rangle = \sum_{e} \langle \nabla G(0,e)\,(\xi+\nabla\phi_{\xi})(e)$$

$$(\mathcal{L}+1)^{-1}\,(\xi+\nabla\phi_{\xi})(e)\,\nabla G(x,e)\rangle$$

$$\nabla G(0,e) \simeq \nabla \mathcal{G}(e) + \sum_{i=1}^{d} \partial_{x_i}\mathcal{G}(e)\,\nabla\phi_{e_i}(e)$$

$$= \sum_{i=1}^{d} \partial_{x_i}\mathcal{G}(\underline{e})\,(e_i+\nabla\phi_{e_i})(e)$$

$$\langle \phi(0)\,\phi(x)\rangle = \sum_{e} \langle \nabla G(0,e)\,(\xi+\nabla\phi_{\xi})(e)\,(\mathcal{L}+1)^{-1}\,(\xi+\nabla\phi_{\xi})(e)\,\nabla G(x,e)\rangle$$

$$\nabla G(0,e) \simeq \nabla \bar{G}(e) + \sum_{i=1}^{d} \partial_{x_i}\bar{G}(e)\,\nabla\phi_{e_i}(e)$$

$$= \sum_{i=1}^{d} \partial_{x_i}\bar{G}(e)\,(e_i+\nabla\phi_{e_i})(e)$$

$$G^{(\xi)}_{i,j} = \sum_{k=1}^{d} \langle (e_i+\nabla\phi_{e_i})(\xi+\nabla\phi_{\xi})(e_k)\,(\mathcal{L}+1)^{-1}\,(e_j+\nabla\phi_{e_j})(\xi+\nabla\phi_{\xi})(e_k)\rangle$$

$$\bar{a}$$

$$(2)\quad \varepsilon^{-\frac{(d-2)}{2}} \int \phi(x/\varepsilon)\, f(x)\,dx \,/\, \sqrt{(\cdot)} \to \mathcal{N}(0,1)$$