History dependent stochastic processes

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I. 2 models of history dependent st. proc.

1. non lin. sigma model

II. relations, results

III. techniques

III techniques

Model I discrete time process on $\mathbb{Z}^d$ $\quad (X_n)_{n\in\mathbb{N}}$

initial weights 'conductances' $\quad (a_e)_{e\in E}$ $\quad E=\{x\sim y \quad x,y\in\mathbb{Z}^d\}$

$i\sim j$ means $|i-j|=1$

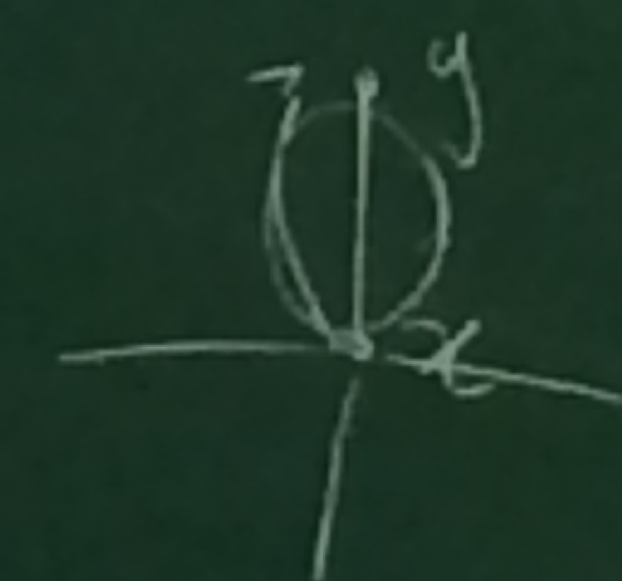
undirected edges

time dependent conductance

$$\omega_e(n) = \underset{\substack{\shortparallel \\ \omega_e(0)}}{a_e} + T_n(e)$$

$T_n(e)$ = # time X_n crossed e (both directions) up to time n

$$\mathbb{P}\left(X_{n+1}=y \mid X_n=x,\ X_{n-1},\ \dots,\ X_0\right) = \frac{\omega_{xy}(n)}{\sum_{z\sim x}\omega_{zx}(n)}$$



reinforcement on the edges

meaning of a_e

first time we visit the edge e

ω_e goes from $a_e \longrightarrow a_e + 1$

$a_e \ll 1$	$a_e+1 \gg a_e$	strong reinf
$a_e \gg 1$	$a_e+1 \sim a_e$	weak reinf

IMP simplification : this process = RW in Rand Env

$\Lambda \subset \mathbb{Z}^d$ finite E_Λ

$$\mathbb{P}_\Lambda^{ERRW}(\ ) = \int \underset{\substack{\uparrow \\ \text{law for} \\ \text{rand environment}}}{d\rho(\omega)} \ \underset{\substack{\uparrow \\ \text{RW environment} \\ \text{with conduct } \omega_e}}{\mathbb{P}_\Lambda^{\omega}(\ )}$$

Model 2 continuous time jump process

$$d\rho(\omega) = \frac{1}{K} \frac{\prod_{e \in E_\Lambda} \omega_e^{a_e}}{\prod_{v \neq v_0} \left(\sum_{e \ni v} \omega_e \right)^{\frac{1}{2} + \frac{\sum_{e \ni v} a_e}{2}}} \sqrt{D[m]} \prod_{\substack{e \in E_\Lambda \\ e \neq e_0}} \frac{d\omega_e}{\omega_e} \delta_{\omega_{e_0} = 1}$$

$D[m]$ = diagonal minor of the matrix $\Lambda \times \Lambda$

$$m_{ij} = \begin{cases} -\omega_{ij} & i \sim j \\ \sum_{k \sim i} \omega_{ik} & i = k \end{cases}$$

$$D[m] = \sum_T \prod_{e \in T} \omega_e$$

T spanning tree on Λ

$$\mathbb{P}_\Lambda^{ERRW}(\ ) = \int \underbrace{d\rho(\omega)}_{\substack{\uparrow \\ \text{law for} \\ \text{rand environment}}} \ \underbrace{\mathbb{P}_\Lambda^{\omega}(\ )}_{\substack{\uparrow \\ \text{RW environment} \\ \text{with conduct } \omega_e}}$$

Model 2 continuous time ~~process~~ jump process $(Y_\tau)_{\tau\geq 0}$ on $\mathbb{Z}^d$

Vertex Reinforced Jump Process

rates of jump over the edge e depend on time $\omega_e(\tau)$

initial rate $\omega_e(0) = \beta_e$

local time $T_x(\tau)$ = total time Y spent at x up to time τ

reinforcement scheme $\omega_{xy}(\tau) = \begin{cases} \beta_e[1+T_y(\tau)] \neq \omega_{yx}(\tau) & \text{Volkov, Davies} \\ \beta_e e^{T_x(\tau)+T_y(\tau)} = \omega_{xy}(\tau) & \text{[Sabot-Tarrès]} \end{cases}$

$x \quad y$

conditioned on $(Y_s)_{s\leq\tau}$

the process jumps from $x = Y_\tau$ to y with rate $\omega_{xy}(\tau)$

meaning of β

$\beta_c >> 1$ initial jump rate from $x \to y$ large

$T_x(\tau)$ does not grow much

$\Rightarrow$ weak reinf

$\beta_c << 1$ takes a long time to do the first jump $\Rightarrow T_x(\tau)$ large

$\Rightarrow$ strong reinf

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discrete time process associated to Y

$Z_n = Y_{\tau_n}$ τ_n: time before n^{th} jump

① $$d\mathfrak{z}(\omega) = \frac{1}{K} \frac{\prod_{e\in E_\Lambda} \omega_e^{a_e}}{\prod_{v\neq v_0} \left(\sum_{e\ni v} \omega_e\right)^{\frac{1}{2}+\frac{\sum_{e\ni v} a_e}{2}}} \sqrt{D[m]} \prod_{\substack{e\in E_\Lambda \\ e\neq e_0}} \frac{d\omega_e}{\omega_e} \, \delta_{\omega_{e_0}=1}$$

$e_0 \ni v_0$

$D[m]$ = diagonal minor of the matrix $\Lambda\times\Lambda$

$$m_{ij} = \begin{cases} -\omega_{ij} & i\sim j \\ \sum_{k\, k\sim i} \omega_{ik} & i=k \end{cases}$$

$$D[m] = \sum_T \prod_{e\in T} \omega_e$$

T spanning tree on Λ

② $$d\mu(t) = \prod_{i\sim j} e^{-\beta_{ij}[\operatorname{ch}(t_i-t_j)-1]} \sqrt{D[m]} \prod_{\substack{j\in\Lambda \\ j\neq 0}} \frac{dt_j}{\sqrt{2\pi}} e^{-t_j} \, \delta_{t_0=0}$$

$D[m]$ = diag minor of

$$m_{ij} = \begin{cases} -\beta_{ij} e^{t_i+t_j} & i\sim j \\ \sum_{k\, k\sim i} e^{t_i+t_k} \beta_{ik} & i=j \end{cases}$$

1, 2 gradient measures
$t_j \to t_j + t \;\; \forall j$ meas does
$\omega_e \to \omega_e c \;\; \forall e$ not change

Model 3 $H^{2|2}$ non-linear sigma model introduced in the context of RMT

Zirnbauer 92

non lin σ-model gradient $e^{-(\nabla\phi)^2}$

ϕ takes values in a non linear manifold

manifold here $U(1,1|2)$ SUSY

$\longrightarrow$ 2 real variables (t_j, s_j) $j \in \mathbb{Z}^d$

$$m_{ij} = \begin{cases} -\beta_{ij} e^{t_i+t_j} & i\sim j \\ \sum_{k\, k\sim i} e^{t_i+t_k} \beta_{ik} & i=j \end{cases}$$

1, 2 gradient measures
$t_j \to t_j + t$ $\forall j$ meas. does
$w_e \to w_e c$ $\forall e$ not change

$$d\mu(t,s) = \prod_{i\sim j} e^{-\beta_{ij}[\ldots]}$$

$$\cdot\ D[m] \prod_{j\neq 0} \frac{dt_j\, ds_j}{2\pi} e^{-t_j}\, \delta_{t_0=0}\, \delta_{s_0=0}$$

$m_{ij} =$

$$\int_c e^{-\frac{1}{2}(s, ms)} ds = \frac{1}{\sqrt{D[m]}}$$

$\Longrightarrow$ VRJP = marginal of $H^{2|2}$ integrate out s

Relation between LERRW and VRJP

[Sabot-Tarrès]

VRJP with β_e indep. r.v. $=$ LERRW

$$\mathbb{P}^{LERRW}_{\Lambda a} = \int \prod_e \underbrace{d\gamma(\beta_e)}_{\frac{e^{-\beta_e}\beta_e^{a_e-1}}{\Gamma(a_e)}} \; \mathbb{P}^{\bar{t}}_{\Lambda,\beta} = \int \prod_e d\gamma(\beta_e)\; d\mu_{\Lambda,\beta}(t) \; \mathbb{P}^{\overbrace{U(t,\beta)}^{\omega}}$$

law for ... RW environment
rand environment with conduct w_e

Results / Conj $\qquad \mathbb{Z}^d$

$H^{2|2}$ ext. localiz. high temp $\equiv \beta_e \le \beta_1 << 1 \quad \forall\, e \in E$

VRJP $\qquad \forall\, d \ge 1$

$$\int d\mu(t)\; e^{\frac{t_x}{2}} \le e^{-|x|c} \longrightarrow \text{positive recurrence}$$

$d=1$ exp loc $\forall\, \beta$

ext states lowT $\quad \beta_e \ge \beta_2 >> 1 \quad \forall e \qquad d \ge 3$

t_x aligned $\qquad \int (\cosh (t_x - t_y))^m\, d\mu(t) < 2 \quad \forall\, x, y$ unif in Λ

Rolles. Tei. K

strip exp local $\forall\, \beta \qquad \longrightarrow$ transience


LERRW
d ≥ 1 strong reinf. a_e ≤ a_1 << 1
positive recurrence ST
d ≥ 3 weak reinf. a_1 >> a_2 >> 1
transience
d = 1, strip local Rollos, Merkl
weak rec in d = 2