Taming infinities

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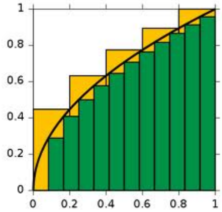
University of Warwick

29.05.2014

Infinities and infinitesimals

Bonaventura Cavalieri (1635): Computes areas by adding infinitesimals. (Even much earlier: Archimedes!)



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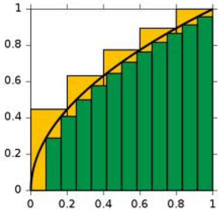

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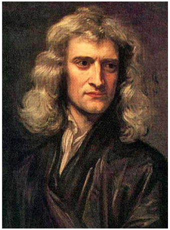
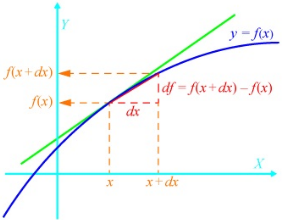
Calculus

Leibniz and Newton (1680's): Compute derivatives by dividing infinitesimals.





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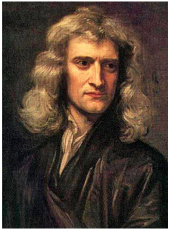

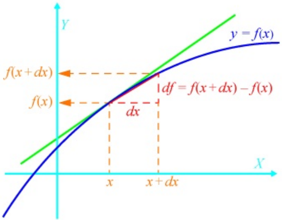

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Resolution

Cauchy, Bolzano, Weierstrass, etc (1820's onward): Rigorous formulation of limits without need for infinitesimals.

Dramatic consequences: Development of a consistent calculus allowing to model many physical processes and underpinning much of modern technology.

Moral of the story: Expressions like $\frac{df}{dx}$ can be given a meaning (as a real number) without having to give meaning to df and dx separately.

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Quantum field theory

General methodology: "Guess" form of Lagrangian, predict outcomes of experiments as function of free parameters, perform experiments to determine them.

Problem: In QFT, when matching theory and experiments, infinities appear!

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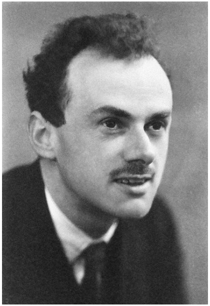
Some reactions

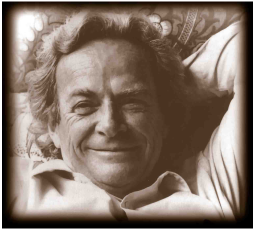
Not everybody liked these techniques...



"This is just not sensible mathematics. Sensible mathematics involves neglecting a quantity when it is small - not neglecting it just because it is infinitely great and you do not want it." – Paul Dirac

More reactions

... not even those who developed them!



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However: Experimental verification of quantum electrodynamics predictions to within 9 digits of accuracy!

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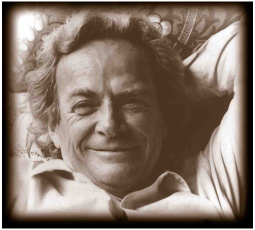

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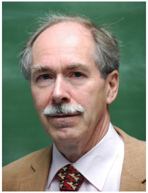
Renormalisability

Some models are perturbatively renormalisable: at every order, parameters can be adjusted (in a diverging way!) to provide consistent answers.

Outcome: Theory with as many parameters as the naïve model.
Moral: "Form" of a model matters, not finiteness of constants.



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Despite investing billions (LHC), that model has not been faulted yet.

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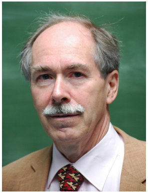

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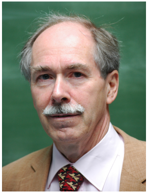

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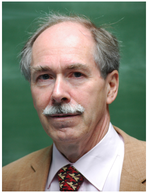

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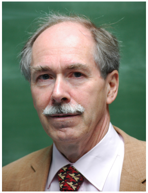

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Example

Try to define distribution "$\eta(x) = \frac{1}{|x|}$".

Problem: Integral of $1/|x|$ diverges, so we need to set "$C = \infty$" to compensate!

Formal definition:

$$\eta_\chi(\varphi) = \int_{\mathbf{R}} \frac{\varphi(x) - \chi(x)\varphi(0)}{|x|}\, dx \;,$$

for some smooth compactly supported cut-off χ with $\chi(0) = 1$. Yields one-parameter family $c \mapsto \eta_c$ of models, but no canonical "choice of origin" for c.

Approximation: $1/(\varepsilon + |x|) - 2|\log \varepsilon|\, \delta(x)$ converges to η_c for some c.

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Stochastics / Finance

Random walk: $W^{\varepsilon}_{t+\varepsilon} = W^{\varepsilon}_t + \sqrt{\varepsilon}\xi_t$ with $\{\xi_t\}$ independent identically distributed random variables, zero mean, unit variance. Donsker's invariance principle: $W^{\varepsilon}_t \to W_t$ with W_t a Brownian motion.

Simple asset price model: $S^{\varepsilon}_{t+\varepsilon} = S^{\varepsilon}_t(1+\sqrt{\varepsilon}\xi_t)$. (So $\delta S^{\varepsilon}_t = S^{\varepsilon}_t\,\delta W^{\varepsilon}_t$.) Formally, one expects in the limit $\varepsilon \to 0$ to have $dS/dt = S\,dW/dt$, so that $S_t = S_0\exp(W_t)$.

Wrong: The limit satisfies $S_t = S_0\exp(W_t - t/2)$. (Can be "guessed" from $\mathbf{E}S_t = \mathbf{E}S_0$.)

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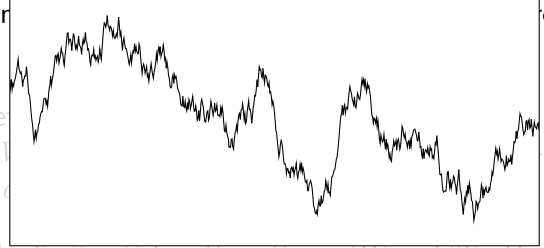

Sample path of Brownian motion

Simple asse… $\delta S^{\varepsilon}_t = S^{\varepsilon}_t\, \delta W$… … 0 to have $dS/dt = S$ …

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What went wrong??

Problem: While S^ε converges to a limit and dW^ε/dt converges to a limit, these are too rough for their product to be well-posed.

In general $(f, \xi) \mapsto f \cdot \xi$ well-posed on $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ if and only if $\alpha + \beta > 0$. Here: just below borderline.

Consequence: Limit depends on details of discretisation. For example, if we set instead $S_{t+\varepsilon} = \frac{S_t + S_{t+\varepsilon}}{2}(1 + \sqrt{\varepsilon}\xi_t)$, then $S_t^\varepsilon \to S_0 \exp(W_t)$ as expected. In general: one-parameter family $S_0 \exp(W_t - ct)$ with $c \in \mathbf{R}$.

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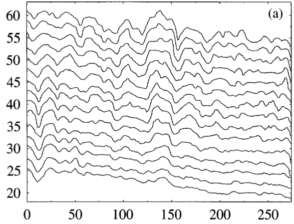
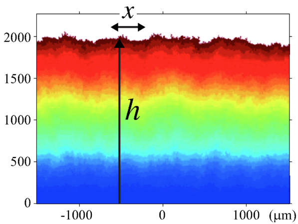
Universality: Many random systems "look the same" and are "scale invariant" when viewed at scales much larger than that of the mechanism producing them, provided that they share some basic features:



Maunuksela & Al, PRL



Takeuchi & Al, Sci. Rep.

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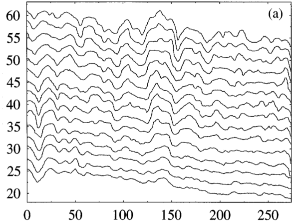

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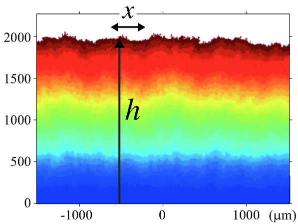

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Crossover regimes

Described by simple "normal form" equations:

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi - C \;, \qquad (\text{KPZ}; d = 1)$$
$$\partial_t \Phi = -\Delta\big(\Delta\Phi + C\Phi - \Phi^3\big) + \nabla\xi \;. \quad (\Phi^4; d = 2, 3)$$

Here ξ is space-time white noise (think of independent random variables at every space-time point).

KPZ: universal model for weakly asymmetric interface growth.

Φ^4: universal model for phase coexistence near mean-field.

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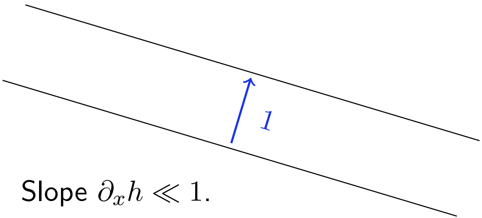
Flat interface propagation



Slope $\partial_x h \ll 1$.

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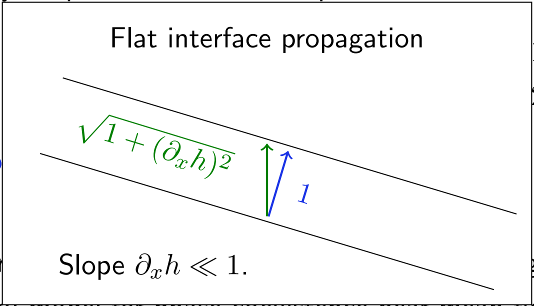

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Well-posedness results

Write ξ_ε for mollified version of space-time white noise. Consider

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(Periodic boundary conditions on torus / circle.)

Theorem (H. 2013): There are $C_\varepsilon \to \infty$ so that solutions converge to a limit independent of the regularisation. (The constants themselves do depend on that choice.) For KPZ, limit coincides with Cole-Hopf solution (if C_ε is chosen appropriately).

Corollary of proof: Rates of convergence, precise local description of limit, suitable continuity, etc.

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Universality result for KPZ

(In progress; joint with J. Quastel.) Consider

$$\partial_t h = \partial_x^2 h + \sqrt{\varepsilon} P(\partial_x h) + \xi \;,$$

with P even polynomial, ξ smooth space-time Gaussian field with compactly supported correlations.

Theorem: As $\varepsilon \to 0$, there is a choice of $C_\varepsilon \sim \varepsilon^{-1}$ such that $\varepsilon^{1/2} h(\varepsilon^{-1}x, \varepsilon^{-2}t) - C_\varepsilon t$ converges to solutions to (KPZ)$_\lambda$ with λ depending in a non-trivial way on all coefficients of P.

Remark: Convergence to KPZ with $\lambda \neq 0$ even if $P(u) = u^4$!!

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Main Idea

Problem: Solutions are not smooth.

Insight: What is "smoothness"? Proximity to polynomials; we know how to multiply these...

Idea: Replace polynomials by a (finite / countable) collection of tailor-made space-time functions / distributions with similar algebraic / analytic properties. Depends on the realisation of the noise, but not on "details" of the equation. (Values of constants, initial condition, boundary conditions, etc.)

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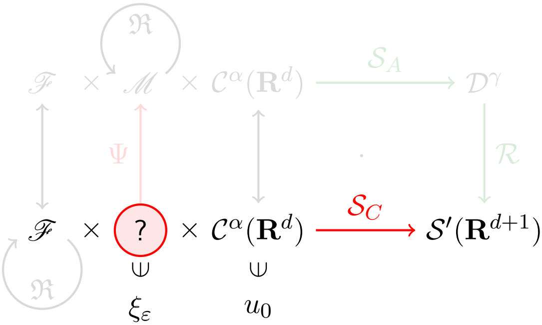
General picture

Method of proof: Build objects for the following diagram:



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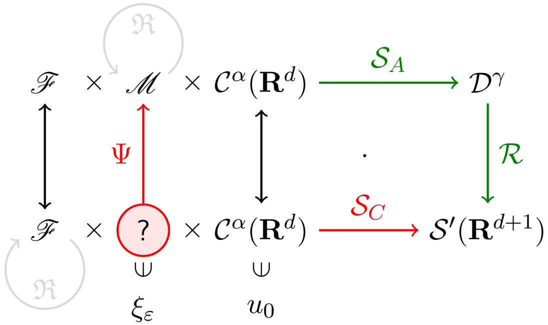

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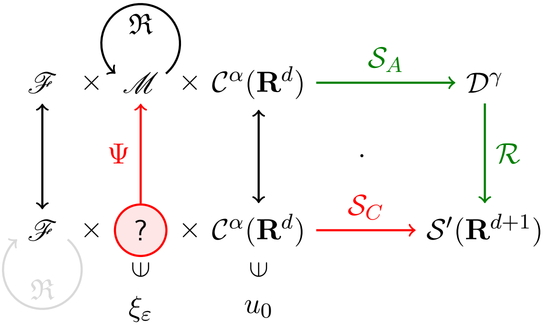

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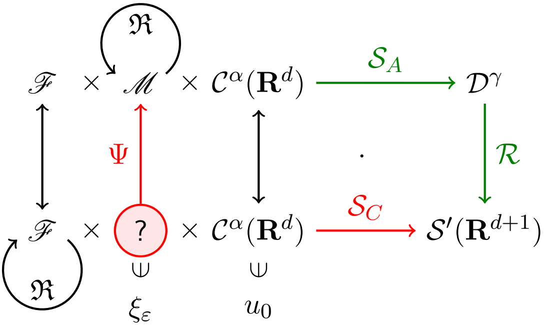

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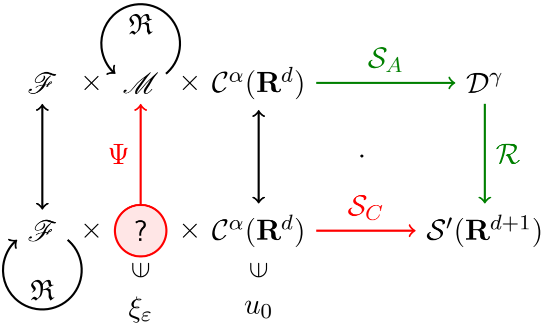

Strategy: find $M_\varepsilon \in \mathfrak{R}$ such that $M_\varepsilon \Psi(\xi_\varepsilon)$ converges.

Some concluding remarks

1. Diverging terms can (sometimes) be cured by suitable counterterms, in probabilistic models, not just in QFT.

2. Forces one to deal with families of models parametrised by constants defined "up to an infinite part". Not a problem as soon as "observables" are finite and answers are consistent...

3. Goal: obtain "universality" results for models from statistical mechanics with tuneable parameters. (Current theory works well for continuous rather than discrete models.)

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