Semiclassical Limit of Liouville Field Theory

Rémi Rhodes

joint work with Hubert Lacoin and Vincent Vargas

Workshop "Gradient random fields"

Plan of the talk

1. Motivations in physics
2. Liouville Field Theory
3. Correlation functions of LFT

(Euclidean) $2d$-Liouville quantum gravity

Theory aimed at constructing "canonical" $2d$-random metrics, taking eventually into account an interaction with matter.

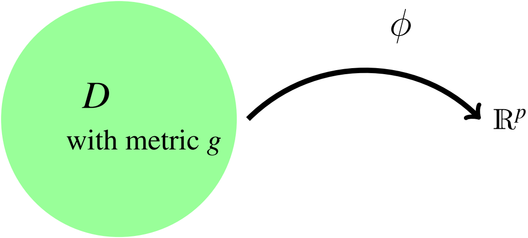
Applications:

- Toy model for $4d$ quantum gravity.
- Conjectural scaling limit of models of statistical physics living on random planar maps at their critical point.

  Refs: Ambjørn, Duplantier, Le Gall, Miermont, Sheffield,...

Construct a couple of random variables (g, ϕ) such that:

$g =$ random metric on D $\qquad$ $\phi =$ matter field on D



The Hamiltonian of (g, ϕ) reflects the weight of the partition function of ϕ in the metric g.

Why $2d$-Liouville field theory?

Fix a random metric g_0 and condition the random metric g on belonging to the conformal equivalence class of g_0, i.e.

$$g = e^{\varphi} g_0$$

where $\varphi : D \subset \mathbb{R}^2 \to \mathbb{R}$ is a random field.

Conjecture (David, Distler, Kawai, Knizhnik, Polyakov, Zamolodchikov,...)

Under some conformal invariance conditions, there is some $\gamma \in [0, 2]$ s.t.

1. φ is a Liouville field, i.e. has the law

$$\tilde{\mathbb{E}}[F(\varphi)] = Z_{\mu,\gamma}^{-1} \mathbb{E}\Big[F(\gamma X) \exp\Big(-\mu \int_D e^{\gamma X}\, dx\Big)\Big] \quad \text{(when } g_0 \text{ is flat)}$$

   and X is a Gaussian Free Field (GFF).

2. γ is determined by the structure of the matter field (central charge).

Why $2d$-Liouville field theory?

Fix a random metric g_0 and condition the random metric g on belonging to the conformal equivalence class of g_0, i.e.

$$g = e^{\varphi} g_0$$

where $\varphi : D \subset \mathbb{R}^2 \to \mathbb{R}$ is a random field.

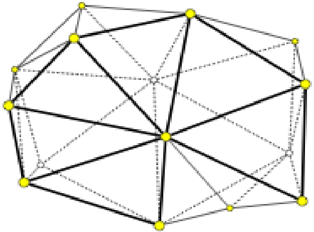
Conjecture (David, Distler, Kawai, Knizhnik, Polyakov, Zamolodchikov,...)

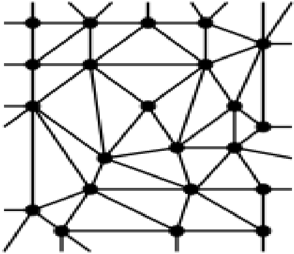
Under some conformal invariance conditions, there is some $\gamma \in [0, 2]$ s.t.

1. φ is a Liouville field, i.e. has the law

$$\tilde{\mathbb{E}}[F(\varphi)] = Z_{\mu,\gamma}^{-1} \mathbb{E}\Big[F(\gamma X) \exp\Big(-\mu \int_D e^{\gamma X}\, dx\Big)\Big] \quad \text{(when } g_0 \text{ is flat)}$$

   and X is a Gaussian Free Field (GFF).
2. γ is determined by the structure of the matter field (central charge).





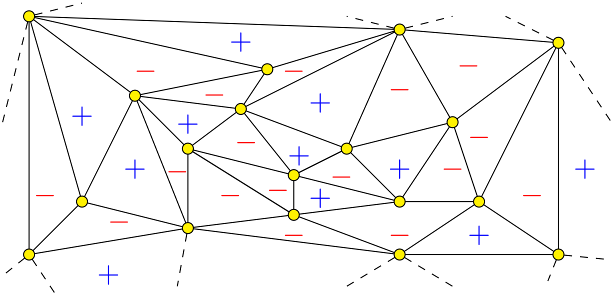
Random planar triangulations:
-uniform law on embeddings of triangulations (with n faces) into the sphere, up to homeomorphisms.

What does the scaling limit of random triangulations look like?

Folklore conjecture

Limit of random planar triangulations is a Liouville Field Theory on the sphere with

$$\gamma = \sqrt{\frac{8}{3}}$$



Random planar triangulations coupled to Ising model:
-probability measure on couplings (T_n, σ) with distribution:

$$\mathbb{P}_J(T,\sigma) \asymp \exp\Big(J \sum_{f\sim f'} \sigma_f \sigma_{f'}\Big)$$

Compute the critical point (Kazakhov) J_c and take the scaling limit...

Coupling matter with quantum gravity

Folklore conjecture

The scaling limit of random planar triangulations weighted with the Ising model is a Liouville Field Theory on the sphere with

$$\gamma = \sqrt{3}$$

Liouville metrics

Study the metric tensor

$$g = e^{\varphi(x)} dx^2$$

where $\varphi : D \subset \mathbb{R}^2 \to \mathbb{R}$ is a random field with law

$$\tilde{\mathbb{E}}[F(\varphi)] = \frac{1}{Z_{\mu,\gamma}} \mathbb{E}_{\mathrm{GFF}}\Big[F(\gamma X)\exp\Big(-\mu \int_D e^{\gamma X}\,dx\Big)\Big]$$

for any F continuous and bounded on $H^{-1}(D)$.

Questions:

1. How to construct this law?
   (studied by Hoeg-Krohn for small γ)
2. Describe how the parameters γ, μ interact to shape the metric g (where do these metrics concentrate on?)

GFF

Gaussian random distribution (Schwartz) $(X_x)_{x\in D}$ on D s.t.:

- a.s. X lives in the Sobolev $H^{-1}(D)$
- X is centered and $\mathbb{E}[X_x X_y] = G(x,y)$ where G = Green function of Laplacian (say with Dirichlet boundary condition)

Let $(\lambda_n)_n$ be the (positive) eigenvalues of $\triangle$ and $(e_n)_n$ the eigenfunctions

$$X(x) = \sum_{k\geq 1} \frac{\alpha_k e_k(x)}{\sqrt{\lambda_k}}$$

where $(\alpha_n)_n$ are i.i.d. with law $\mathcal{N}(0,1)$.

Write

$$X_n(x) = \sum_{k=1}^{n} \frac{\alpha_k e_k(x)}{\sqrt{\lambda_k}}$$

and define

$$\int_A e^{\gamma X(x)}\, dx = \lim_{n\to\infty} \int_A e^{\gamma X_n(x) - \frac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]}\, dx$$

Gaussian multiplicative chaos (Kahane 1985):

- positive martingale $\Rightarrow$ a.s. convergence
- uniformly integrable $\Leftrightarrow \gamma < 2$.
- the law of the limit does not depend on the chosen cut-off approximation

Write

$$X_n(x) = \sum_{k=1}^{n} \frac{\alpha_k e_k(x)}{\sqrt{\lambda_k}}$$

and define

$$\int_A e^{\gamma X(x)}\,dx = \lim_{n\to\infty} \int_A e^{\gamma X_n(x) - \frac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]}\,dx$$

Gaussian multiplicative chaos (Kahane 1985):

- positive martingale $\Rightarrow$ a.s. convergence
- uniformly integrable $\Leftrightarrow \gamma < 2$.
- the law of the limit does not depend on the chosen cut-off approximation

Take X a GFF under the proba $\mathbb{P}$ and define a new proba ($\mu > 0$ and $\gamma \in [0, 2[$)

$$\mathbb{P}_{\mu,\gamma}(\cdot) = \frac{1}{Z_{\mu,\gamma}} \exp\Big(-\mu \int_D e^{\gamma X(x)}\, dx \Big)\, d\mathbb{P}, \quad Z_{\mu,\gamma} = \mathbb{E}\big[e^{-\mu \int_D e^{\gamma X(x)}\, dx}\big].$$

Liouville field

The law of the Liouville field is the law of γX under $\mathbb{P}_{\mu,\gamma}$.

Semi-classical limit

Compute exact asymptotics for the functionals of the Liouville field (or the metric $e^{\gamma X} dx^2$) when

$$\gamma \to 0, \mu \to \infty, \quad \text{with} \quad \Lambda = \mu\gamma^2 \text{ fixed.}$$

Physics review: Nakayama 2004, Witten 2011,...

Take X a GFF under the proba $\mathbb{P}$ and define a new proba ($\mu > 0$ and $\gamma \in [0,2[$)

$$\mathbb{P}_{\mu,\gamma}(\,\cdot\,) = \frac{1}{Z_{\mu,\gamma}} \exp\Big(-\mu \int_D e^{\gamma X(x)}\,dx \Big)\, d\mathbb{P}, \quad Z_{\mu,\gamma} = \mathbb{E}\big[e^{-\mu \int_D e^{\gamma X(x)}\,dx}\big].$$

Liouville field

The law of the Liouville field is the law of γX under $\mathbb{P}_{\mu,\gamma}$.

Semi-classical limit

Compute exact asymptotics for the functionals of the Liouville field (or the metric $e^{\gamma X} dx^2$) when

$$\gamma \to 0, \mu \to \infty, \quad \text{with} \quad \Lambda = \mu\gamma^2 \text{ fixed.}$$

Physics review: Nakayama 2004, Witten 2011,...

Construction of the Liouville field

Take X a GFF under the proba $\mathbb{P}$ and define a new proba ($\mu > 0$ and $\gamma \in [0,2[$)

$$\mathbb{P}_{\mu,\gamma}(\,\cdot\,) = \frac{1}{Z_{\mu,\gamma}} \exp\Big(-\mu \int_D e^{\gamma X(x)}\,dx \Big)\, d\mathbb{P}, \quad Z_{\mu,\gamma} = \mathbb{E}\big[e^{-\mu \int_D e^{\gamma X(x)}\,dx}\big].$$

Liouville field

The law of the Liouville field is the law of γX under $\mathbb{P}_{\mu,\gamma}$.

Semi-classical limit

Compute exact asymptotics for the functionals of the Liouville field (or the metric $e^{\gamma X} dx^2$) when

$$\gamma \to 0, \mu \to \infty, \quad \text{with} \quad \Lambda = \mu\gamma^2 \text{ fixed.}$$

Physics review: Nakayama 2004, Witten 2011,...

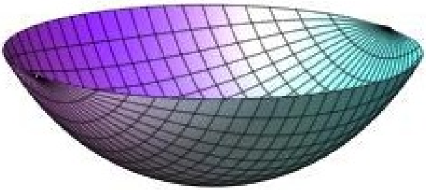
Consider the Liouville equation (LE)

$$\triangle U = 2\pi\Lambda\, e^{U}, \qquad U_{|\partial D} = 0.$$

- Reminder: (LE) determines the metrics on D conformally equivalent to the Euclidean metric with constant scalar curvature $-2\pi\Lambda$.
- Variational formulation:

$$U = \mathrm{Argmin}_{V \in H_0^1(D)}\, \mathcal{E}(V),$$

$$\mathcal{E}(V) = \frac{1}{4\pi}\int_D \left(|\partial V|^2 + 4\pi\Lambda e^{V}\right) dx$$



Consider the Liouville equation (LE)

$$\triangle U = 2\pi\Lambda\, e^{U}, \qquad U_{|\partial D} = 0.$$

- Reminder: (LE) determines the metrics on D conformally equivalent to the Euclidean metric with constant scalar curvature $-2\pi\Lambda$.
- Variational formulation:

$$U = \text{Argmin}_{V\in H^1_0(D)}\, \mathcal{E}(V),$$

$$\mathcal{E}(V) = \frac{1}{4\pi}\int_D \Big(|\partial V|^2 + 4\pi\Lambda e^{V}\Big)\, dx$$

Partition function

When $\gamma \to 0$ and $\Lambda = \mu\gamma^2$ fixed

$$\mathbb{E}[e^{-\frac{\Lambda}{\gamma^2}\int_D e^{\gamma X}\, dx}] \sim e^{-\frac{\mathcal{E}(U)}{\gamma^2}}\, C_\Lambda$$

Hint of proof

- First step

$$\mathbb{E}\Big[\exp\Big(-\frac{\Lambda}{\gamma^2}\int_D e^{\gamma X(x)}\,dx\Big)\Big]$$
$$= e^{-\frac{1}{\gamma^2}\mathcal{E}(U)}\mathbb{E}\Big[\exp\Big(-\frac{\Lambda}{\gamma^2}\int_D e^{U(x)}(e^{\gamma X(x)}-1-\gamma X(x))\,dx\Big)\Big].$$

- Naive conclusion:

$$e^{\gamma X}-1-\gamma X \geq \quad \text{and} \quad e^{\gamma X}-1-\gamma X \sim \frac{\gamma^2}{2}X^2$$

imply,

$$\mathbb{E}\Big[\exp\Big(\dots\Big)\Big] \to \mathbb{E}\Big[\exp\Big(-\frac{\Lambda}{2}\int_D e^{U(x)}\,dx\Big)\Big]$$

- First step

$$\mathbb{E}\Big[\exp\Big(-\frac{\Lambda}{\gamma^2}\int_D e^{\gamma X(x)}\,dx\Big)\Big]$$
$$= e^{-\frac{1}{\gamma^2}\mathcal{E}(U)}\mathbb{E}\Big[\exp\Big(-\frac{\Lambda}{\gamma^2}\int_D e^{U(x)}(e^{\gamma X(x)}-1-\gamma X(x))\,dx\Big)\Big].$$

- Naive conclusion:

$$e^{\gamma X}-1-\gamma X\geq 0 \quad \text{and} \quad e^{\gamma X}-1-\gamma X\sim\frac{\gamma^2}{2}X^2$$

imply,

$$\mathbb{E}\Big[\exp\Big(\dots\Big)\Big]\to\mathbb{E}\Big[\exp\Big(-\frac{\Lambda}{2}\int_D e^{U(x)}X^2\,dx\Big)\Big]$$

- First step

$$\mathbb{E}\Big[\exp\Big(-\frac{\Lambda}{\gamma^2}\int_D e^{\gamma X(x)}\,dx\Big)\Big]$$
$$= e^{-\frac{1}{\gamma^2}\mathcal{E}(U)}\mathbb{E}\Big[\exp\Big(-\frac{\Lambda}{\gamma^2}\int_D e^{U(x)}(e^{\gamma X(x)}-1-\gamma X(x))\,dx\Big)\Big].$$

- Naive conclusion: renormalisation effects

$$e^{\gamma X}-1-\gamma X \geq -\frac{\gamma^2}{2}\mathbb{E}[X^2] = -\infty \quad \text{and} \quad e^{\gamma X}-1-\gamma X \sim \frac{\gamma^2}{2}X^2-\frac{\gamma^2}{2}\mathbb{E}[X^2]$$

imply, by defining $:X^2: = X^2 - \mathbb{E}[X^2]$

$$\mathbb{E}\Big[\exp\Big(\dots\Big)\Big] \to \mathbb{E}\Big[\exp\Big(-\frac{\Lambda}{2}\int_D e^{U(x)} :X^2:\,dx\Big)\Big]$$

-$(\lambda_n)_n$=eigenvalues of $\triangle$
-$(e_n)_n$=eigenfunctions of $\triangle$
-$(\alpha_n)_n$ sequence of i.i.d. $\mathcal{N}(0,1)$

$$GFF \longrightarrow \quad X(x) = \sum_{k\geq 1} \frac{\alpha_k e_k(x)}{\sqrt{\lambda_k}}.$$

Partial sum $\quad X_n(x) = \sum_{k=1}^{n} \frac{\alpha_k e_k(x)}{\sqrt{\lambda_k}}.$

Field $:X^2:$

$$\int_D :X^2: f(x)\,dx = \lim_{n\infty} \int_D \big(X_n(x)^2 - \mathbb{E}[X_n^2(x)]\big) f(x)\,dx.$$

A.s. limit of a martingale bounded in L^2.

Massive Free Field in the metric $g = e^{U(x)}dx^2$.

Centered Gaussian random distribution with covariance the Green function of the operator $(2\pi)^{-1}(m - \triangle_g)$ on D.

Property

Under the probability measure

$$d\tilde{P} = Z_\alpha^{-1} e^{-\frac{\alpha}{2}\int_D :X^2: e^U \, dx} d\mathbb{P},$$

the field X has the law of a Massive GFF with α in the metric $e^{U(x)}dx^2$.

LLN

The field γX converges in $\mathbb{P}_{\mu,\gamma}$-probability towards U in $H^{-1}(D)$.

Proof:

$$\mathbb{E}\Big[F(\gamma X)e^{-\frac{\Lambda}{\gamma^2}\int_D e^{\gamma X(x)}\,dx}\Big] \sim F(U)e^{-\frac{\mathcal{E}(U)}{\gamma^2}}\mathbb{E}[e^{-\frac{\Lambda}{2}\int_D :X^2:\,e^{U(x)}\,dx}]$$

Fluctuations

The field $X - U/\gamma$ converges in law as $\gamma \to 0$ towards a Massive MFF with mass Λ in the metric $e^{U(x)}dx^2$.

Proof:

$$\mathbb{E}\Big[F(X - U/\gamma)e^{-\frac{\Lambda}{\gamma^2}\int_D e^{\gamma X(x)}\,dx}\Big] \sim \mathbb{E}\Big[F(X)e^{-\frac{\Lambda}{2}\int_D :X^2:\,e^{U(x)}\,dx}\Big].$$

LLN

The field γX converges in $\mathbb{P}_{\mu,\gamma}$-probability towards U in $H^{-1}(D)$.

Proof:

$$\mathbb{E}\Big[F(\gamma X)e^{-\frac{\Lambda}{\gamma^2}\int_D e^{\gamma X(x)}\,dx}\Big] \sim F(U)e^{-\frac{\mathcal{E}(U)}{\gamma^2}}\mathbb{E}[e^{-\frac{\Lambda}{2}\int_D :X^2:\,e^{U(x)}\,dx}]$$

Fluctuations

The field $X - U/\gamma$ converges in law as $\gamma \to 0$ towards a Massive MFF with mass Λ in the metric $e^{U(x)}dx^2$.

Proof:

$$\mathbb{E}\Big[F(X - U/\gamma)e^{-\frac{\Lambda}{\gamma^2}\int_D e^{\gamma X(x)}\,dx}\Big] \sim \mathbb{E}\Big[F(X)e^{-\frac{\Lambda}{2}\int_D :X^2:\,e^{U(x)}\,dx}\Big].$$

Exact asymptotics of Laplace transforms

Set $Y_\gamma = \gamma X$. Study the limit as $\gamma \to 0$ of

$$\mathbb{E}\Big[e^{\frac{\int_D Y_\gamma f\,dx}{\gamma^2}} e^{-\frac{\Lambda}{\gamma^2}\int_D e^{\gamma X(x)}\,dx}\Big]$$

Same strategy

$$\mathbb{E}\Big[e^{\frac{Y_\gamma(f)}{\gamma^2}} e^{-\frac{\Lambda}{\gamma^2}\int_D e^{\gamma X(x)}\,dx}\Big] \sim e^{-\frac{T(V,f)}{\gamma^2}}\mathbb{E}\Big[e^{-\frac{\Lambda}{2}\int_D e^{V(x)}:X^2(x):\,dx}\Big]$$

where V is the solution to

$$\triangle V = 2\pi\Lambda\, e^V - 2\pi f, \quad V_{|\partial D} = 0$$

Remark: kind of Varadhan's lemma for (some) non continuous functionals of the GFF

Set $Y_\gamma = \gamma X$. Study the limit as $\gamma \to 0$ of

$$\mathbb{E}\Big[e^{\frac{\int_D Y_\gamma f\,dx}{\gamma^2}} e^{-\frac{\Lambda}{\gamma^2}\int_D e^{\gamma X(x)}\,dx}\Big]$$

Same strategy

$$\mathbb{E}\Big[e^{\frac{Y_\gamma(f)}{\gamma^2}} e^{-\frac{\Lambda}{\gamma^2}\int_D e^{\gamma X(x)}\,dx}\Big] \sim e^{-\frac{T(V,f)}{\gamma^2}}\,\mathbb{E}\Big[e^{-\frac{\Lambda}{2}\int_D e^{V(x)}:X^2(x):\,dx}\Big]$$

where V is the solution to

$$\triangle V = 2\pi\Lambda\, e^V - 2\pi f, \quad V_{|\partial D} = 0$$

Remark: kind of Varadhan's lemma for (some) non continuous functionals of the GFF

Define the "good rate function"

$$I^*(h) = \begin{cases} \mathcal{E}(h) - \mathcal{E}(U), & \text{if } h \in H_0^1(D), \\ +\infty, & \text{sinon}, \end{cases}$$

where U satisfies $\triangle U = 2\pi \Lambda e^U$ and

$$\forall h \in H_0^1(D), \quad E(h) = \frac{1}{4\pi} \int_D (|\partial h(x)|^2 + 4\pi \Lambda e^{h(x)}) dx.$$

Theorem

Fix $\Lambda = \mu\gamma^2$. γX satisfies a LDP in $H^{-1}(D)$

$$-\inf_{h \in \mathring{A}} I^*(h) \leq \liminf_{\gamma \to 0} \; \gamma^2 \ln \mathbb{P}_{\mu,\gamma}(\gamma X \in A)$$

$$\leq \limsup_{\gamma \to 0} \; \gamma^2 \ln \mathbb{P}_{\mu,\gamma}(\gamma X \in A) \leq -\inf_{h \in \overline{A}} I^*(h).$$

Correlation functions of the Liouville field

$$\mathbb{E}\Big[\prod_{i=1}^{p} e^{\alpha_i X(z_i)}\; e^{-\frac{\Lambda}{\gamma^2}\int_D e^{\gamma X(x)}\,dx}\Big]$$

where $(z_i)_i$ are given points in D and $(\alpha_i)_i$ given "weights".

Heavy matter: choose the $\alpha_i' s$ in such a way that they will affect the saddle point of the action as $\gamma \to 0$, i.e.

$$\alpha_i = \frac{\chi_i}{\gamma} \quad \text{avec} \quad \chi_i \in]0,2[.$$

Correlation functions of the Liouville field

$$\mathbb{E}\Big[\prod_{i=1}^{p} e^{\alpha_i X(z_i)}\; e^{-\frac{\Lambda}{\gamma^2}\int_D e^{\gamma X(x)}\,dx}\Big]$$

where $(z_i)_i$ are given points in D and $(\alpha_i)_i$ given "weights".

Heavy matter: choose the $\alpha_i' s$ in such a way that they will affect the saddle point of the action as $\gamma \to 0$, i.e.

$$\alpha_i = \frac{\chi_i}{\gamma} \quad \text{avec} \quad \chi_i \in]0,2[.$$

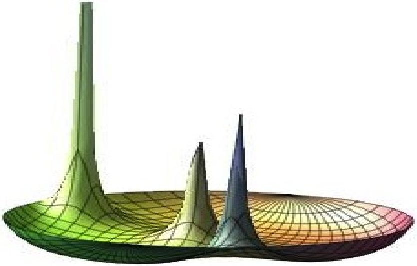
Asymptotic of the field perturbed with heavy matter

Same concentration results (LLN, fluctuations, LDP) around

Singular Liouville equation

$$\triangle V = \Lambda e^V - 2\pi \sum_i \chi_i \, \delta_{z_i}.$$

The metric $e^{V(x)}dx^2$ then stands for a surface with negative curvature $-\Lambda$ and conical singularities at the points z_i with deficit angle χ_i.



Thanks!