Statistical Mechanics arising from Random Matrix Theory

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May 29, 2014

Introduction

Many problems in mathematical physics are best understood using by using a "dual" representation. A classic example of this is the circle method widely used in number theory. For example if P(N) is the number of partitions of the integer N then

$$P(N) = \frac{1}{2\pi i} \oint \frac{1}{\prod_m (1 - z^m)} z^{-N-1} dz \approx \frac{1}{4\sqrt{3}N} e^{\pi\sqrt{2N/3}}$$

The spectral properties of Random matrices can also be obtained by studying dual statistical mechanics systems. The spins are given by 4×4 matrices and the action is invariant under the supersymmetries $U(1,1|2)$. This is a noncompact hyperbolic symmetry combined with a compact $U(2)$ group.
The advantage of this representation is that universality and phenomenology can be formally deduced by studying the saddle manifold and fluctuations about it. Non perturbative.

Outline of Talk

A) Sub-quadratic actions: Example: $\sum_j \sqrt{1+(\nabla\phi_j)^2}$

D. Brydges and T.S.**: JMP 53** (2012)

Free Field bounds on $\langle(\phi_0 - \phi_x)^{2m}\rangle\,,\ \langle e^{\alpha(\phi_0-\phi_x)}\rangle$

Remarks:

Action *non-covex* but it is a *superposition* of free fields

Analyze *non-uniformly elliptic* Green's functions

M. Zirnbauer and T.S. **: CMP 252** (2004)

Hyperbolic sigma model. See also Les Houches.

B) Strong CLT in Two Dimensions

J. Conlon and T.S. **: CMP 325** (2014)

$$g(\alpha) = \ln\langle e^{\alpha(\phi_0 - \phi_x)}\rangle, \quad |x| \gg 1$$

In 2 dimensions for suitable convex actions of $\nabla\phi$:

Theorem: $g''(\alpha) \approx C \ln|x|$, **but** $|g'''(\alpha)| \leq Const$

In 1D: $g''(\alpha) \approx C|x|$, **but** $|g'''(\alpha)| \approx C'|x|, \;\; C' > 0$

Motivation: Dimers heights , fluctuation of CUE eigenvalues

C) Conjecture: Mean Field Theory Univeral for $D \geq 3$

O(n) invariant interacting spins: eg. X-Y model

$s_j \in \mathbb{R}^n$, $j \in \Lambda_L \cap Z^d$, periodic box of side L, $h \in R^n$

Conjecture:

$$\langle e^{\sum_{j\in\Lambda} h\cdot s_j/|\Lambda|}\rangle(\beta) \; as \; L \Rightarrow \infty$$

$$= \int_{S^{n-1}} e^{h\cdot S_0 M(\beta)} d\mu(S_0) \times [1 + O(\frac{1}{\beta L^{d-2}})]$$

$M(\beta) =$ Magnetization, $d\mu(S_0)$ is uniform measure on S^{n-1} .

The leading term is like a **Law of Large Numbers** and the correction is **CLT.**

Theorem (J. Fröhlich and T.S.) Holds for **O(2) symmetric systems**

Main ideas: Pure states are parmetrized by S^1 , IR bounds.

Explanation of Conjectured Universality of Wigner-Dyson statistics: Apply SUSY statistical mechanics, (Kravstov and Mirlin (1994))

$$U(1,1|2)/U(1|1) \times U(1|1), \quad M(\beta) \approx \rho(E), \; DOS$$

Related Example with SU(2) symmetry:

Universality of Characteristic Polynomial Gaussian **Band** Matrices T. Shcherbina, **CMP 328** (2014):

Let H be an $N \times N$ Gaussian matrix such that

$$\langle H_{ij} \bar{H}_{i'j'} \rangle = \frac{e^{-|i-j|/W}}{W} \, \delta_{i,i'} \, \delta_{j,j'} \quad 1 \le i,j \le N$$

Define

$$F_N(E,u) = \frac{\langle \det(H - E + u/N) \det(H - E - u/N) \rangle}{\langle \det(H - E)^2 \rangle}$$

If the width $W \gg \sqrt{N} \gg 1$ then

$$F_N(E,u) \Rightarrow \int_{S^2} e^{iu\rho(E) S_0^{(3)}} d\mu = \frac{\sin(2u\rho(E))}{2u\rho(E)}$$

$$\beta \approx W^2, \ \ \rho(E) = \sqrt{1 - (E/2)^2}, \ \ L = N$$

Proof of A)

Consider the action given by the quadratic form:

$$A(\phi, t) = \sum_{j\sim k} \left(1 + \frac{1}{2}\beta(\phi_j - \phi_k)^2\right) e^{t_{jk}} + \frac{1}{2}\sum_{j\in\Lambda} \epsilon\phi_j^2 \,, \qquad t_{jk} \in \mathbb{R}$$

$e^{t_{jk}}$ are local conductances across edge jk. The t_{jk} are independent:

$$e^{-A(\phi,t)} \prod_{j\sim k} \left(e^{-e^{-t_{jk}} - \frac{1}{2}t_{jk}} dt_{jk}\right)$$

Integration over the t_{jk} gives:

$$\prod_{j\sim k} e^{-\sqrt{(1+(\phi_j-\phi_k)^2}} \,.$$

Let

$$\langle \phi, D(t)\phi \rangle = \sum_{j\sim k} \beta(\phi_j - \phi_k)^2 e^{t_{jk}} + \epsilon \sum_{j\in\Lambda} \phi_j^2$$

We can now compute integrals in ϕ in terms of $G(t) = D(t)^{-1}$, the Green's function of D. Note $G(0) = (-\beta\Delta + \epsilon)^{-1}$.

After integration over t:

$$Z^{-1} \int [G(t)(0,0) - G(t)(0,x)] \; Det[D(t)]^{-1/2} \prod_{j\sim k} \left(e^{-e^{-t_{jk}} - \frac{1}{2} t_{jk}} dt_{jk} \right)$$

$$= \langle [\phi(0) - \phi(x)]^2 \rangle \quad \textit{Subquadratic expectation}$$

Elements of Proof of A)

A) By Matrix Tree Theorem

$$\log Det[D(t)] \textit{ is } \textbf{jointly convex} \textit{ in } t_{jk}$$

So we can apply Brascamp-Lieb.

B) $\langle v;\, G(t)v\rangle \leq \sum_{j\sim k} \left[(G(0)v)_j - (G(0)v)_k\right]^2 e^{-t_{jk}}$

$$v = \delta_0 - \delta_x$$

C) Ward identity: $t_{jk} \to t_{jk} + c$ get bounds on $\langle t\rangle$

Ideas for proof of Strong CLT

Action $= V(\nabla\phi)$ and suppose: $\quad \lambda \le V'' \le \Lambda, \quad \Lambda/\lambda < 2$

$$g'''(\alpha) = \langle\, [X - \langle X \rangle]^3 \,\rangle_{V,\alpha} \quad \textit{where } X = (\phi_0 - \phi_x)$$

Integrate by parts using **Helffer - Sjöstrand, twice**.

Then use bounds on *singular integral operators* on Weighted spaces.

$$\nabla G_0(j-k) \approx |j-k|^{-1}$$

belongs to $\ell_2(W)$ if W decays.

Singular integral operator: $\nabla\Delta^{-1}\nabla$, dipole kernel.

Helffer-Sjöstrand Formula for $\sum_j V(\nabla\phi_j)$

Let F be smooth function of ϕ_j and let $dF(j) = \partial F/\phi_j$

$$\langle (F_1 - \langle F_1\rangle)\, F_2\rangle \;=\; \langle dF_1 \cdot [d^* d + \nabla^* V''(\nabla\phi)\nabla]^{-1} dF_2\rangle$$

Note $\nabla^* V''(\nabla\phi)\nabla$ is the Hessian,

$d^* d$ denotes the operator corresponding to the Dirichlet form.

$$d_j = \partial/\partial\phi_j \quad d_j^* = -d_j + \partial V/\partial\phi_j.$$

SUSY Hyperbolic Sigma Model

Hyperbolic sigma model in Horosperical coordinates: $t_j, s_j \in \mathbb{R}$

$$A(s,t) = \beta \sum_{j \sim j'} \{\cosh(t_j - t_j) + \frac{1}{2}(s_j - s_{j'})^2 e^{t_j + t_{j'}}\}$$

After integrating out the s variables we get the partition function:

$$e^{-\beta \sum_{j \sim j'} \cosh(t_j - t_j)} \times Det^{-1/2}[D(t)] \times \prod_j e^{+t_j} dt_j.$$

Note A is invariant under $t_j \to t_j + c$. **Convex Gradient Model**

To obtain the effective SUSY Hyperbolic model: flip signs in red.

The partition function $Z(\beta, \epsilon) = 1$ and $\langle e^{\alpha\, t_0} \rangle = \langle e^{(1-\alpha)\, t_0} \rangle$

Convexity **fails** and there is a transition in 3D. (DSZ '10).

Model due to M. Zirnbauer ('91). Multifractal exponents at β_c

Sabot and Tarres (2012) proved that the SUSY Hyperbolic sigma model is mixing measure for Vertex Reinforced Jump Process. β represents the speed of the jumps.

Transition is from recurrence (localization) to transience (diffusion) .