Loop Weighted Walk

Plan (i) Model def$^{\underline{n}}$

(ii) Main difficulty: Lack of a repulsion

(iii) Sol$^{\underline{n}}$

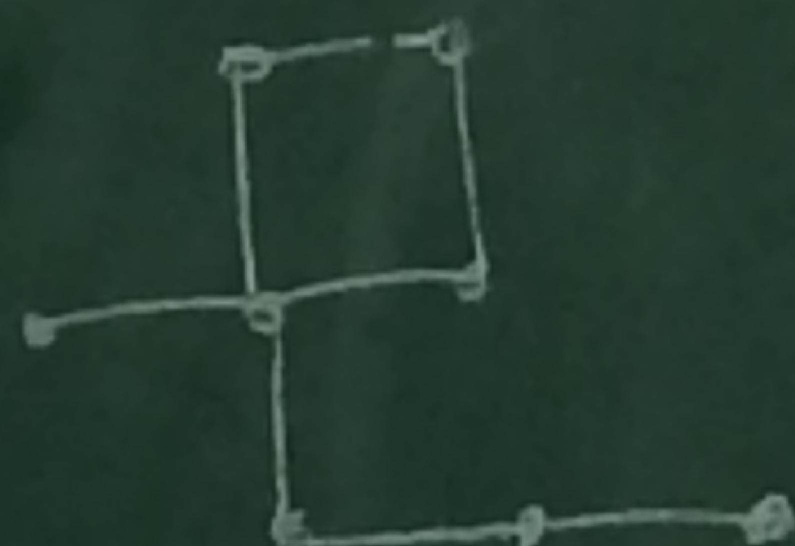
Model: λ-LWW (on $\mathbb{Z}^d$) is the

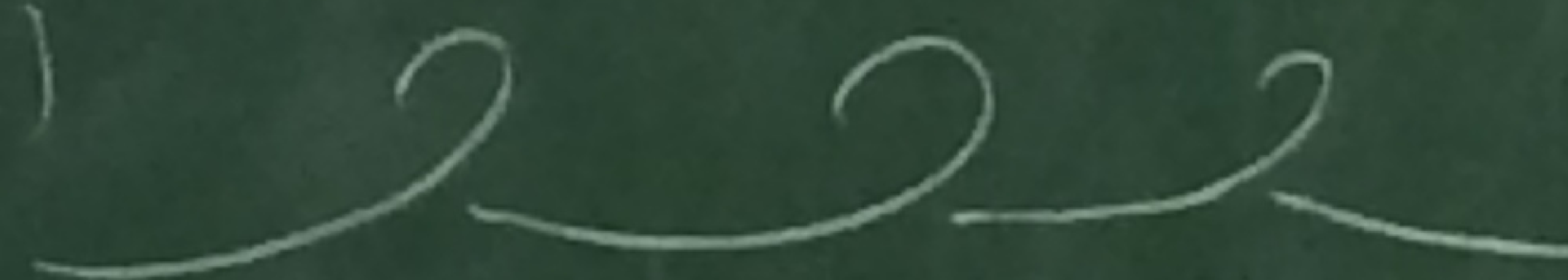
weight $W_{\lambda,\mu}(\omega) = \mu^n \lambda^{n_L(\omega)}$

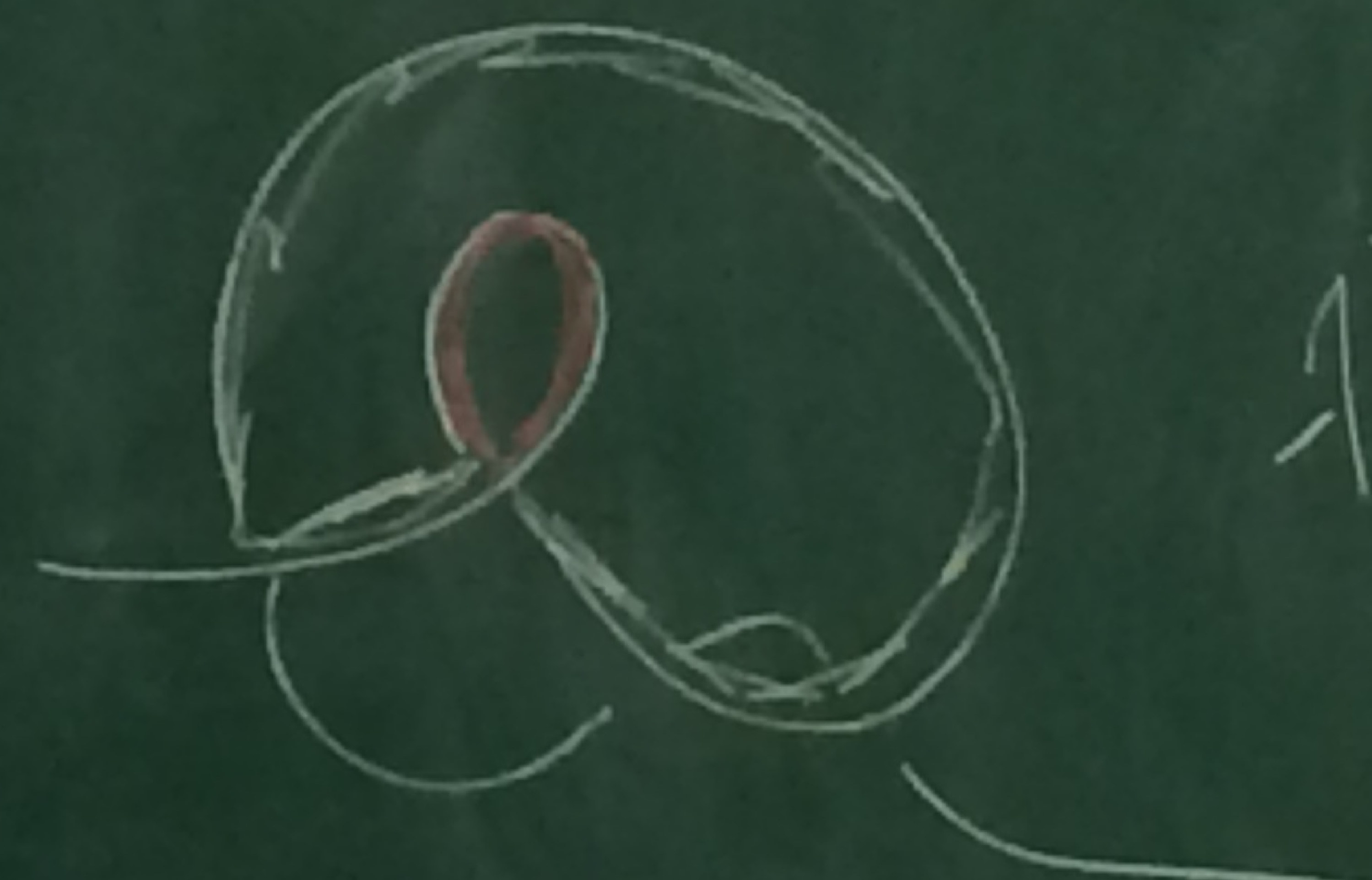
- $\mu > 0$ is the activity per step

- $\lambda > 0$ is the loop activity, $n_L(\omega)$ # loops erased by loop erasure.

• $\lambda \geq 0$ is the loop activity, $n_L(\omega)$ # loops erased by loop erasure.

(i)  $z^8 \lambda$

(ii)  λ^3

(iii)  λ^2

Let $G_{\lambda,z}(x) = \sum_{\omega: 0 \to x} w_{\lambda,z}(\omega)$, $\chi_\lambda(z) = \sum_x G_{\lambda,z}(x)$.

Let z_c = radius of conv. of $\chi_\lambda(z)$.

<u>Thm</u>: Fix $\lambda \geq 0$. $\exists\, d_0 = d_0(\lambda)$ s.t. if $d \geq d_0$, the lace expansion for λ-LWW converges.

In particular, $\exists\, \overline{K} > 0$ s.t.

$$\hat{G}_{\lambda,z_c}(k) \leq \overline{K}\, \hat{C}(k)$$

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<u>Rmk</u>. This is closely related to the loop $O(N)$ model

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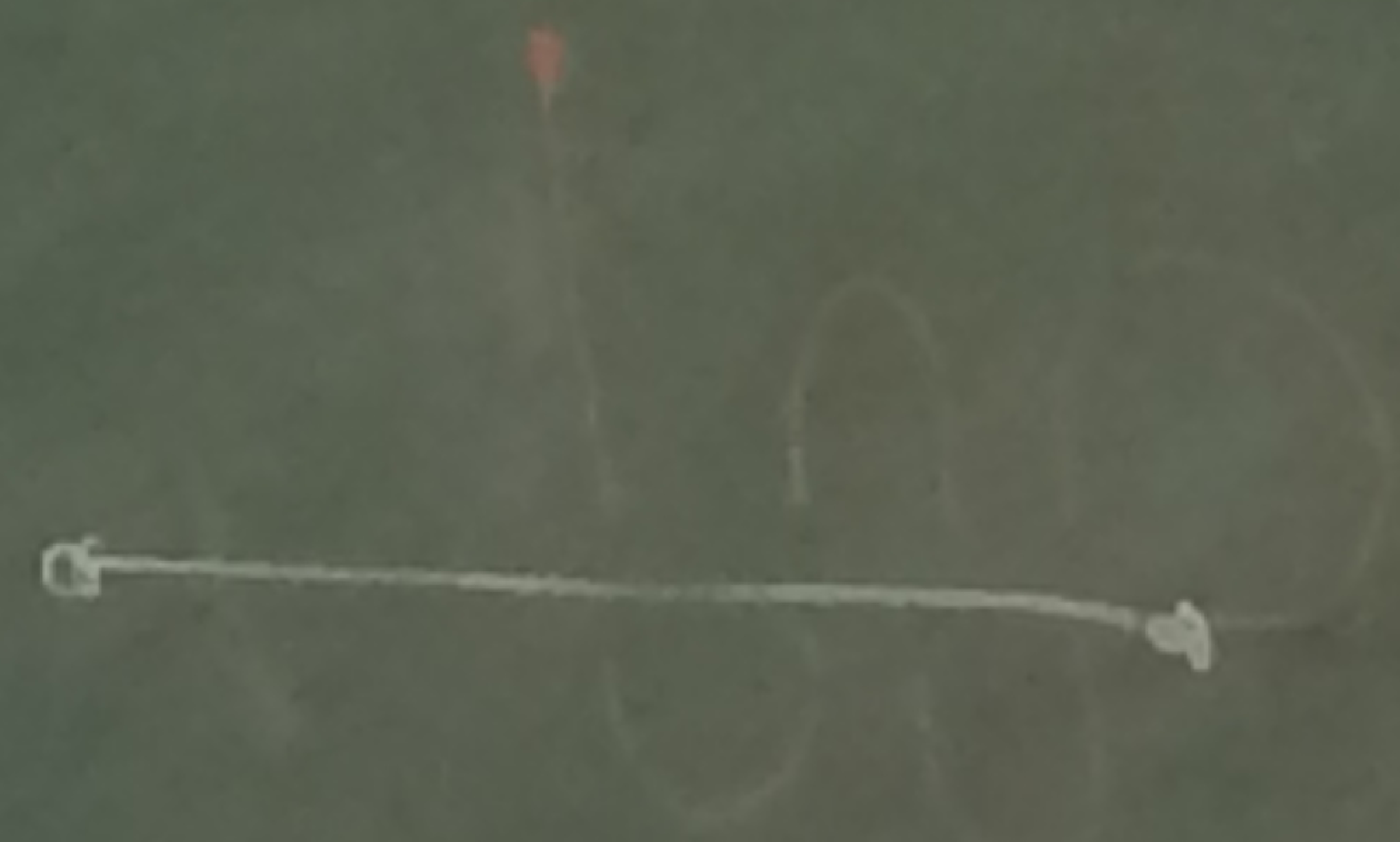
<u>Difficulty</u>. Call a walk model <u>repulsive</u> if

$$w(\eta^{(1)} \circ \eta^{(2)}) \leq w(\eta^{(1)})\, w(\eta^{(2)})$$

<u>E.g.</u> SRW, SAW.

λ-LWW is <u>not</u> repulsive:

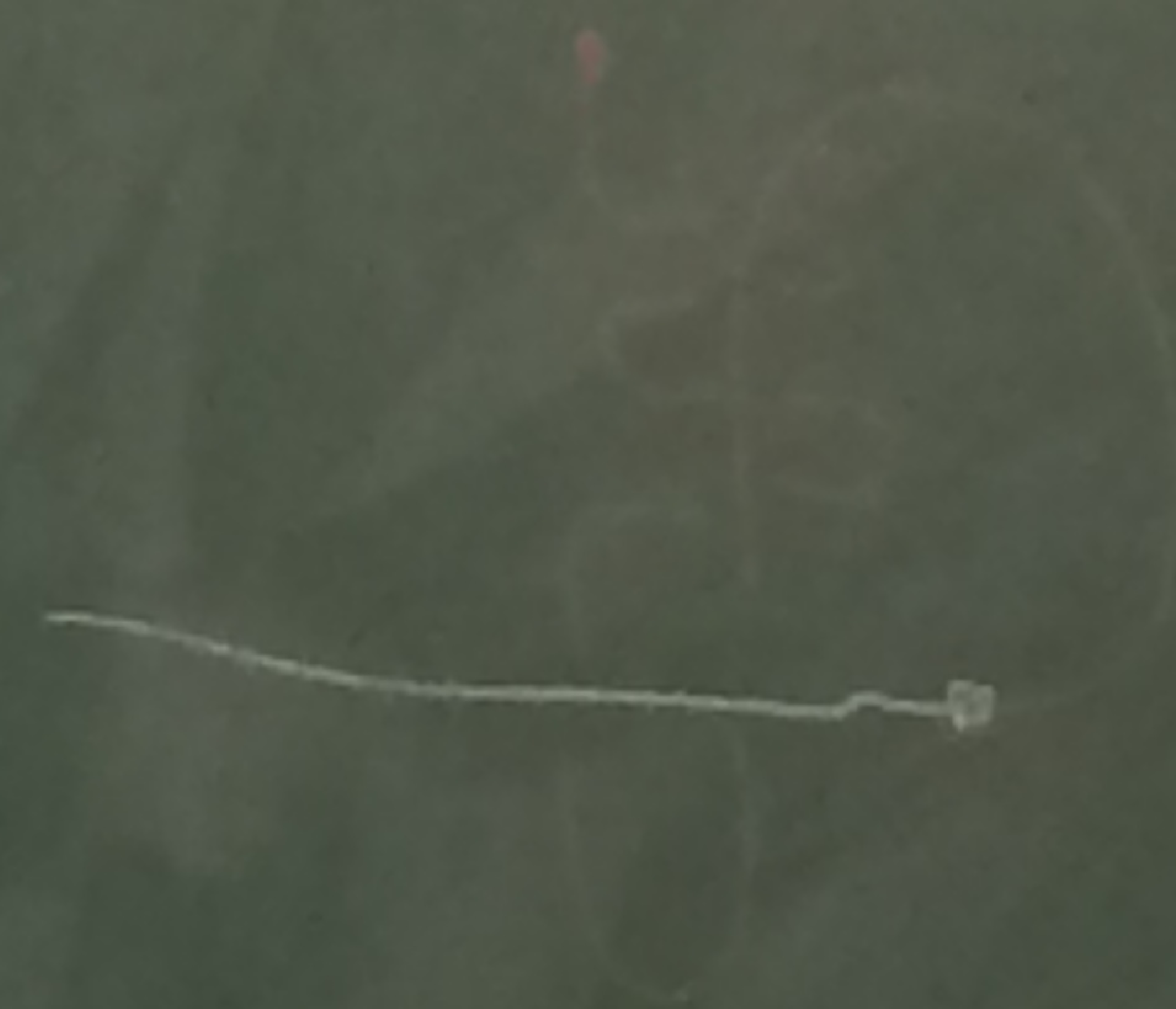
(i)



<u>together</u> λ^4

<u>apart</u>. λ^0

(ii)



<u>tog</u>. λ^2

<u>apart</u>. λ^4

$$W_{\lambda,\Omega_1}(\eta^{(1)}\circ\eta^{(2)}) \not\lessgtr W_{\lambda}(\Omega_2)(\eta^{(1)})\, W_{\lambda,\partial}(\eta^{(2)})$$

<u>Rmk.</u> AFAIK, only prev. use of lace ERW w/o repulsion was by Vellteh.

$$\prod_{0\le s<t\le n}\left(1-U(\omega_s-\omega_t)\right), \quad \bar{\omega} \qquad U(x)=\begin{cases}0 & |x|\geq 1\\ -k & |x|=1\\ 1 & |x|=0\end{cases}$$

<u>Sol</u>^{n}

$$\sum_{\substack{\omega: 0\to x\\ LE(\omega)=\eta}} w_{\lambda,z_1}(\omega) = z_1^{|\eta|}\exp\left(\sum_{y\in\mathbb{Z}^d}\sum_{\substack{\omega: y\to y\\ |\omega|\geq 1}} \frac{w_{\lambda,z_1}(\omega)}{|\omega|}\mathbb{1}_{\{\omega\cap\eta\neq\emptyset\}}\right)$$

<u>Rmk</u>. $\lambda=1$ is known (Lawler–Limic)

$$\prod_{0 \le s < t \le n} \left(1 - U(\omega_s - \omega_t)\right) \quad , \quad \bar{\omega} \qquad U(x) = \begin{cases} 0 & |x| \ge 1 \\ -k & |x| = 1 \\ 1 & |x| = 0 \end{cases}$$

Sol^{n}

$$\overbrace{\sum_{\substack{\omega : 0 \to x \\ LE(\omega) = \eta}} W_{\lambda,\kappa}(\omega)}^{\bar{W}_{\lambda,\kappa}(\eta)} = z_1^{|\eta|} \exp\left(\sum_{y \in \mathbb{Z}^d} \sum_{\substack{\omega : y \to y \\ |\omega| \ge 1}} \frac{W_{\lambda,\kappa}(\omega)}{|\omega|} \mathbb{1}_{\{\omega \cap \eta \neq \emptyset\}} \right)$$

Rmk. $\lambda = 1$ is known (Lawler-Limic)

If $\lambda \neq 1$ cannot reroot walks, so different proof

The formula $\Rightarrow$ $\bar{W}_{\lambda,\kappa}(\eta^{(1)} \circ \eta^{(2)}) \le$

Rmk. $\lambda = 1$ is known

If $\lambda \neq 1$ cannot reroot walks, so different proof

The formula $\Rightarrow$ $\bar{w}_{\lambda,R}(\eta^{(1)} \circ \eta^{(2)}) \leq \bar{w}_{\lambda,R}(\eta^{(1)})\, \bar{w}_{\lambda,R}(\eta^{(2)})$.

and this is enough for a lace exp analysis

Proof of the identity. Key tool is Viennot's theory of heaps of pieces

We formalize a heap of things.

Take (B, R), B is a set of piece types. aRb means that a and b overlap.

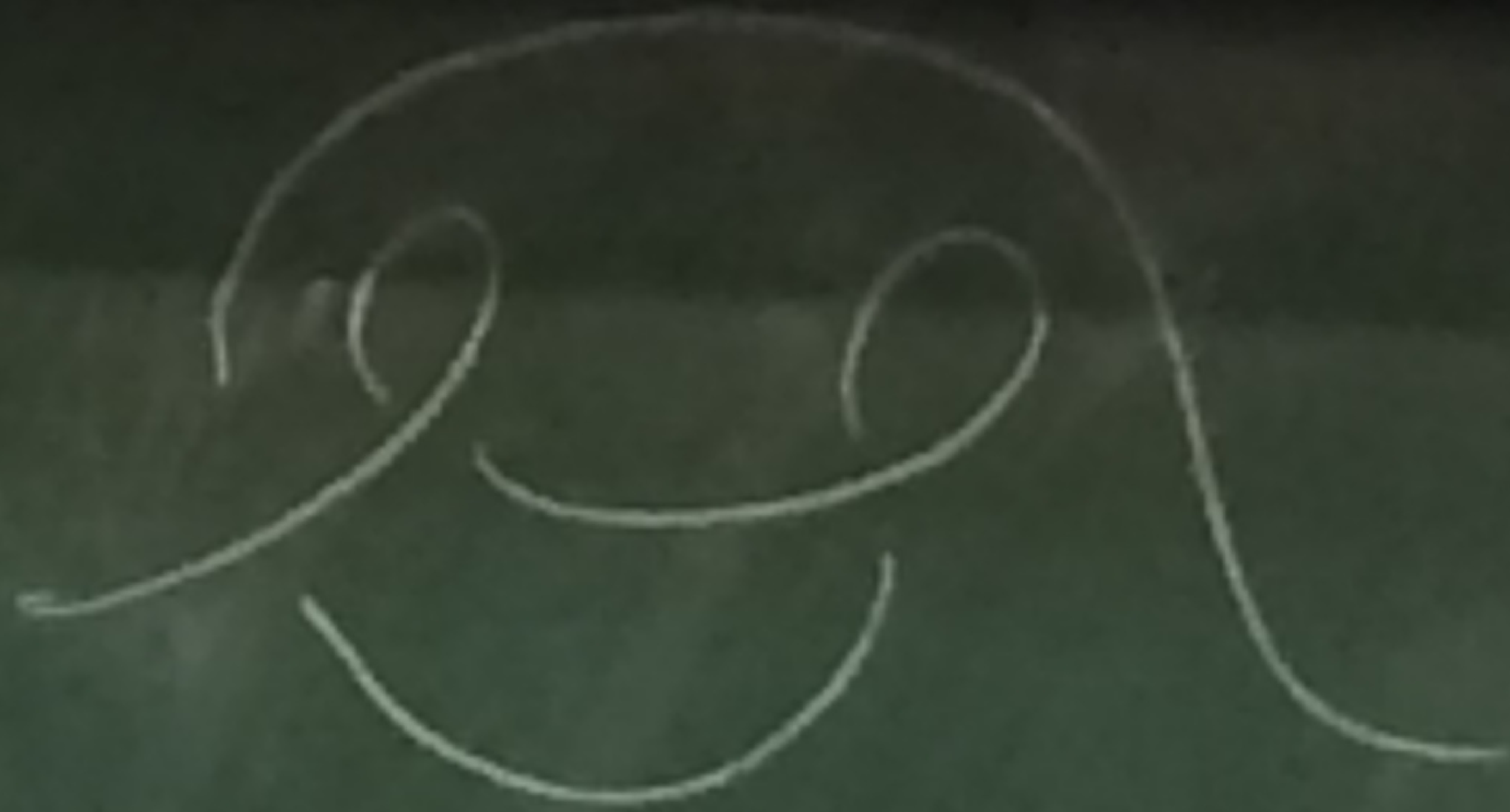
e.g. $B = \{$ pairs of adj. squares of a checkerboard $\}$

aRb if $a \cap b \neq \emptyset$.

A heap $(H, l, \leq)$ $l : H \to B$, $\leq$ a partial order, A heap $(H, \ell, \leq)$ $\quad \ell : H \to B$, $\leq$ a partial order,

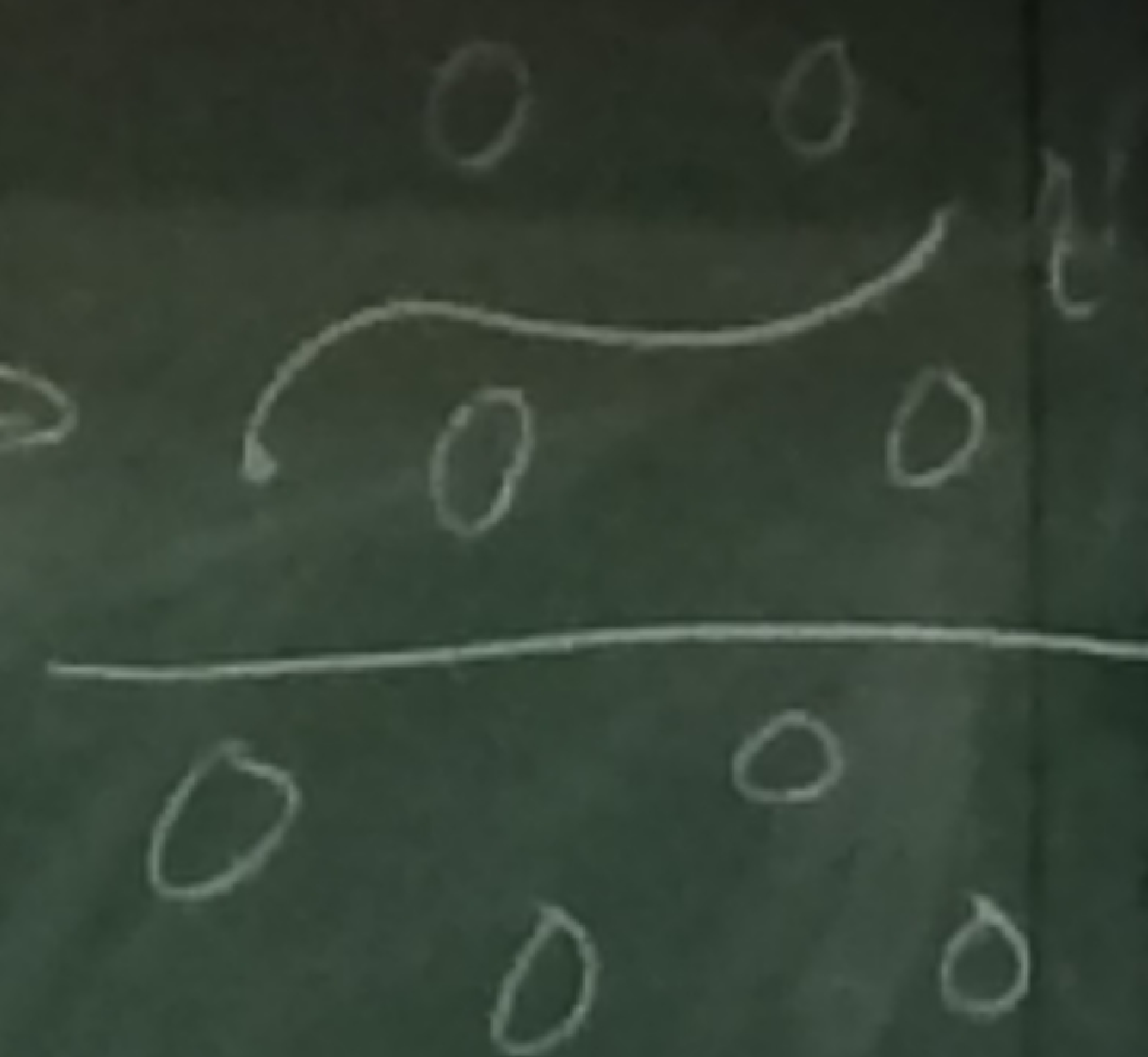
if $\ell(x) \, R \, \ell(y)$ either $x \leq y$ $\quad y \leq x$.

e.g. H a bag of dominoes, $\leq$ rules to build a structure



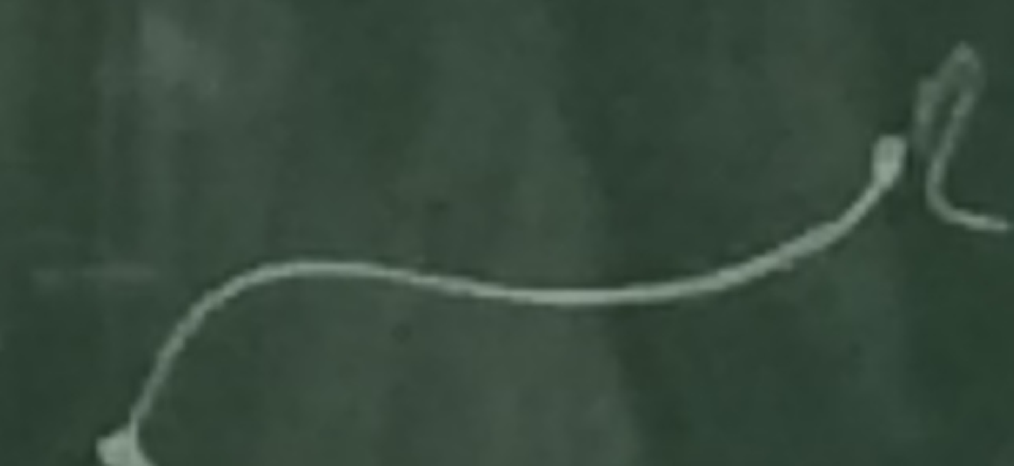
λ-LWW

(A)



SAW in a gas of loops

(B)



w/ self interaction

$\exp(\text{—} \cdot \text{—})$

(A) Take ω, loop erase it. At each erasure, let the loop fall to the floor $(= \mathbb{Z}^d)$

this takes $\omega \longmapsto (\eta, H_\eta^\omega)$

where $\eta = LE(\omega)$

H_η^ω is a heap of simple loops, loops on top of H_η^ω share a vertex $\overline{\omega}$ η.

$\prod \left(1 - U(\omega_s - \omega_t)\right)$ $\quad \overline{\omega}$ $\quad U(x) = \begin{cases} 0 & |x| \geq 1 \\ -k & |x| = 1 \end{cases}$

Thm (Viennot) $\{w : LE(w) = \eta\} \cong \{H_\eta^w\}$

Heap thm

$$\sum_{H_\eta^w} w_{\Lambda,R}(H_\eta^w) = \frac{\sum_{T \in \mathcal{T}_\eta} (-1)^{|T|} w_{\Lambda,R}(T)}{\sum_{T \in \mathcal{T}_\emptyset} (-1)^{|T|} w_{\Lambda,R}(T)}$$

$\mathcal{T}_\eta = \{$collections of disj. oriented cycles disj from $\eta\}$

Rmk


This is Loop O(-2λ)
(B) exp ∫ d/dz log Σ w(T)
on.