# Complex Gaussian multiplicative chaos

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# Outline

# Gaussian multiplicative Chaos

We want to give a precise meaning to distributions $M^{\gamma,\beta}$ defined formally by:

$$M^{\gamma,\beta}(A) = \int_A e^{\gamma X(x) + i\beta Y(x)} dx, \quad A \subset D$$

where:

- $\gamma, \beta \geq 0$
- $D \subset \mathbb{C}$ a bounded domain
- $X, Y$ two centered **independent** GFF with covariance given by:

$$E[X(x)X(y)] = G_D(x,y) \underset{|x-y|\to 0}{\sim} \ln \frac{1}{|x-y|}$$

  where $G_D$ is the Green kernel:

$$-\Delta_y G_D(x,y) = 2\pi\delta_x$$

# Framework

We consider a family of centered Gaussian processes $(X_\varepsilon(x))_{x\in D}$ $(\varepsilon \leq 1)$:

- Covariance: $E[X_\varepsilon(x)X_\varepsilon(y)] \sim \ln \frac{1}{|x-y|+\varepsilon} \underset{\varepsilon\to 0}{\to} G_D(x,y)$
- Variance: $E[X_\varepsilon(x)^2] = \ln \frac{1}{\varepsilon} + \ln C(x,D) + o(1)$ where $C(x,D)$ conformal radius.
- $\varepsilon \mapsto X_\varepsilon$ independent increments

Same for $(Y_\varepsilon(x))_{x\in D}$ $(\varepsilon \leq 1)$ independent from $X$.

# Gaussian multiplicative Chaos: notations

We define:

$$M_\varepsilon^{\gamma,\beta}(A) = \int_A e^{\gamma X_\varepsilon(x) + i\beta Y_\varepsilon(x)} dx, \quad A \subset D$$

Observe that:

$$M_\varepsilon^{\gamma,0}(A) = \int_A e^{\gamma X_\varepsilon(x)} dx, \quad A \subset D$$

# Other Frameworks

One can also work with other "smooth" cut-offs

- 1985: Kahane, $H^1$-basis decomposition
- 2006-2008: Robert, V., general convolutions
- 2008: Duplantier, Sheffield , circle averages

In fact, one can work with any log-correlated field in any dimension (Kahane, 1985, Robert, V., 2006, 2008): see our review with Rhodes.

# Gaussian multiplicative chaos: $\beta = 0$

**Theorem (Kahane, 1985)**

*There exists a random measure $M^{\gamma,0}$ such that following limit exists almost surely in the space of Radon measures:*

$$\varepsilon^{\frac{\gamma^2}{2}} M_\varepsilon^{\gamma,0}(dx) \underset{\varepsilon \to 0}{\to} M^{\gamma,0}(dx).$$

*$M^{\gamma,0}$ is called Gaussian multiplicative chaos associated to the Green kernel.*

**Remark**

*When J.P. Kahane meets Paul Levy...*

# Gaussian multiplicative chaos: $\beta = 0$

**Theorem (Kahane, 1985)**

*The measure $M^{\gamma,0}$ is different from 0 if and only if $\gamma < 2$.*

**Theorem (Kahane, 1985)**

*For $\gamma < 2$, the measure $M^{\gamma,0}$ "lives" almost surely on a set of Hausdorff dimension $2 - \frac{\gamma^2}{2}$ (the set of $\gamma$-thick points).*

# Density of Gaussian multiplicative chaos with respect to $\gamma$

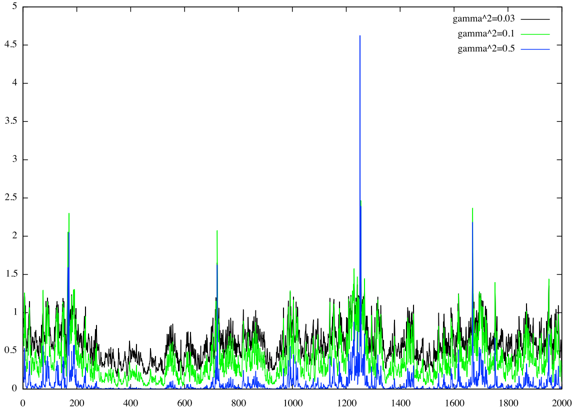
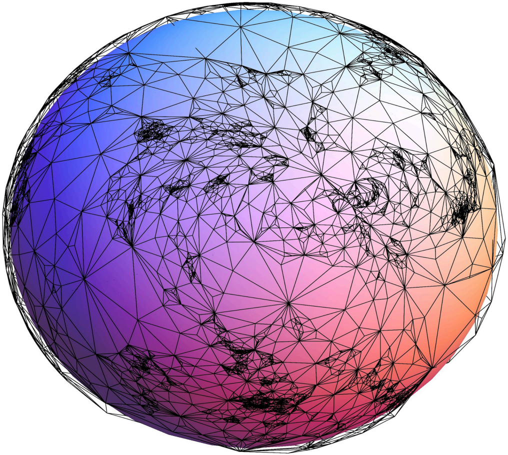



Figure: Density of Gaussian multiplicative chaos

# Uniformisation of a uniform triangulation: courtesy of N. Curien



# Liouville quantum gravity

It is conjectured to be the limit of random planar maps weighted by a critical statistical physics system (CFT with central charge $c \leq 1$) and conformally mapped to a domain $D$:

- Ambjorn-Durhuus-Jonsson (2005): Quantum geometry: A Statistical Field Theory Approach.
- Sheffield (2010): Conformal weldings of random surfaces: SLE and the quantum gravity zipper. Precise math conjectures.
- Curien (2013): A glimpse of the conformal structure of random planar maps ($c = 0$). First step in a mathematical proof.
- Miller, Sheffield (2014): Quantum Loewner evolution.

# Critical Gaussian multiplicative chaos: $\gamma = 2, \beta = 0$

**Theorem** (Duplantier, Rhodes, Sheffield, V., 2012)

*There exists a random measure $M$ such that following limit exists almost surely in the space of Radon measures:*

$$\varepsilon^2 (2 \ln \frac{1}{\varepsilon} - X_\varepsilon(x)) M_\varepsilon^{2,0}(dx) \underset{\varepsilon \to 0}{\to} M^{'}(dx).$$

*$M^{'}$ is called critical Gaussian multiplicative chaos associated to the Green kernel.*

# Critical Gaussian multiplicative chaos: $\gamma = 2, \beta = 0$

**Theorem** (Duplantier, Rhodes, Sheffield, V., 2012)

*The following limit exists almost surely (along suitable subsequences) in the space of Radon measures:*

$$\sqrt{\ln \frac{1}{\varepsilon}}\varepsilon^2 M_\varepsilon^{2,0}(dx) \underset{\varepsilon \to 0}{\to} \sqrt{\frac{2}{\pi}} M'(dx).$$

**Theorem** (Barral, Kupiainen, Nikula, Saksman, Webb, 2013)

*The measure $M'$ lives on a set of Hausdorff dimension* $0$.

# Complex Gaussian multiplicative chaos: Phase diagram

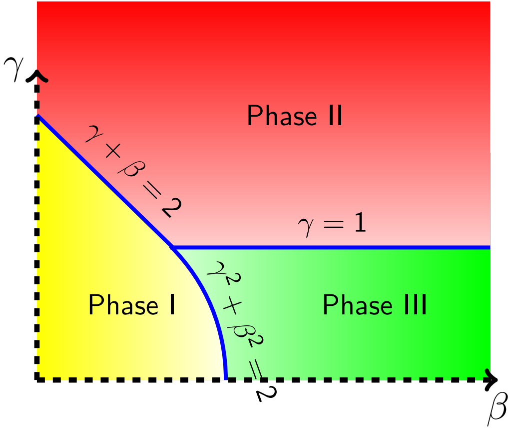



Figure: Phase diagram

# Previous works on the topic

Previous work on the complex case:

- Computation of the Free Energy of complex multiplicative cascades: Derrida, Evans, Speer, 1993. In our context:

$$\lim_{\varepsilon \to 0} \frac{\ln |M_{\varepsilon}^{\gamma,\beta}([0,1]^2)|}{\ln \frac{1}{\varepsilon}}$$

- Complex multiplicative cascades: series of works in dimension 1 by Barral, Jin, Mandelbrot, 2010. Essentially investigated phase I. Partial results in phase III.

# Convergence in phase I and it's frontier I/II (excluding extremal points)

**Theorem** (Lacoin, Rhodes, V., 2013)

*On phase I and it's frontier I/II (excluding the extremal points), the $\mathcal{D}'(D)$-valued distribution:*

$$M_\varepsilon^{\gamma,\beta} : \varphi \to \varepsilon^{\frac{\gamma^2}{2}-\frac{\beta^2}{2}} \int_D \varphi(x) M_\varepsilon^{\gamma,\beta}(dx)$$

*converges almost surely in the space $\mathcal{D}'_2(D)$ of distributions of order 2 towards a non trivial limit $M^{\gamma,\beta}$.*

# The Sine Gordon model

The probability measure is $e^{-S(Y)}dY$ where $S(Y)$ is the action:

$$S(Y) = \frac{1}{4\pi}\int_D |\nabla Y(x)|^2 dx + \mu \int_D \cos(\beta Y(x))dx$$

Different regimes ($\varepsilon \to 0$):

- $\beta^2 < 2$: non trivial convergence of $\mathbb{E}[e^{-\mu\varepsilon^{-\beta^2/2}\int_D \cos(\beta Y_\varepsilon(x))dx}]$
- $2 \leq \beta^2 < \beta_c^2$: $\varepsilon^{-\beta^2/2}\int_D \cos(\beta Y_\varepsilon(x))dx \approx \sigma_\varepsilon N + O(1)$ where $\sigma_\varepsilon \to \infty$ and $N$ Gaussian variable.
- $\beta^2 > \beta_c^2$: more and more cumulants to take out...

# Convergence in the inner phase III and it's frontier I/III

**Theorem** (Lacoin, Rhodes, V., 2013)

- *When* $\gamma \in [0,1[$ *and* $\beta^2+\gamma^2>2$, *we have*

$$\left(\varepsilon^{\gamma^2-1} M_\varepsilon^{\gamma,\beta}(A)\right)_{A\subset D} \Rightarrow \left(W_{\sigma^2 M^{2\gamma,0}}(A)\right)_{A\subset D}. \qquad (1)$$

*where* $\sigma^2 := \sigma^2(\beta^2+\gamma^2)>0$ *and* $W$ *is a complex Gaussian measure on* $D$ *with intensity* $\sigma^2 M^{2\gamma,0}$.

- *When* $\gamma \in [0,1[$ *and* $\beta^2+\gamma^2=2$, *we have*

$$\left(\varepsilon^{\gamma^2-1} |\log\varepsilon|^{-1/2} M_\varepsilon^{\gamma,\beta}(A)\right)_{A\subset D} \Rightarrow \left(W_{\sigma^2 M^{2\gamma,0}}(A)\right)_{A\subset D}. \qquad (2)$$

*where* $\sigma^2>0$ *and* $W$ *is a complex Gaussian measure on* $D$ *with intensity* $\sigma^2 M^{2\gamma,0}$.

# Convergence in the frontier phase II/III

**Theorem (Lacoin, Rhodes, V., 2013)**

*When $\gamma = 1$ and $\beta^2 + \gamma^2 > 2$, we have*

$$\Big( |\ln \varepsilon|^{1/4} M_\varepsilon^{\gamma,\beta}(A) \Big)_{A \subset D} \Rightarrow \big( W_{\sigma^2 M'}(A) \big)_{A \subset D} \,.$$

*with $\sigma^2 := \sigma^2(\beta) > 0$ and $W_{\sigma^2 M'}(\cdot)$ is a complex Gaussian random measure with intensity $\sigma^2 M'$.*

# Conformal Field theory $c = 1$ coupled to Liouville Quantum Gravity

Recall that, on phase I and it's frontier I/II (excluding the extremal points), the $\mathcal{D}'(D)$-valued distribution:

$$M_\varepsilon^{\gamma,\beta} : \varphi \to \varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} \int_D \varphi(x) M_\varepsilon^{\gamma,\beta}(dx)$$

converges almost surely in the space $\mathcal{D}'_2(D)$ of distributions of order 2 towards a non trivial limit $M_{X,Y}^{\gamma,\beta}$.

# Setup

In fact, we must denote

$$M_{X,Y}^{\gamma,\beta}(dx) = e^{\gamma X(x) + i\beta Y(x) - \frac{\gamma^2}{2}\mathbb{E}[X(x)^2] + \frac{\beta^2}{2}\mathbb{E}[Y(x)^2]} C(x,D)^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} dx,$$

where $C(x,D)$ is the conformal radius. This is because we do not renormalize by the mean!

# CFT with central charge $c = 1$ coupled to Gravity

- Polyakov action on a domain $D$

$$S(X, Y) = \frac{1}{4\pi} \int_D |\nabla Y(x)|^2 dx + \frac{1}{4\pi} \int_D |\nabla X(x)|^2 + QR(x)X(x) dx,$$

  $R$ is the curvature and $Q = 2$

- Equivalence class of random surfaces:

$$(X, Y) \to (X \circ \psi + 2 \ln |\psi'|, Y \circ \psi),$$

  where $\psi : \tilde{D} \to D$ is a conformal map. See Ginsparg, Moore (1993), Lectures on $2D$ gravity and $2D$ string theory.

# The Tachyon fields

Under the above equivalence class ($\psi : \tilde{D} \to D$)

$$M^{\gamma,\beta}_{X\circ\psi+2\ln|\psi'|,Y\circ\psi}(\varphi) = |\psi' \circ \psi^{-1}|^{2\gamma-\frac{\gamma^2}{2}+\frac{\beta^2}{2}-2} M^{\gamma,\beta}_{X,Y}(\varphi \circ \psi^{-1}),$$

for every function $\varphi \in C^2_c(\tilde{D})$

Tachyon Fields are conformally invariant. One must solve

$$2\gamma - \frac{\gamma^2}{2} + \frac{\beta^2}{2} - 2 = 0 \leftrightarrow \gamma \pm \beta = 2, \;\; \gamma \in ]1,2[.$$

# The Tachyon field for $(\gamma = 2, \beta = 0)$

At the special point $(\gamma = 2, \beta = 0)$, we recover the special tachyon field, i.e. the background measure

$$M_{X,Y}^{\gamma,\beta}(A) = M'(A)$$

where $M'$ is critical Gaussian multiplicative chaos.

# Sine-Gordon model

The probability measure is $e^{-S(Y)}dY$ where $S(Y)$ is the action:

$$S(Y) = \frac{1}{4\pi}\int_D |\nabla Y(x)|^2 dx + \mu \int_D \cos(\beta Y(x))dx$$

Representation of the density of charge $\rho$ of the Coulomb gas:

$$< \rho(x)\rho(y) >= \mathbb{E}[\sin(\beta Y(x))\sin(\beta Y(y))e^{-\mu\int_D \cos(\beta Y(z))dz}]$$

# Sine-Gordon model coupled to gravity?

The probability measure is $e^{-S(X,Y)}dXdY$ where $S(X,Y)$ is the action:

$$S(X,Y) = \frac{1}{4\pi}\int_D |\nabla Y(x)|^2 dx + \frac{1}{4\pi}\int_D |\nabla X(x)|^2 dx + \mu_1 \int_D \cos(\beta Y(x)) e^{\gamma X(x)} dx + \mu_2 \int_D e^{2X(x)} dx$$

where $\gamma + \beta = 2$ (see G. Moore, Gravitational Phase transitions and the Sine-Gordon model).

The problem is linked to defining the Coulomb gas on a random lattice.