MCMC sampling colourings and independent sets of $G(n, d/n)$ near the uniqueness threshold.

Charis Efthymiou
efthymiou@math.uni-frankfurt.de

Goethe University, Frankfurt

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Phase Transitions in Discrete Structures and Computational Problems
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Gibbs Distributions and the Sampling Problem

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Gibbs Distribution

Given a graph $G = (V, E)$ and some integer $k > 0$ and $\lambda > 0$ we let

Colouring Model: For each proper k-colouring σ we have

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Hard-Core Model: For each independent set σ

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Sampling Problem

Input: A graph $G = (V, E)$ and a target distribution $\mu(\cdot)$, e.g. Colouring or Hard-Core Model.

Output: A configuration distributed as in $\mu(\cdot)$.

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Remark

... for "*typical instances*" of $G(n, d/n)$ we do not expect to have exact algorithms, too.

Markov Chain Monte Carlo Sampling

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  - It is ergodic, i.e. it converges to a unique stationary distribution
  - The stationary distribution should be the Gibbs distribution, $\mu(\cdot)$
- The algorithm simulates the chain and outputs X_T, for sufficiently large T.

The Markov Chain

The Markov Chain

"Glauber Block Dynamics"

- We are given a partition of the vertex set $\mathcal{B} = \{B_1, \ldots, B_N\}$.
- $X_0 = \sigma$ for arbitrary σ.
- Given X_t, we get X_{t+1} as follows:
  - Choose block B *uniformly at random* among all the blocks in $\mathcal{B}$
  - Set $X_{t+1}(u) = X_t(u)$, for every vertex $u \notin B$
  - Set $X_{t+1}(B)$ according to distribution μ conditional on $X_{t+1}(V \backslash B)$.

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Remark

For the chains we consider here ergodicity is well known to hold [DFFV'05].

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Total Variation Distance

For two distributions ν, μ over Ω, we define their total variation distance as follows:

$$||\nu - \mu||_{TV} = \max_{A \subseteq \Omega} |\nu(A) - \mu(A)|.$$

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Rapid Mixing

The mixing time τ_{mix} is polynomial in n, the number of the vertices of G.

- If $T(err)$ is the minimum number of transitions to get within error err from μ, then

$$T(err) \leq \ln\left(\frac{1}{err}\right)\tau_{mix}.$$

Rapid Mixing and Maximum Degree Δ

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Maximum Degree Bounds for colourings

Vigoda (1999) $k > \frac{11}{6}\Delta$ for general G

Hayes,Vera,Vigoda (2007) $k = \Omega(\Delta/\log\Delta)$ for planar G

Goldberg, Martin, Paterson (2004) $k \geq (1.763 + \epsilon)\Delta$ for G triangle free and amenable

Dyer, Frieze, Hayes, Vigoda (2004) $k \geq (1.48 + \epsilon)\Delta$ for G of girth $g \geq 7$

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Hard-Core

The situation is very similar for the parameter λ in the Hard-Core Model .

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Remark

It seems "natural" to have the bounds on k, λ for rapid mixing depending on the *expected degree d* rather than maximum degree Δ.

Statistical Physics Perspective

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- For *colouring* we need $k > d$
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Otherwise

... there are exceptional initial states, from which the mixing is slow or there is no mixing at all

Past Work

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- Mossel, Sly (2008): $k \geq f(d)$ and $\lambda \leq h(d)$.
  - ... $f(d) = d^c$ and $h(d) = d^{-c'}$, for some $c, c' > 4$.

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Result for Rapid Mixing

W.h.p. over the instances of $G(n, d/n)$ the graph admits a partition of the vertex set into a set of "simple structured" blocks $\mathcal{B}$ s.t. the following holds: Let $\mathcal{M}_c$ and $\mathcal{M}_{hc}$ denote the Glauber block dynamics for the *colouring model* and the *hard core model*, respectively, with set of blocks $\mathcal{B}$.

- For $k \geq \frac{11}{2}d$ the mixing time of $\mathcal{M}_c$ is $O(n \ln n)$
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For efficient sampling we need to have efficient...

- construction of $\mathcal{B}$
- implementation of the updates
- algorithms that provide initial configurations for both chains.

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It is all about creating an appropriate set of blocks.

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It is all about creating an appropriate set of blocks.

- ... it is highly non-trivial!

Block Construction I

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Weights for Vertices and Paths

- We assign weight to each vertex u of degree deg_u as follows:

$$W(u) = \begin{cases} (1+\gamma)^{-1} & deg_u \leq (1+\epsilon)d \\ d^c \cdot deg_u & \text{otherwise} \end{cases}$$

- Every path L is assigned $\prod_{u \in L} W(u)$

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"Break Points"

Let $\mathbb{P}(v)$ denote the set of paths of length at most $\frac{\ln n}{d^{2/5}}$ that emanate from v. We call "break point" every vertex v s.t.

$$\max_{L \in \mathbb{P}(v)} \left\{ \prod_{u \in L} W(u) \right\} \leq 1.$$

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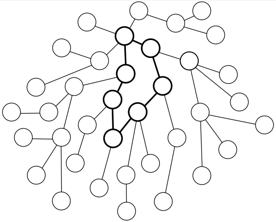

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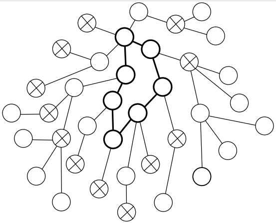

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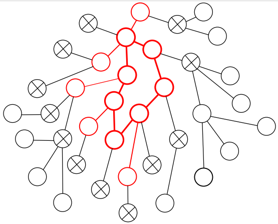

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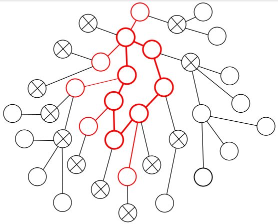

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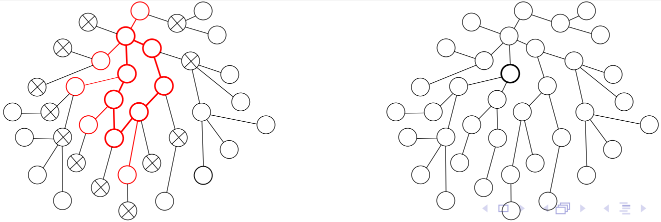

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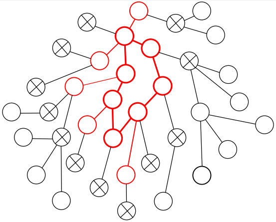

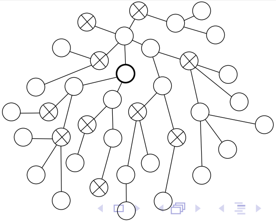

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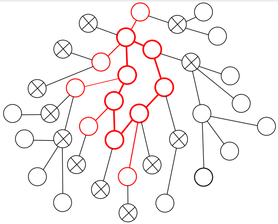

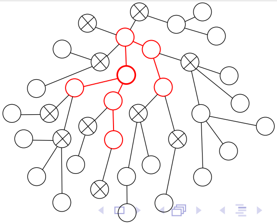

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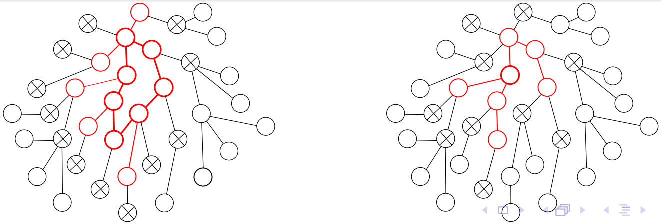

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W.h.p. over the graph instances $G(n, d/n)$ the following is true:

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  - No cycles in $\mathcal{C}$ end up in the same block
- The creation of $\mathcal{B}$ can be implemented in polynomial time
  - We can check in polynomial time whether some vertex is break-point.

Technique for Rapid Mixing

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Path Coupling, [Bubley, Dyer 1997]

- Consider two copies of the chain at configuration X_0 and Y_0 such that $H(X_0, Y_0) = 1$
- Couple the transitions of the two chains
- For rapid mixing it suffices to have that
$$E[H(X_1, Y_1)|X_0, Y_0] = 1 - \Theta(1/n).$$

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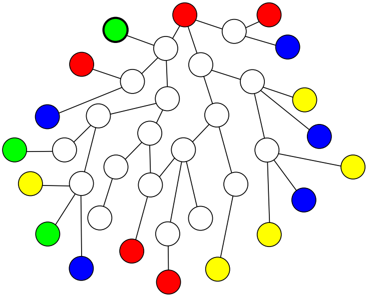

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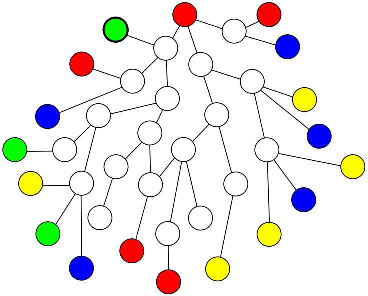
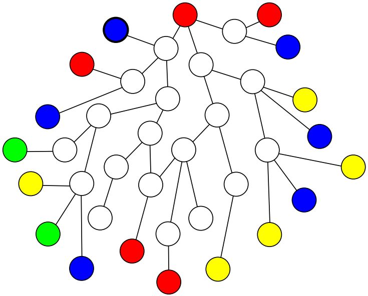
- $\forall B \in \mathcal{B}$ consider arbitrary $\sigma(\partial B)$ and $\tau(\partial B)$ s.t. $H(\sigma(\partial B), \tau(\partial B)) = 1$
- Take $X(B) \sim \mu(\cdot|\sigma(\partial B))$ and $Y(B) \sim \mu(\cdot|\tau(\partial B))$





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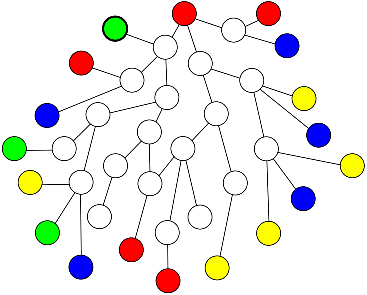

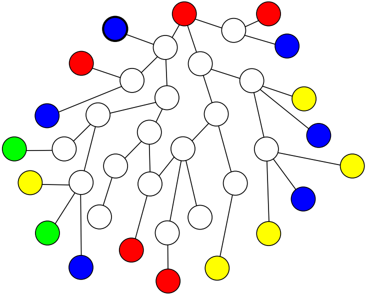

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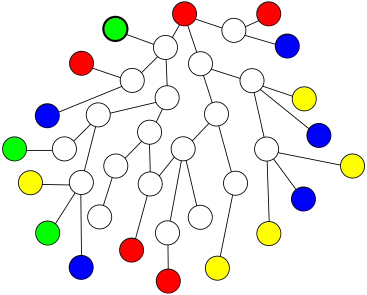

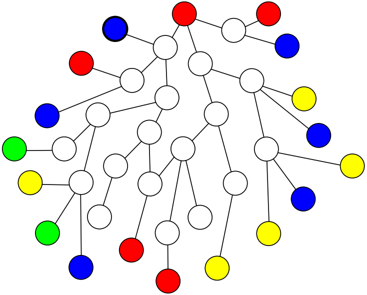

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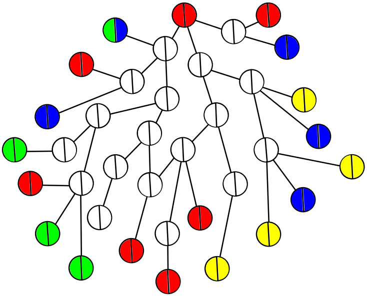

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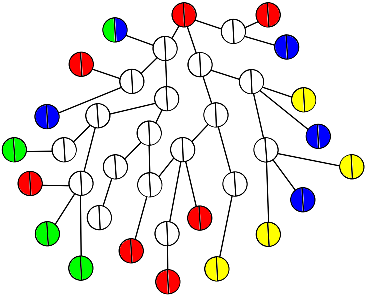

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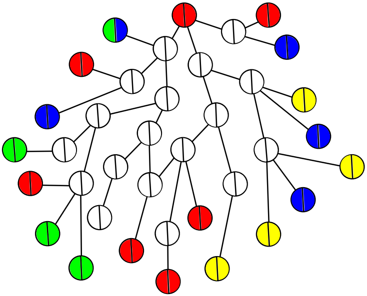

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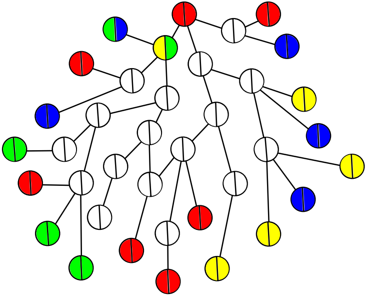

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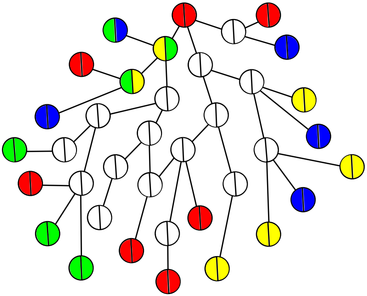

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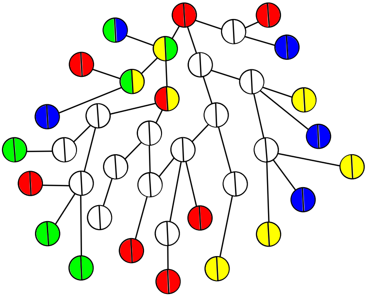

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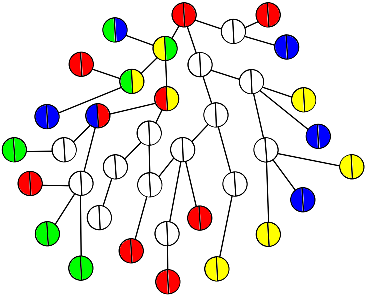

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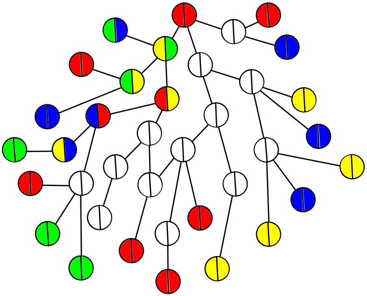

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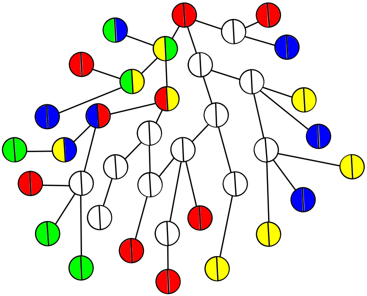

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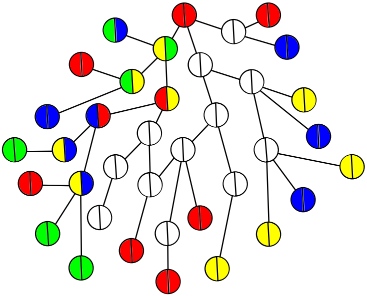

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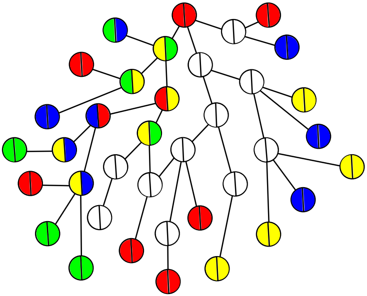

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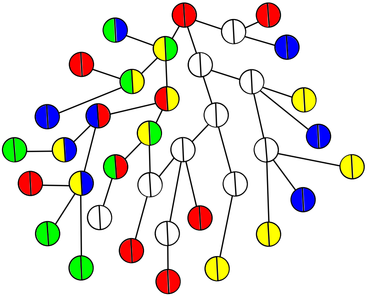

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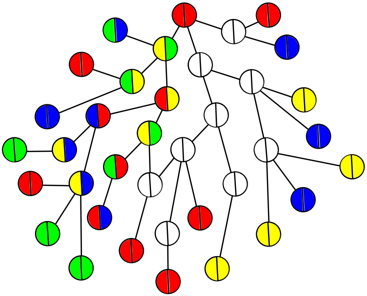

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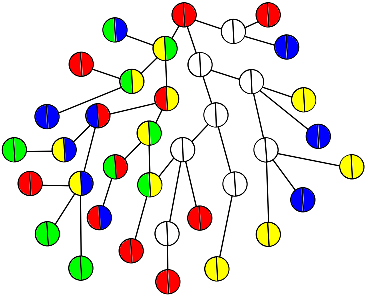

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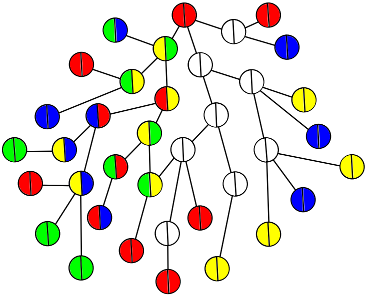

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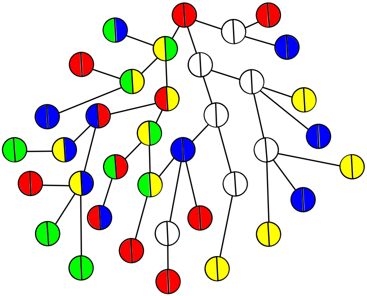

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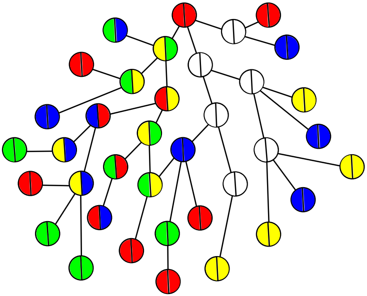

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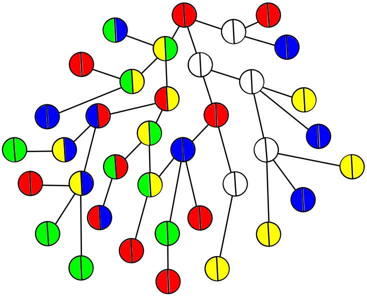

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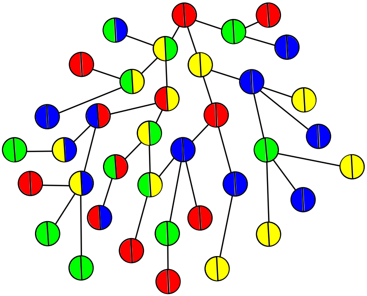

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- If the root is of degree s, the condition reduces to the subtrees of r

$$L_{i-1}^{T'} \leq c(1-\delta)^i/(s \cdot \varrho_{root})$$

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- Using appropriate parameters for the weighting schema as well as appropriate k (or λ) the above condition is satisfied.

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    - the lower bound on k is off by a factor $\frac{11}{2}$
  - For the "hard-cores" we need to have $\lambda \leq \frac{1-\epsilon}{2d}$
    - λ if off by a factor of $\frac{2}{e}$
- We introduced a new technique of creating blocks
  - Weighting Schema
- Is it possible to prove rapid mixing with site updates?
  - Comparison techniques
- How can we improve the bounds?

Concluding Remarks

- Considered Glauber Block Dynamics for sampling "k-colouring" and "hard-cores" with underlying graph $G(n, d/n)$
  - The structure of the blocks is simple
  - For the colourings we need to have $k \geq \frac{11}{2}d$ for rapid mixing
    - the lower bound on k is off by a factor $\frac{11}{2}$
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- We introduced a new technique of creating blocks
  - Weighting Schema
- Is it possible to prove rapid mixing with site updates?
  - Comparison techniques
- How can we improve the bounds?
  - We will need to speak about "spatial mixing"

The End

THANK YOU!