Phase transitions and large deviations in geophysical fluid dynamics

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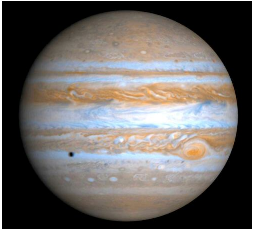
Collaborators

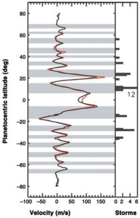
- Random changes of flow topology in the 2D Navier-Stokes equations: E. Simonnet (INLN-Nice) (ANR Statocean)
- Asymptotic stability and inviscid damping of the 2D-Euler equations: H. Morita (Tokyo university) (ANR Statflow)
- Instantons and large deviations for the 2D Navier-Stokes equations: J. Laurie (Post-doc ANR Statocean), O. Zaboronski (Warwick Univ.)
- Large deviations for systems with connected attractors: H. Touchette (Queen Mary Univ, London)
- Stochastic Averaging and Jet Formation in Geostrophic Turbulence: C. Nardini and T. Tangarife (ENS-Lyon)

Earth and Jupiter's Zonal Jets

We look for a theoretical description of zonal jets



Jupiter's atmosphere

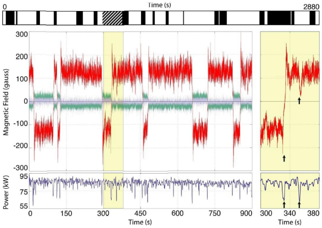


Jupiter's zonal winds (Voyager and Cassini, from Porco et al 2003)

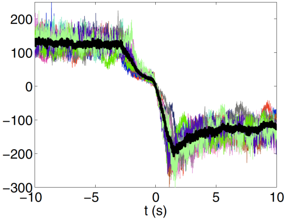
How to theoretically predict such a velocity profile?

Random Transitions in Turbulence Problems

Magnetic Field Reversal (Turbulent Dynamo, MHD Dynamics)



Magnetic field timeseries



Zoom on reversal paths

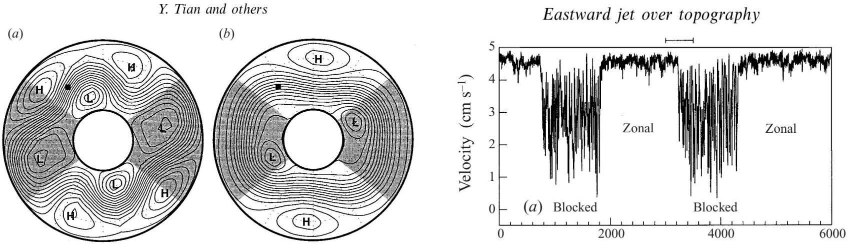
(VKS experiment)

In turbulent flows, transitions from one attractor to another often occur through a predictable path.

Phase Transitions in Rotating Tank Experiments

The rotation as an ordering field (Quasi Geostrophic dynamics)

Transitions between blocked and zonal states



Y. Tian and col, J. Fluid. Mech. (2001) (groups of H. Swinney and M. Ghil)

The Main Issues for Physicists

- Why do the large scales of geophysical flows self-organize?
- Can we predict the statistics of the large scales of geophysical flows?
- Can we predict phase transitions for geophysical turbulent flows and their statistics?

The Main Mathematical Questions

- How to characterize and predict attractors in turbulent geophysical flows?
- When is Freidlin–Wentzell theory relevant for turbulent flows?
- Large deviation results beyond Freidlin–Wentzell theory?

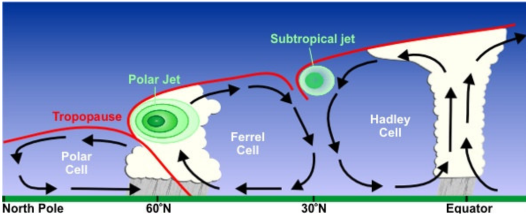
Non-Equilibrium Stat. Mech.

1. Stochastic averaging technics (kinetic theory in a stochastic framework).
2. Large deviation for transition probabilities of rare events (through path integrals, or the Freidlin–Wentzell theory).
3. Tools from field theory in statistical physics.

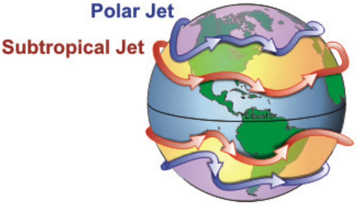
Outline

1. Basics of mid-latitude atmosphere dynamics.
2. 2D Euler and Quasi-Geostrophic Langevin dynamics: Large Deviations and Instantons.
    - Path integral representation of transition probabilities and time reversal symmetry.
    - Instantons for Langevin quasi-geostrophic dynamics (F.B., J. Laurie, and O. Zaboronski).
    - Non-Equilibrium Instantons for the 2D Navier–Stokes equations (F.B. and J. Laurie)
3. Stochastic averaging and jet formation in geostrophic turbulence.
    - The stochastic quasi-geostrophic equations.
    - Stochastic averaging (with C. Nardini and T. Tangarife).
    - Validity of this approach, and the main technical points.

Atmosphere Convection and Jet Streams

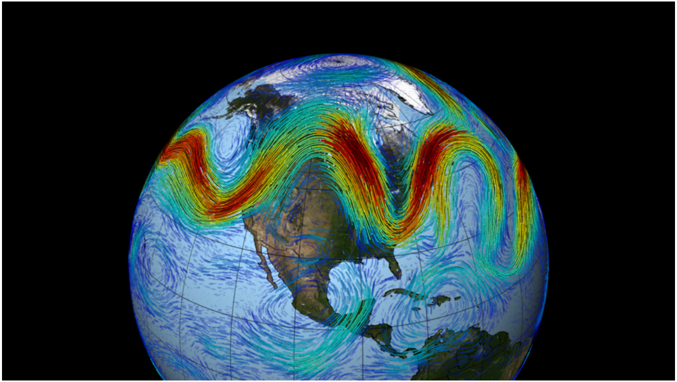


Atmosphere convection cells



Schematic jet stream configuration

Atmosphere Jet Dynamics



Upper atmosphere velocity

The Barotropic Quasi-Geostrophic Equations

- The simplest model for geostrophic turbulence.
- Geostrophic balance: $\mathbf{v} = -\nabla\psi \wedge \mathbf{e}_z$ with ψ proportional to the pressure. $\beta = 2\Omega\cos y$ is the Coriolis parameter.
- Quasi-Geostrophic equations with random forces

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \nu_d \Delta\omega - \lambda\omega + \sqrt{2\varepsilon} f_s,$$

where $\omega = (\nabla \wedge \mathbf{v}).\mathbf{e}_z$ is the vorticity, $q = \omega + \beta y$ is the Potential Vorticity (PV), f_s is a random Gaussian field with correlation $\langle f_S(\mathbf{x},t) f_S(\mathbf{x}',t')\rangle = C(\mathbf{x}-\mathbf{x}')\delta(t-t')$, ε is the average energy input rate, λ is the Rayleigh friction coefficient.

- Quasi-Geostrophic models: the basic models for midlatitude large scale dynamics.

The 2D Stochastic-Navier-Stokes (SNS) Equations

- The simplest model for two dimensional turbulence.
- Navier Stokes equations with random forces

$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla\omega = \nu\Delta\omega - \alpha\omega + \sqrt{\sigma} f_s,$$

  where $\omega = (\nabla \wedge \mathbf{v}).\mathbf{e}_z$ is the vorticity, f_s is a random force, α is the Rayleigh friction coefficient.
- An academic model with experimental realizations (Sommeria and Tabeling experiments, rotating tanks, magnetic flows, and so on). Analogies with geophysical flows (Quasi Geostrophic and Shallow Water layer models).

The 2D Stochastic Navier-Stokes Equations

$$\frac{\partial \omega}{\partial t} + \mathbf{u}.\nabla\omega = \nu\Delta\omega + \sqrt{\nu} f_s$$

- Some recent mathematical results: Kuksin, Shirikyan, Bricmont, Kupiainen, Hairer, Mattingly, Sinai, and so on;
  - Existence of a stationary measure μ_ν. Existence of $lim_{\nu\to 0}\ \mu_\nu$,
  - In this limit, almost all trajectories are solutions of the 2D Euler equations.

Kuksin, S. B., & Shirikyan, A. (2012). Mathematics of two-dimensional turbulence. Cambridge University Press.

- We would like to describe the invariant measure:
  - Is it concentrated close to steady solutions of the 2D Euler (quasi-geostrophic) equations?
  - Can we describe the dynamics among these states?

Large Deviations for the Stochastic Navier-Stokes Eq.

Mathematical results

- Mathematicians established large deviation principles, for instance for the 2D stochastic Navier-Stokes equations:
- M. Gourcy, A large deviation principle for 2D stochastic Navier-Stokes equation, Stochastic Process. Appl. 117 (2007), no. 7, 904-927.
- V. Jaksic, V. Nersesyan, C.-A. Pillet, A. Shirikyan, Large deviations from a stationary measure for a class of dissipative PDE's with random kicks, preprint.

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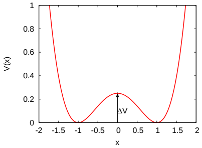
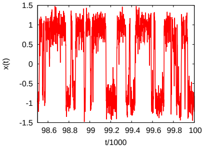
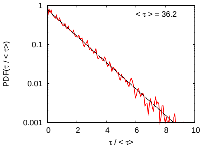
Kramers' Problem: a Pedagogical Example for Bistability

Historical example: Computation by Kramer of Arrhenius' law for a bistable mechanical system with stochastic noise

$$\frac{dx}{dt} = -\frac{dV}{dx}(x) + \sqrt{2k_B T}\eta(t) \text{ Rate : } \lambda = \frac{1}{\tau}\exp\left(-\frac{\Delta V}{k_B T}\right).$$







The problem was solved by Kramers (30'). Modern approach: path integral formulation (instanton theory, physicists) or large deviation theory (Freidlin-Wentzell, mathematicians).

Path Integrals for ODE – Onsager Machlup (50')

- Path integral representation of transition probabilities:

$$P(x_T, T; x_0, 0) = \int_{x(0)=x_0}^{x(T)=x_T} \mathrm{e}^{-\frac{\mathscr{A}_T[x]}{2k_B T}} \mathscr{D}[x]$$

$$\text{with } \mathscr{A}_T[x] = \int_0^T \mathscr{L}[x,\dot{x}]\ \mathrm{d}t \text{ and } \mathscr{L}[x,\dot{x}] = \frac{1}{2}\left[\dot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}(x)\right]^2.$$

- The most probable path from x_0 to x_T is the minimizer of

$$A_T(x_0, x_T) = \min_{\{x(t)\}} \left\{ \mathscr{A}_T[x] \,|\, x(0) = x_0 \text{ and } x(T) = x_T \right\}.$$

- We may consider the low temperature limit, using a saddle point approximation (WKB), Then we obtain the large deviation result

$$\log P(x_T, T; x_0, 0) \underset{\frac{k_B T}{\Delta V} \to 0}{\sim} -\frac{A_T(x_0, x_T)}{2k_B T}.$$

Relaxation Paths Minimize the Action

$$\mathscr{A}_T[x] = \int_0^T \mathscr{L}[x,\dot{x}]\ \mathrm{d}t \text{ and } \mathscr{L}[x,\dot{x}] = \frac{1}{2}\left[\dot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}(x)\right]^2.$$

- A relaxation path $\{x_r(t)\}_{0\leq t\leq T}$ is a solution of

$$\dot{x} = -\frac{\mathrm{d}V}{\mathrm{d}x}.$$

  Then we see that

$$\mathscr{A}_T[x_r] = 0.$$

- Interpretation: if one follows the deterministic dynamics, no noise is needed and the cost is zero.
- Because for any path $\mathscr{A}_T[x_r] \geq 0$, any relaxation path minimizes the action.

Fluctuation Paths and Instantons

- The most probable path from an attractor of the system x_0 to a state x is called a fluctuation path. It solves

$$A_\infty(x_0,x) = \min_{\{x(t)\}} \{\mathscr{A}_\infty[x]\,|x(-\infty) = x_0 \text{ and } x(0) = x\}.$$

- When the WKB limit is justified (low temperature), most of the paths leading to a rare fluctuation x concentrate close to the fluctuation path. The probability to observe x is

$$P(x) \sim Ce^{-\frac{A_\infty(x)}{2k_BT}}.$$

- In bistable systems (more than one attractor), fluctuation paths from one attractor x_1 to a saddle point x_s play an important role. They lead to a change of basin of attraction. They are called instantons. The transition rate is

$$P(x_{-1},T;x_1,0) \sim Ce^{-\frac{A(x_{-1},x_1)}{2k_BT}},$$

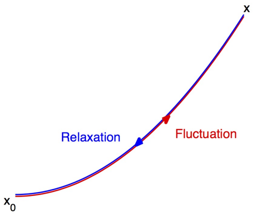
with $A(x_{-1},x_1) = \min_{\{x(t)\}} \{\mathscr{A}_\infty[x]\,|x(-\infty) = x_1 \text{ and } x(+\infty) = x_s\}$.

Time Reversed Relaxation Paths Minimize the Action

We have the symmetry relation

$$\mathscr{A}_T\left[R\left[x\right]\right]=\mathscr{A}_T\left[x\right]+2V(x(T))-2V(x(0)).$$

We conclude that the time reversed relaxation paths also minimizes the action.



- The minimizer of the action from an attractor of the system to any point of its basin of attraction is the reversed of the relaxation path.
- This is an extended Onsager-Machlup relation. For time reversible systems, the most probable way to get a fluctuation is through the reversal of the relaxation path from this fluctuation.

Instantons are Time Reversed Relaxation Paths

- We have the symmetry relation

$$\mathscr{A}_T[R[x]] = \mathscr{A}_T[x] + 2V(x(T)) - 2V(x(0)).$$

- Using this equation, we can conclude that instantons are time reversed relaxation paths from a saddle to an attractor. Then we obtain the large deviation result

$$\log P(x_1, T; x_{-1}, 0) \underset{\frac{k_B T}{\Delta V} \to 0}{\sim} -\frac{\Delta V}{k_B T}.$$

- The computation of the prefactor is more tricky

$$P(x_{-1}, T; x_1, 0) \underset{t \ll 1/\lambda}{\simeq} \frac{T}{\tau} \exp\left(-\frac{\Delta V}{k_B T}\right) \text{ with } \tau = 2\pi \left(\frac{d^2 V}{dx^2}(x_0) \frac{d^2 V}{dx^2}(x_{-1})\right)^{-1/2}.$$

This is the subject of Langer theory (70'), see also Caroli, Caroli, and Roulet, J. Stat. Phys., 1981, for a computation through path integrals.

Time Reversal and Action Symmetry: Conclusions

- We consider a path $x = \{x(t)\}_{0\leq t\leq T}$ and its reversed path $R[x] = \{x(T-t)\}_{0\leq t\leq T}$. We have

$$\mathscr{A}_T[R[x]] = \mathscr{A}_T[x] + 2V(x(T)) - 2V(x(0)).$$

- This implies detailed balance.
- This implies that the most probable path to reach a state x (a fluctuation) is the time reversal of a relaxation path starting from x (dissipation).
- This is a generalized Onsager-Machlup relation, that explains quite easily and naturally fluctuation-dissipation relations.
- For dynamics with time reversal symmetry, instantons are time reversed relaxation paths.

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Langevin Dynamics In a General Framework

$$\frac{\partial q}{\partial t} = \mathscr{F}[q](\mathbf{r}) - \alpha \int_{\mathscr{D}} C(\mathbf{r},\mathbf{r}') \frac{\delta \mathscr{G}}{\delta q(\mathbf{r}')}[q]\,\mathrm{d}\mathbf{r}' + \sqrt{2\alpha\gamma}\eta,$$

- Assumptions: i) $\mathscr{F}$ verifies a Liouville theorem

$$\nabla.\mathscr{F} \equiv \int_{\mathscr{D}} \frac{\delta \mathscr{F}}{\delta q(\mathbf{r})}\,\mathrm{d}\mathbf{r} = 0 \quad \left(\text{Generalization of } \nabla.\mathscr{F} \equiv \sum_{i=1}^{N} \frac{\partial \mathscr{F}}{\partial q_i} = 0\right),$$

- ii) The potential $\mathscr{G}$ is a conserved quantity of $\frac{\partial q}{\partial t} = \mathscr{F}[q](\mathbf{r})$:

$$\int_{\mathscr{D}} \mathscr{F}[q](\mathbf{r}) \frac{\delta \mathscr{G}}{\delta q(\mathbf{r})}[q]\,\mathrm{d}\mathbf{r} = 0.$$

- iii) η a Gaussian process, white in time, with covariance

$$\mathbb{E}\left[\eta(\mathbf{r},t)\eta(\mathbf{r}',t')\right] = C(\mathbf{r},\mathbf{r}')\delta(t-t').$$

- For most classical Langevin dynamics:

$$\mathscr{F}[q](\mathbf{r}) = \{q,\mathscr{H}\} \text{ and } \mathscr{G} = \mathscr{H}.$$

Time Reversal and Action Duality: Conclusions

- We consider a path $q=\{q(t)\}_{0\leq t\leq T}$ and its reversed path $q_r=\{I[q(T-t)]\}_{0\leq t\leq T}$. We have

$$\mathscr{A}_T[q_r]=\mathscr{A}_T[q]+2V(q(T))-2V(q(0)).$$

- Transition probabilities of the direct process are related to transition probabilities of the dual process (a generalization of detailed balance).
- This implies that the most probable path to reach a state x (a fluctuation) is the time reversal of a relaxation path starting from $I[x]$ for the dual process (dissipation).
- This is a generalized Onsager-Machlup relation, that justifies generalization of fluctuation-dissipation relations.
- Instantons are the time reversed relaxation paths of the dual process.

Langevin Dynamics for the Quasi-Geostrophic Eq.

$$\frac{\partial q}{\partial t} = \mathbf{v}[q-h].\nabla q - \alpha \int_{\mathscr{D}} C(\mathbf{r},\mathbf{r}') \frac{\delta \mathscr{G}}{\delta q(\mathbf{r}')}[q]\,\mathrm{d}\mathbf{r}' + \sqrt{2\alpha\gamma}\eta.$$

- Assumptions: i) $\mathscr{F} = -\mathbf{v}[q-h].\nabla q$ verifies a Liouville theorem.
- ii) The potential $\mathscr{G}$ is a conserved quantity of $\frac{\partial q}{\partial t} = \mathscr{F}[q](\mathbf{r})$ with

$$\mathscr{G} = \mathscr{C} + \beta \mathscr{E},$$

with a Casimir functionals

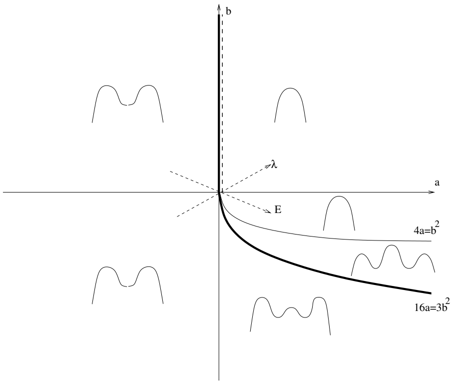
$$\mathscr{C}_c = \int_{\mathscr{D}} \mathrm{d}\mathbf{r}\, c(q),$$

and energy

$$\mathscr{E} = -\frac{1}{2}\int_{\mathscr{D}} \mathrm{d}\mathbf{r}\, [q - H\cos(2y)]\,\psi = \frac{1}{2}\int_{\mathscr{D}} \mathrm{d}\mathbf{r}\, \nabla\psi^2.$$

Tricritical Points

Bifurcation from a second order to a first order phase transition



Tricritical point corresponding to the normal form $s(m) = -m^6 \text{-} \frac{3b}{2} m^4 - 3am^2$.

A Quasi-Geostrophic Potential with A Tricritical Point

$$\mathscr{G} = (1-\varepsilon)\frac{1}{2}\int_{\mathscr{D}} \mathbf{dr}\,[q - H\cos(2y)]\,\psi + \int_{\mathscr{D}} \mathbf{dr}\left[\frac{q^2}{2} - a_4\frac{q^4}{4} + a_6\frac{q^6}{4}\right] \text{ with } h(y) = H\cos(2y).$$

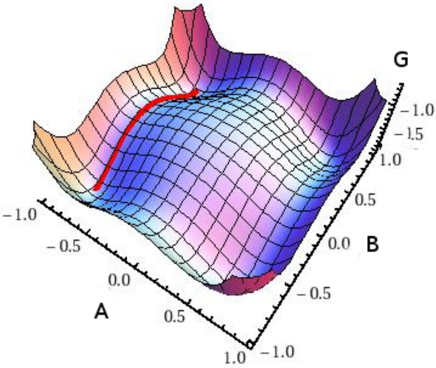
- There is a tricritical transition (transition from first order to second order) close to $\varepsilon = 0$ and $a_4 = 0$ for small H.
- Close to the transition the stochastic dynamics can be reduced to a two-degrees of freedom stochastic dynamics, which is a gradient dynamics with potential

$$G(A,B) = -\frac{H^2}{3} + \varepsilon\left[A^2+B^2\right] - \frac{3a_4}{2}\left[A^2+B^2\right]^2 + \frac{a_6}{6}\gamma\left[A^2+B^2\right]^3 + \frac{5\pi}{144}a_6 H^2\left(A^2-B^2\right)^2.$$

- And the potential vorticity field is

$$q(y) \simeq A\cos(y) + B\sin(y).$$

The Reduced Potential and the Instanton



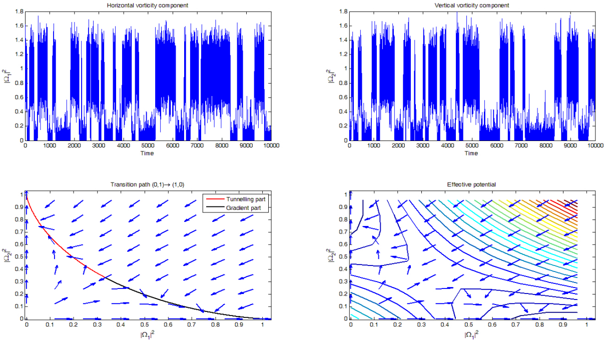
The reduced potential and one instanton/relaxation path.

Quasi-Geostrophic Tricritical Point

- For this turbulent dynamics, we can predict the phase diagram (a tricritical point). For a range of parameter, we have first order phase transitions.
- Using large deviations, we can compute transition probabilities.
- We can compute the transition rate between two attractors.
- Most transitions concentrate close to the optimal one, it is describe by an instanton that is easily computed.
- Sufficiently close to the tricritical point, the dynamics reduces to a two degrees of freedom stochastic dynamics.

Bistability Between Horizontal and Vertical Parallel Flows

A further example for the 2D Navier-Stokes equations



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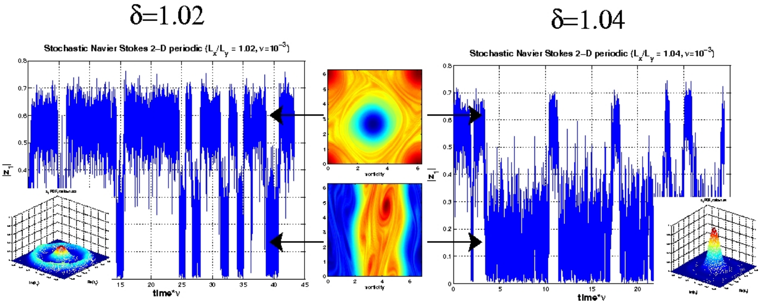
2D Stochastic Navier-Stokes Eq. and 2D Euler Steady States

$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla \omega = \nu \Delta \omega - \alpha \omega + \sqrt{2\alpha} f_s$$

- This is no more a Langevin dynamics.
- Time scale separation: magenta terms are small.

Non-Equilibrium Phase Transition (2D Navier–Stokes Eq.)

The time series and PDF of the Order Parameter

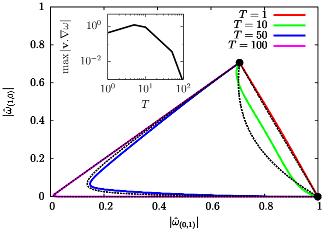


Order parameter : $z_1 = \int dxdy \exp(iy)\omega(x,y)$.

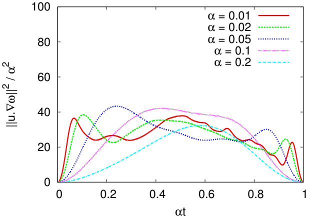
For unidirectional flows $|z_1| \simeq 0$, for dipoles $|z_1| \simeq 0.6 - 0.7$

F. Bouchet and E. Simonnet, PRL, 2009.

Instantons from Dipole to Parallel Flows



Comparison of numerical instantons with analytical ones



Instantons are close to the set of attractors

- In the limit of weak forces and dissipations, instantons follows the set of attractors of the 2D Euler equations (with J. Laurie).

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$$\frac{\partial q}{\partial t}+\mathbf{v}.\nabla q=\nu\Delta\omega-\alpha\omega+\sqrt{2\alpha}f_s,$$

with $q=\omega+\beta y$.

The 2D Stochastic Navier-Stokes Equations

$$\frac{\partial \omega}{\partial t} + \mathbf{u}.\nabla\omega = \nu\Delta\omega + \sqrt{\nu} f_s$$

- Some recent mathematical results: Kuksin, Shirikyan, Bricmont, Kupiainen, Hairer, Mattingly, Sinai, and so on;
  - Existence of a stationary measure μ_ν. Existence of $lim_{\nu\to 0}\ \mu_\nu$,
  - In this limit, almost all trajectories are solutions of the 2D Euler equations.

Kuksin, S. B., & Shirikyan, A. (2012). Mathematics of two-dimensional turbulence. Cambridge University Press.

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Jet Formation in the Barotropic QG Model

In the weak forces and dissipation limit

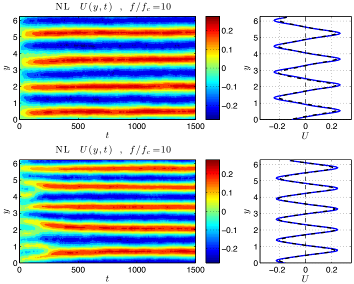


Figure by P. Ioannou (Farrell and Ioannou).

The Inertial Limit

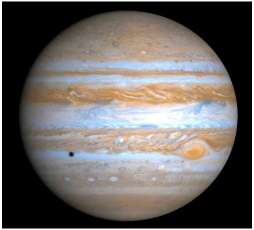
- The non-dimensional version of the barotropic QG equation.
- Quasi-Geostrophic equations with random forces

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \nu \Delta \omega - \alpha \omega + \sqrt{2\alpha} f_s,$$

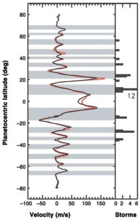
with $q = \omega + \beta' y$.

- Spin up or spin down time $= 1/\alpha \gg 1 =$ jet inertial time scale.

Weak Fluctuations around Jupiter's Zonal Jets



Jupiter's atmosphere.



Jupiter's zonal winds (Voyager and Cassini, from Porco et al 2003).

We will treat those weak fluctuations perturbatively.

Orbital Stability of the Euler or QG Equations?

- The inertial equations is an Hamiltonian system.
- Quasi-Geostrophic equations with no forces and dissipation

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = 0,$$

with $q = \omega + \beta' y$.

- It conserves energy

$$E = \frac{1}{2}\int_{\mathscr{D}} \mathbf{v}^2 \mathrm{d}\mathbf{x},$$

enstrophy, and an infinite number of Casimir invariants.

- Zonal jets or other steady states of the inertial dynamics are attractors. Orbital stability?

Outline

Averaging out the Turbulence

$$\frac{\partial q}{\partial t}+\mathbf{v}.\nabla q=\nu\Delta\omega-\alpha\omega+\sqrt{2\alpha}f_s.$$

- $P[q]$ is the PDF for the Potential Vorticity field q (a functional). Fokker–Planck equation:

$$\frac{\partial P}{\partial t}=\int \mathrm{d}\mathbf{r}\,\frac{\delta}{\delta q(\mathbf{r})}\left\{\left[\mathbf{v}.\nabla q-\nu\Delta\omega+\alpha\omega+\int \mathrm{d}\mathbf{r}'\,C(\mathbf{r},\mathbf{r}')\frac{\delta}{\delta q(\mathbf{r})}\right]P\right\}.$$

- Time scale separation. We decompose into slow (zonal flows) and fast variables (eddy turbulence)

$$q_z(y)=\langle q\rangle\equiv\frac{1}{2\pi}\int_{\mathscr{D}}\mathrm{d}x\,q \text{ and } q=q_z+\sqrt{\alpha}q_m.$$

- Stochastic reduction (Van Kampen, Gardiner, ...) using the time scale separation.
- We average out the turbulent degrees of freedom.

A New Fokker–Planck Equation for the Zonal Jets

- $R[q_z]$ is the PDF to observe the Zonal Potential Vorticity q_z:

$$\frac{1}{\alpha}\frac{\partial R}{\partial t} = \int \mathrm{d}y_1 \frac{\delta}{\delta q_z(y_1)} \left\{ \left[\frac{\partial}{\partial y} \mathbb{E}_{q_z} \left\langle v_{m,y} q_m \right\rangle + \omega_z(y_1) - \frac{\nu}{\alpha} \frac{\partial^2 q_z}{\partial y^2}(y_1) + \right.\right.$$
$$\left.\left. + \int \mathrm{d}y_2\, C_z(y_1,y_2) \frac{\delta}{\delta q_z(y_2)} \right] R \right\}.$$

- This new Fokker–Planck equation is equivalent to the stochastic dynamics

$$\frac{1}{\alpha}\frac{\partial q_z}{\partial t} = -\frac{\partial}{\partial y} \mathbb{E}_{q_z} \left\langle v_{m,y} q_m \right\rangle - \omega_z + \frac{\nu}{\alpha}\frac{\partial^2 q_z}{\partial y^2} + \eta_z,$$

with $\langle \eta_z(y,t)\eta_z(y',t')\rangle = C_z(y,y')\delta(t-t')$.

The Deterministic Part and the Quasilinear Approximation

$$\frac{1}{\alpha}\frac{\partial q_z}{\partial t} = F[q_z] - \omega_z + \frac{\nu}{\alpha}\frac{\partial^2 q_z}{\partial y^2}.$$

- $F[q_z] = -\frac{\partial}{\partial y}\mathbb{E}_{q_z}\langle v_{m,y} q_m\rangle$. The average of the Reynolds stress is over the statistics of the quasilinear dynamics:

$$\partial_t q_m + U(y)\frac{\partial q_m}{\partial x} + v_{m,y}\frac{\partial q_z}{\partial y} = \nu\Delta q_m - \alpha\omega_m + \sqrt{2\alpha} f_s.$$

and

$$\langle v_{m,y} q_m\rangle = \frac{1}{L_y}\int \mathrm{d}y\, \mathbb{E}_{q_z}[v_{m,y} q_m].$$

- We identify SSST by Farrell and Ioannou (JAS, 2003); quasilinear theory by Bouchet (PRE, 2004); CE2 by Marston, Conover and Schneider (JAS, 2008); Sreenivasan and Young (JAS, 2011).

Dynamics of the Relaxation to the Averaged Zonal Flows

The turbulence has been averaged out

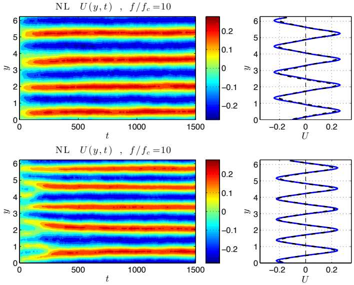


Figure by P. Ioannou (Farrell and Ioannou).

- Extremely efficient numerical simulation of the average jet dynamics.

The Stochastic Dynamics of the Zonal Jet

The turbulence has been averaged out

- We can now go further. What is the effect of the noise term?

$$\frac{1}{\alpha}\frac{\partial q_z}{\partial t} = F\left[q_z\right] - \omega_z + \frac{\nu}{\alpha}\frac{\partial^2 q_z}{\partial y^2} + \eta_z.$$

- $R\left[q_z\right]$ is the PDF to observe the Zonal Potential Vorticity q_z:

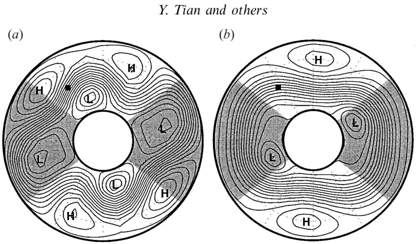
$$\frac{1}{\alpha}\frac{\partial R}{\partial t} = \int \mathrm{d}y_1 \, \frac{\delta}{\delta q_z(y_1)} \left\{ \left[\frac{\partial}{\partial y}\mathbb{E}_{q_z}\left\langle v_{m,y} q_m \right\rangle + \omega_z(y_1) - \frac{\nu}{\alpha}\frac{\partial^2 q_z}{\partial y^2}(y_1) + \right.\right.$$
$$\left.\left. + \int \mathrm{d}y_2 \, C_z(y_1,y_2) \frac{\delta}{\delta q_z(y_2)} \right] R \right\}.$$

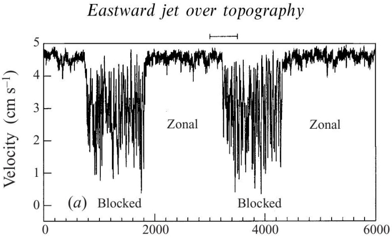
- This equation describes the zonal jet statistics and not only the mean zonal flow.
- This statistics can be nearly Gaussian, but can also be strongly non-Gaussian.

Rare Transitions in Real Flows?

Rotating tank experiments (Quasi Geostrophic dynamics)

Transitions between blocked and zonal states:





Y. Tian and col, J. Fluid. Mech. (2001) (groups of H. Swinney and M. Ghil).

Can such multiple attractors and rare transitions exists for geostrophic turbulence?

Theory based on non-equilibrium statistical mechanics?

Multiple Attractors Do Exist for the Barotropic QG Model

Two attractors for the same set of parameters

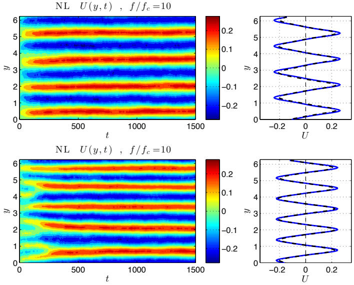


Figure by P. Ioannou (Farrell and Ioannou).

- Two attractors for the mean zonal flow for one set of parameters.
- What is the dynamics for the transition? What is the rate?

Work in Progress : Zonal Flow Instantons

Onsager Machlup formalism (50'). Statistical mechanics of histories

$$\frac{1}{\alpha}\frac{\partial q_z}{\partial t} = F[q_z] - \omega_z + \frac{\nu}{\alpha}\frac{\partial^2 q_z}{\partial y^2} + \eta_z.$$

- Path integral representation of transition probabilities:

$$P(q_{z,0}, q_{z,T}, T) = \int_{q(0)=q_{z,\mathbf{0}}}^{q(T)=q_{z,T}} \mathscr{D}[q_z] \exp\left(-\mathscr{S}[q_z]\right) \text{ with}$$

$$\mathscr{S}[q_z] = \frac{1}{2}\int_{\mathbf{0}}^{T} \mathrm{d}t \int \mathrm{d}y_{\mathbf{1}}\mathrm{d}y_{\mathbf{2}} \left[\frac{\partial q_z}{\partial t} - F[q_z] + \omega_z - \frac{\nu}{\alpha}\frac{\partial^{\mathbf{2}} q_z}{\partial y^{\mathbf{2}}}\right](y_{\mathbf{1}}) C_Z(y_{\mathbf{1}}, y_{\mathbf{2}}) \left[\frac{\partial q_z}{\partial t} - F[q_z] + \omega_z - \frac{\nu}{\alpha}\frac{\partial^{\mathbf{2}} q_z}{\partial y^{\mathbf{2}}}\right](y_{\mathbf{2}}).$$

- Instanton (or Freidlin-Wentzel theory): the most probable path with fixed boundary conditions

$$S(q_{z,0}, q_{z,T}, T) = \min_{\left\{q_z \middle| q_z(0)=q_{z,0} \text{ and } q_z(T)=q_{z,T}\right\}} \left\{\mathscr{S}[q_z]\right\}.$$

Outline

1. Basics of mid-latitude atmosphere dynamics.
2. 2D Euler and Quasi-Geostrophic Langevin dynamics: Large Deviations and Instantons.
   - Path integral representation of transition probabilities and time reversal symmetry.
   - Instantons for Langevin quasi-geostrophic dynamics (F.B., J. Laurie, and O. Zaboronski).
   - Non-Equilibrium Instantons for the 2D Navier–Stokes equations (F.B. and J. Laurie)
3. Stochastic averaging and jet formation in geostrophic turbulence.
   - The stochastic quasi-geostrophic equations.
   - Stochastic averaging (with C. Nardini and T. Tangarife).
   - Validity of this approach, and the main technical points.

The Real Issue was to Cope with UltraViolet Divergences

We have proven that they are no such divergences

$$\partial_t q_m + U(y)\frac{\partial q_m}{\partial x} + v_{m,y}\frac{\partial q_z}{\partial y} = \nu \Delta q_m - \alpha \omega_m + \sqrt{2\alpha} f_s$$

- We need to prove that the Gaussian process has an invariant measure which is well behaved in the limit $\nu \to 0$, and $\alpha \to 0$.
- This is true because the linearized Quasi-Geostrophic or Euler dynamics is non-normal.
- The result is based on asymptotics of the linearized equations:

$$v_{m,x}(y,t) \underset{t\to\infty}{\sim} \frac{v_{m,x,\infty}(y)}{t}\exp(-ikU(y)t) \text{ and } v_{m,y}(y,t) \underset{t\to\infty}{\sim} \frac{v_{m,y,\infty}(y)}{t^2}\exp(-ikU(y)t).$$

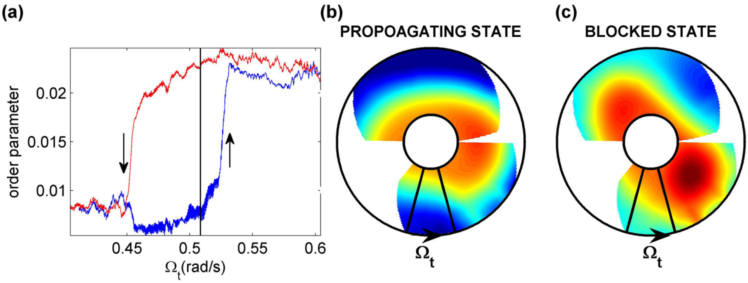
F. Bouchet and H. Morita, 2010, Physica D.

Stat. Mech. of Zonal Jets: Conclusions

- Stochastic averaging for the barotropic Quasi-Geostrophic equation leads to a non-linear Fokker-Planck equation.
- This Fokker-Planck equation predicts the Reynolds stress and jet statistics. Related to Quasilinear theory and SSST.
- For some parameters, multiple attractors are observed.
- Path integral, instanton and large deviation theories can predict rare transitions between attractors.

Bistability in a Rotating Tank Experiment

Rotating tank with a single-bump topography

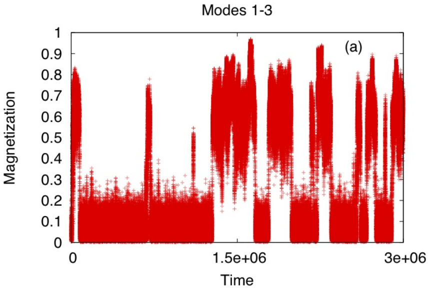


Bistability (hysteresis) in rotating tank experiments

M. MATHUR, and J. SOMMERIA, to be submitted to J. Geophys. Res., M. MATHUR, J. SOMMERIA, E. SIMONNET, and F. BOUCHET, in preparation.

Non-Equilibrium Phase Transitions for the Stochastic Vlasov Eq.

with a theoretical prediction based on non-equilibrium kinetic theory



Time series for the order parameter for the 1D stochastic Vlasov Eq.

C. NARDINI, S. GUPTA, S. RUFFO, T. DAUXOIS, and F. BOUCHET, 2012, J. Stat. Mech., L01002, and 2012 J. Stat. Mech., P12010.

Summary and Perspectives

- Non-equilibrium statistical mechanics and large deviations can be applied to geophysical turbulence and climate.

Ongoing projects and perspectives:

- Large deviations and non-equilibrium free energies for particles with long range interactions (with K. Gawedzki).
- Microcanonical measures for the Shallow Water equations (with M. Potters and A. Venaille) and for the 3D axisymmetric Euler equations (with S. Thalabard).
- Instantons for zonal jets in the quasi-geostrophic dynamics (with C. Nardini, T. Tangarife and O. Zaboronski).

F. Bouchet, and A. Venaille, Physics Reports, 2012, Statistical mechanics of two-dimensional and geophysical flows.

F. Bouchet, C. Nardini and T. Tangarife
http://hal.archives-ouvertes.fr/hal-00819779.